State complexity of the multiples of the Thue-Morse set

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- Disjoint states : $L(q) \cap L(q') = \emptyset$
- Coaccessible state, automaton

Let $b \in \mathbb{N}_{\geq 2}$

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- $u = u_1 \ldots u_n \in A^*, v = v_1 \ldots v_n \in B^*$,

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- $\text{rep}_b(n_1, n_2) = \left(0^{\ell - |\text{rep}_b(n_1)|}\,\text{rep}_b(n_1), 0^{\ell - |\text{rep}_b(n_2)|}\,\text{rep}_b(n_2)\right)$
  where $\ell = \max\{|\text{rep}_b(n_1)|, |\text{rep}_b(n_2)|\}$

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Proposition

Let $b \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}$. If $X \subseteq \mathbb{N}$ is b-recognizable, so is mX.

Theorem [Alexeev, 2004]

The state complexity of the base-b representations of $m\mathbb{N}$ is

$$\min_{N\geq 0}\left\{\frac{m}{\gcd\left(m,b^N\right)}+\sum_{n=0}^{N-1}\frac{b^n}{\gcd\left(b^n,m\right)}\right\}$$

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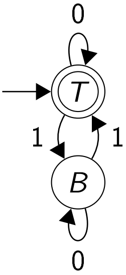

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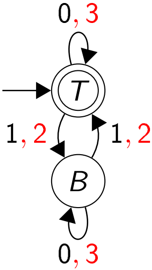

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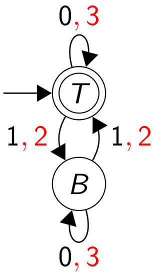

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$.
Then the state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T})$ is

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

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$\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}$	$\{(t, mt) : t \in \mathscr{T}\}$

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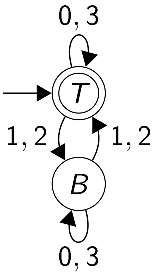
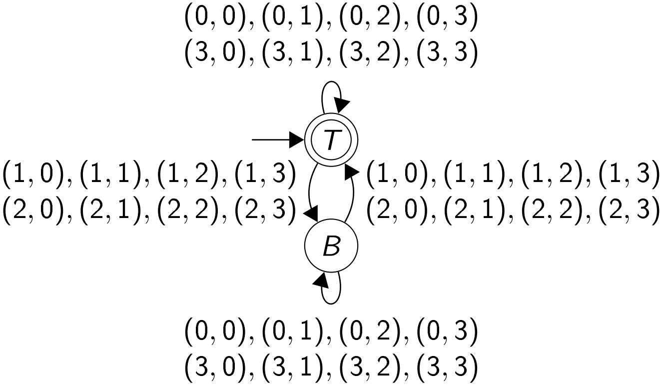
$\pi\left(\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}\right)$	$m\mathscr{T}$

The automaton $\mathscr{A}_{\mathscr{T},2^p}$: $\mathscr{T} \times \mathbb{N}$ in base 2^p



The automaton $\mathscr{A}_{\mathscr{T},2^p}$: $\mathscr{T} \times \mathbb{N}$ in base 2^p



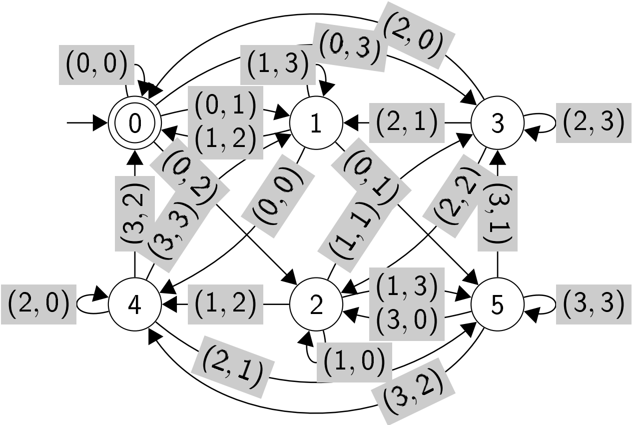
$$\delta_{\mathscr{T},2^p}(X,(d,e)) = \begin{cases} X \text{ if } d \in \mathscr{T} \\ \overline{X} \text{ otherwise} \end{cases}$$

