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# A lubricant boundary condition for a biological body lined by a thin heterogeneous biofilm

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We study the asymptotic behavior of an incompressible viscous fluid flow in a biological body lined by a thin biological film with a cellular microstructure, varying thickness, and a heterogeneous viscosity regulated by a time random process. Letting the thickness of the film tend to zero, we derive an effective biological slip boundary condition on the boundary of the body. This law relates the tangential fluxes to the tangential velocities via a proportional coefficient corresponding to the energy of some local problem. This law describes the ability of the biological film to function as a lubricant reducing friction at the wall of the body. The tangential velocities are functions of the random trajectories of a finely concentrated biological particle.

*Keywords*: Biological body; incompressible viscous flow; thin biofilm flow; random structural evolution; asymptotic analysis; lubricant effective boundary condition.

Mathematical Subject Classification 2010: 92C99

## 1. Introduction

The main purpose of this work is to study the asymptotic behavior of viscous flows in a biological body lined by a thin heterogeneous biofilm. Biofilms have complex structures in which cells exhibit different patterns of gene expression. Biofilms, such as the thin fluid films that line the epithelium of the eye, respiratory, gastrointestinal and urogenital tracts, are subject to considerable uncertainties and intrinsic spatial variability due to the chemical heterogeneity and the variability of the mucus concentrations (see for instance [7, 15], and [27]). Mucus is a viscous complex biogel which possesses important length-scale and shear-dependent rheological

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characteristics that enable it to function both as a lubricant at the mucosal surfaces and as a selective filter that limits the effective exposure of epithelial cells to foreign pathogens and allows rapid passage of selected gases, ions, nutrients, and many proteins (see for example [9] and [15]). Mucus is essentially composed of water, immunoglobulins, cholesterol, lipids, inorganic salts, proteins, and the high molecular weight glycoproteins known as mucins. The mucus layer is continuously produced, secreted, and shed by different mechanisms of clearance [11]. The thickness of mucus layer varies by location in the conducting airways; being anywhere from $8.3\,\mu$m in the trachea compared to about $1.8\,\mu$m in small bronchioles [20]. Physicochemical characteristics like composition, pH, ionic strength, conformation are important in the formation, function and rheological properties of mucus [16]. At acidic pH, mucins in gastric mucus change conformation from random coil to extended conformation and form a gel phase in mucus [5]. Thus, mucus exhibit a variety of rheological complexities which cannot be described even qualitatively using the Newtonian fluid behavior. Among many other works, Smith *et al.* [23] developed a mathematical model of the transport of mucus and periciliary liquid in the airways and observed that the mucus velocity based on the Newtonian fluid model is only slightly larger than the mucus velocity based on the non-Newtonian fluid model. Recently, Georgiades *et al.* [9] showed that stomach mucin solutions of up to 20 mg/mL behave as purely viscous Newtonian liquids with increasing viscosity with respect to mucin concentration, and a critical concentration is reached at the physiologically relevant concentration of 25 mg/mL, in which the solution behaves as a complex liquid exhibiting both viscous and elastic behavior. Xu *et al.* [27] examined theoretically a thin liquid biological film composed of heterogeneous solute, having a spatial distribution with prescribed statistical features, assuming that the film viscosity is determined by the concentration of the solute.