Lemma

The automaton $\mathscr{A}_{\mathscr{T},2^p}$

- recognizes $\mathscr{T} \times \mathbb{N}$ in base 2^p
- is minimal
- is complete
- is trim
- has disjoint states

The automaton $\mathscr{A}_{m,b}$: $\{(n, mn) : n \in \mathbb{N}\}$ in base b



$$\delta_{m,b}(i,(d,e)) = j \qquad \Leftrightarrow \qquad bi + e = md + j$$

Proposition

The automaton $\mathscr{A}_{m,b}$

- is accessible
- is coaccessible
- has disjoint states
- is minimal
- is trim

The product $\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}$: $\{(t, mt) : t \in \mathscr{T}\}$ in base 2^p

$$(0, T), \ldots, (m-1, T) \qquad (0, B), \ldots (m-1, B)$$

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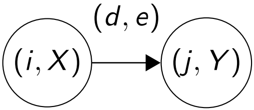
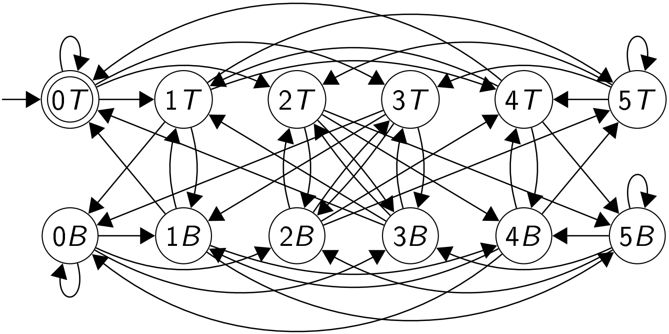
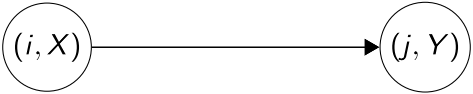


$$2^p i + e = md + j$$

$$Y = \begin{cases} X \text{ if } d \in \mathscr{T} \\ \overline{X} \text{ otherwise} \end{cases}$$





Reading (d, e) :

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Reading (u, v) :

$2^{p|(u,v)|} i + \mathrm{val}_{2^p}(v) = m\,\mathrm{val}_{2^p}(u) + j$

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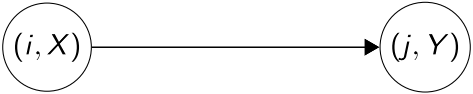

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Remark

Given i, X, v, there exist unique j, Y, u such that we have a transition labeled by (u, v) from (i, X) to (j, Y).

Proposition

The automaton $\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}$

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\begin{aligned}
(0, T) \to (i, T) &: \quad \text{rep}_{2^p}(0, i) \\
(0, T) \to (i, B) &: \quad \text{rep}_{2^p}(1, m + i)
\end{aligned}
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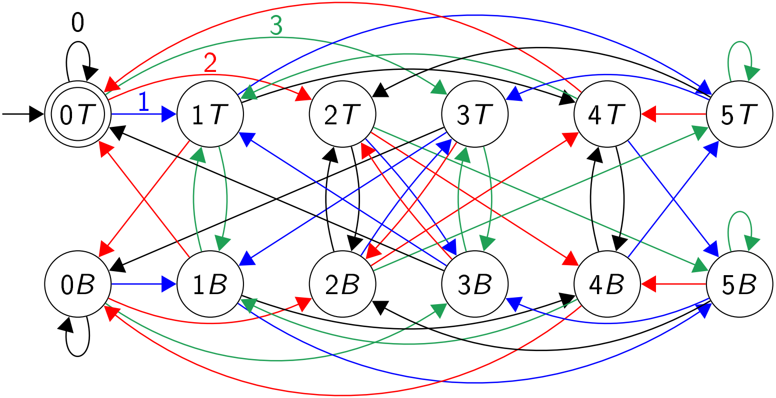
$$(0,B) \to (0,T): \quad \mathrm{rep}_{2^p}(1,m) \text{ if } p|z$$
$$\mathrm{rep}_{2^p}\left(2^{p-r}, k2^{q+1}\right) \text{ if } z = qp + r \text{ with } q \in \mathbb{N} \text{ and } r \in \{1,\dots,p-1\}$$

The automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$: $m\mathscr{T}$ in base 2^p



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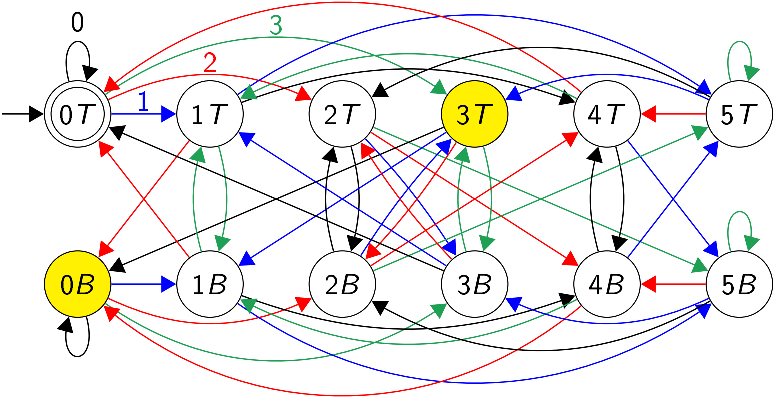
In the automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$, the states (i, T) and (i, B) are disjoint for all $i \in \{0, \ldots, m-1\}$.

The automaton $\pi(\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4})$



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Definition

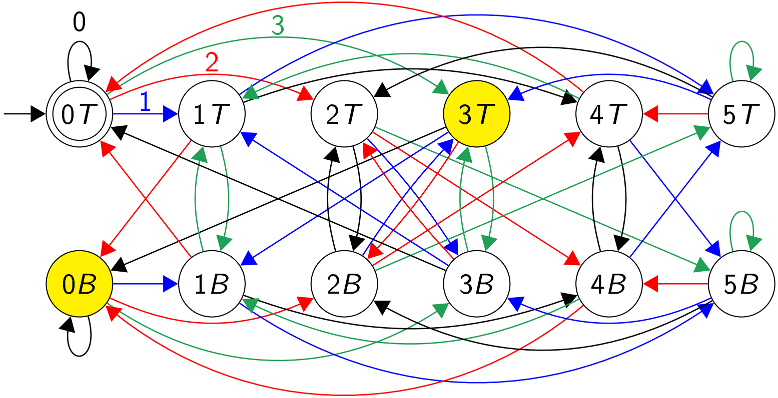
For all $j \in \{1, \ldots, k-1\}$, we set

$$[(j,T)] := \{(j + k\ell, X_\ell) : 0 \leq \ell \leq 2^z - 1\}$$
$$[(j,B)] := \left\{\left(j + k\ell, \overline{X_\ell}\right) : 0 \leq \ell \leq 2^z - 1\right\}$$

We also set

$$[(0,T)] := \{(0,T)\}$$
$$[(0,B)] := \left\{\left(k\ell, \overline{X_\ell}\right) : 0 \leq \ell \leq 2^z - 1\right\}$$

$p = 2, m = 6 = 3 \times 2^1 \Rightarrow k = 3, z = 1$



$$[(0, B)] = \left\{ \left(3\ell, \overline{X_\ell}\right) : 0 \le \ell \le 2 - 1 \right\}$$
$$= \left\{ \left(0, \overline{X_0}\right), \left(3, \overline{X_1}\right) \right\} = \{(0, B), (3, T)\}$$

Definition

For all $\alpha \in \{0, \ldots, z-1\}$, we set

$$C_\alpha := \left\{ \left(k2^{z-\alpha-1} + k2^{z-\alpha}\ell, \overline{X_\ell} \right) : 0 \leq \ell \leq 2^\alpha - 1 \right\}$$

For all $\beta \in \left\{0, \ldots, \left\lceil \frac{z}{p} \right\rceil - 2\right\}$, we set

$$\Gamma_\beta := \bigcup_{\alpha \in \{\beta p, \ldots, (\beta+1)p-1\}} C_\alpha$$