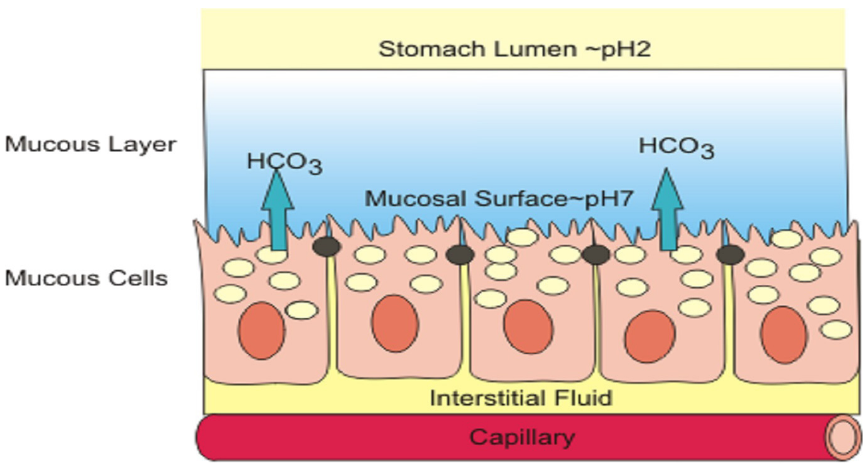


Fig. 1. A view of a stomach mucus film [25].

We consider here a biological film with a spatial heterogeneity, cellular microstructure and varying thickness $\varepsilon \in (0,1)$. We suppose, without loss of generality, that the solid surface $\Sigma$ of the biofilm lies on the plane $x_3 = 0$. We suppose that the biological fluid in the film has Newtonian rheology where, taking into account the microscale characteristics of the film the stress tensor $\tau$ is given by

$$\begin{cases} \tau = -p\mathrm{Id}_{\mathbb{R}^3} + \varepsilon \mathbf{c}\left(\dfrac{x}{\varepsilon}, \xi_{t/\varepsilon}\right) Du, \\ Du = \dfrac{1}{2}(\nabla u + \nabla^t u), \end{cases} \tag{1.1}$$

where $\mathrm{Id}_{\mathbb{R}^3}$ is the identity matrix on $\mathbb{R}^3$, $p$ is the pressure, $\nabla u$ is the gradient velocity tensor, and $\mathbf{c}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon})$ is a spatially periodic viscosity regulated by a time random process $\xi_{t/\varepsilon}$ which can be related to the random structural evolution of the biofilm. The cell-diffusion time $t/\varepsilon$ describes the time scale over which a finely concentrated particle of biological constituents will spread over a spatial microscopic periodic cell.

The main objective of the present work is to study the asymptotic behavior, as $\varepsilon$ tends to zero, of the fluid in the biofilm. Using the classical asymptotic and homogenization methods (see for instance [3]), we derive (see Theorem 2.1) the following boundary condition, which holds almost surely on the surface $\Sigma$:

$$\nu \frac{\partial u_\beta^0}{\partial x_3}\bigg|_{\Sigma} = \overline{\langle \mathcal{C} \rangle} u_\beta^0(x' + \widetilde{\sigma} W_t, 0, t) \quad \text{on } \Sigma \times (0,T); \;\; \beta = 1,2, \tag{1.2}$$

where $\nu$ is a constant parameter representing the cinematic viscosity of the Newtonian fluid in the biological body lined by the biofilm, $\overline{\langle \mathcal{C} \rangle}$ is defined in (A.34), $\widetilde{\sigma}$ is the matrix defined in $(\text{A.26})_2$, and $W_t$ is the standard 2-dimensional Brownian motion. If $\xi_t$ is a diffusion process with

$$d\xi_t = \mathbf{b}(\xi_t)dt + \lambda(\xi_t)dW_t,$$

then (see Remark A.9), replacing condition (A2) by some effectively verifiable sufficient condition in terms of the coefficients of the generator, we obtain the same result. The effective slip boundary condition (1.2) relates the shear stress $\nu \frac{\partial u_\beta^0}{\partial x_3}|_\Sigma$ to the sliding speed $u_\beta^0(x' + \widetilde{\sigma} W_t, 0, t)$; $\beta = 1, 2$, through the coefficient $\overline{\langle \mathcal{C} \rangle}$ which depends on the lubricant viscosity of the flow in the biological film via relation (A.34). This boundary condition can be interpreted as a time random lubricant law on the wall of the biological body. This law confirms the slip model proposed by [24] and the study of thin film hydrodynamic lubrication models in [10]. This boundary slippage condition is a result of the differences in the affinity of the biological fluid with the solid surface, the viscous forces of the fluid flows, and the random trajectories of the biological fluid particles. Here, the slip velocity $u_\beta^0(x' + \widetilde{\sigma} W_t, 0, t)$; $\beta = 1, 2$, depends on the random trajectory $x' + \widetilde{\sigma} W_t$ of a finely concentrated biological particle on $\Sigma$ released initially from some point $x'$. Due to the presence of the Brownian motion $W_t$, the velocity $u^0_\beta(x' + \widetilde{\sigma} W_t, 0, t)$; $\beta = 1, 2$, is characterized by the following Itô formula:

$$du^0_\alpha = \left( \frac{\partial u^0_\alpha}{\partial t} + \frac{1}{2} \sum_{\substack{l,m=1,2 \\ j=1,2,3}} \sigma_{lj}\sigma_{mj} \frac{\partial^2 u^0_\alpha}{\partial x_l \partial x_m} \right) dt + \overline{\sigma} \nabla_{x'} u_\alpha . dW_t; \quad \alpha = 1, 2,$$

where $\sigma$ is the square root of the matrix $\sigma^2$ defined in (A.25) and $\overline{\sigma}$ is the matrix defined in $(\text{A.26})_1$. The effects of mechanical stress on lung airway epithelia have been studied by several authors, among which [8] and [21]. In [8], the author showed that, during normal breathing, airflow across the surface of the airway epithelium produces a wall shear stress with varying magnitude from the large to the small generations of the airways. In [21], the authors studied a model airflow-related shear stress, which depends on the velocity and viscosity of the air, during heterogeneous constriction and mechanical ventilation. They concluded that elevated airflow-related shear stress on the epithelial cell layer can occur during heterogeneous constriction and conjecture that this may constitute a mechanism contributing to ventilator-induced lung injury. The present work can be extended to biofilms which exhibit a viscous non-Newtonian behavior, which obey the power law:

$$\tau = -p\mathrm{Id}_{\mathbb{R}^3} + \varepsilon^{r-1} \mathbf{c}\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon}\right) |Du|^{r-2} Du, \tag{1.3}$$

where $\tau$ is the Cauchy stress and $r \in (1, 2)$. The parameter $r$ may depend on the temperature (see [1]) or on the pressure (see [19]) in the fluid flows. With slight modifications in the auxiliary problems introduced in Sec. A.3 of the Appendix, we can derive the following nonlinear lubricant law on $\Sigma$:

$$\nu \left.\frac{\partial \overline{u}^0}{\partial x_3}\right|_\Sigma = K_r |\overline{u}^0|^{r-1} \overline{u}^0(x' + \widetilde{\sigma} W_t, 0, t) \quad \text{on } \Sigma \times (0, T), \tag{1.4}$$

where $\overline{u}^0 = \left(u^0_1, u^0_2\right)$ and $K_r$ is a constant depending on $r$.

The paper is organized as follows: in Sec. 2, we introduce necessary notations, describe the structure of the biofilm, pose the problem to be studied, make assumptions on the random fields under consideration and formulate the main result of this work. In Sec. 3, we present our conclusion. The last section, which is devoted to the appendix, contains 3 Subsections. Appendix A.1, contains intermediate technical results and their proofs. In Appendix A.2, we establish compactness results for the solution of the original problem. In Appendix A.3, we study some local problems which are crucial in constructing test-functions, which depend on the trajectory of the process $\xi_t$ after time $t/\varepsilon$, in order to pass to the limit in the original problem.