We also set

$$\Gamma_{\left\lceil \frac{z}{p} \right\rceil - 1} := \bigcup_{\alpha \in \left\{ \left(\left\lceil \frac{z}{p} \right\rceil - 1 \right) p, \ldots, z-1 \right\}} C_\alpha$$

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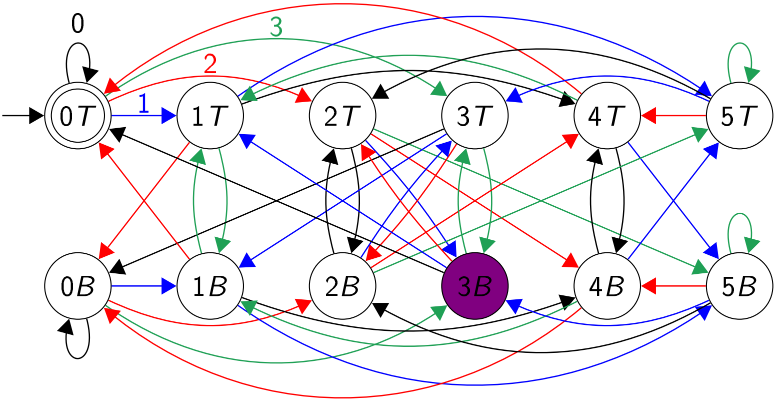
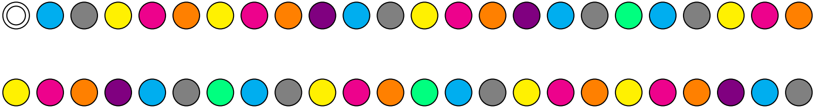


$$\Gamma_{\lceil \frac{1}{2} \rceil - 1} = \Gamma_0 = C_0 = \left\{ \left(3 \times 2^{1-0-1} + 3 \times 2^{1-0}\ell, \overline{X_\ell} \right) : 0 \leq \ell \leq 2^0 - 1 \right\}$$
$$= \{(3, B)\}$$

The automaton $\pi\left(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4}\right)$



We can build a new automaton

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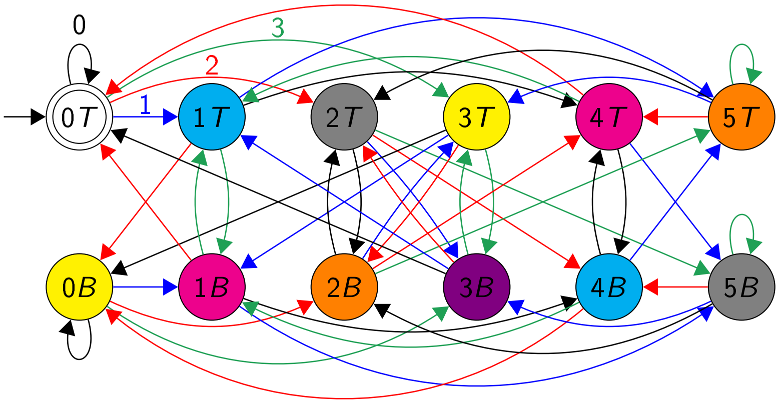
Automaton recognizing $6\mathscr{T}$ in base 4



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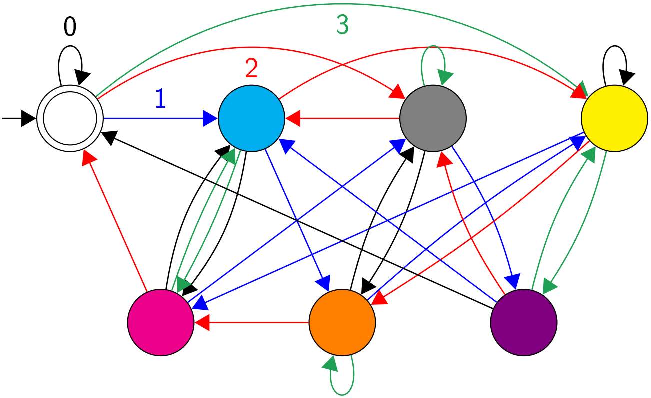

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$2 \times 3 + \left\lceil \frac{1}{2} \right\rceil = 7$

The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$