## 2. Statement of the Problem and the Main Result

Define the set

$$\mathbb{R}^{3-} = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3; x_3 < 0\},$$

and let $\Omega$ be a bounded open subset in $\mathbb{R}^{3-}$ with Lipschitz continuous boundary $\partial\Omega$. We suppose that the set $\Sigma = \partial\Omega \cap \{x_3 = 0\}$ is a smooth surface. Furthermore, assume that the boundary $\partial\Omega\backslash\Sigma$ can be represented by a smooth negative function $x' = (x_1, x_2) \to a(x')$. Let $Y = (-\frac{1}{2}, \frac{1}{2})^2$ be the 2-dimensional unit reference cell and $Z = Y \times (-1, 0)$. For every $\varepsilon \in (0, 1)$, we consider the cells:

$$Y_{ij}^{\varepsilon} = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)^2 + (i\varepsilon, j\varepsilon), \quad Z_{ij}^{\varepsilon} = Y_{ij}^{\varepsilon} \times (-\varepsilon, 0) \quad \forall\, i, j \in \mathbb{Z}. \tag{2.1}$$

and define the set $I_\varepsilon \subset \mathbb{Z}^2$ as follows:

$$I_\varepsilon = \left\{(i,j) \in \mathbb{Z}^2 \,|\, Y_{ij}^{\varepsilon} \subset \Sigma\right\}. \tag{2.2}$$

Moreover, we define

$$Y^{\varepsilon} \bigcup_{(i,j)\in I_\varepsilon} Y_{ij}^{\varepsilon}, \quad \Sigma_\varepsilon = Y^{\varepsilon} \times (-\varepsilon, 0). \tag{2.3}$$

We also set

$$\Omega_\varepsilon = \Omega\backslash\overline{\Sigma}_\varepsilon, \quad \Gamma_\varepsilon = \partial\Omega_\varepsilon \cap \partial\Sigma_\varepsilon. \tag{2.4}$$

For convenience, we suppose that for every $\varepsilon \in (0, 1)$,

$$\Omega = \Omega_\varepsilon \cup (Y^{\varepsilon} \times [-\varepsilon, 0)). \tag{2.5}$$

Finally, we suppose that $\Omega$ is filled with a slow viscous incompressible fluid flow and that the inertia effects are negligible in $\Omega$ in such a way that the flow is governed by the unsteady Stokes equations:

$$\begin{cases} \dfrac{\partial u^{\varepsilon}}{\partial t} - \nu\Delta u^{\varepsilon} + \nabla p^{\varepsilon} = f & \text{in } \Omega_\varepsilon \times (0,T), \\ \dfrac{\partial u^{\varepsilon}}{\partial t} - \varepsilon \operatorname{div}\left(\mathbf{c}\left(\dfrac{x}{\varepsilon}, \xi_{t/\varepsilon}\right)\nabla u^{\varepsilon}\right) + \nabla p^{\varepsilon} = f & \text{in } \Sigma_\varepsilon \times (0,T), \\ \operatorname{div} u^{\varepsilon} = 0 & \text{in } \Omega, \\ u^{\varepsilon}(x, 0) = \mathbf{u}(x) & \text{on } \Omega, \end{cases} \tag{2.6}$$

with the transmission and boundary conditions

$$\begin{cases} [u^{\varepsilon}]_{\Gamma_\varepsilon} = 0 & \text{on } \Gamma_\varepsilon \times (0,T), \\ \nu\dfrac{\partial u^{\varepsilon}}{\partial n} - \varepsilon\mathbf{c}\left(\dfrac{x}{\varepsilon}, \xi_{t/\varepsilon}\right)\dfrac{\partial u^{\varepsilon}}{\partial n} = 0 & \text{on } \Gamma_\varepsilon \times (0,T), \\ u^{\varepsilon} = 0 & \text{on } (\partial\Omega\backslash\Sigma) \times (0,T), \\ u_3^{\varepsilon} = 0 & \text{on } \Sigma, \end{cases} \tag{2.7}$$

where $f \in L^2(\Omega, \mathbb{R}^3)$, $[u^{\varepsilon}]_{\Gamma_\varepsilon}$ is, for almost every $t \in [0,T]$, the jump of $u^{\varepsilon}$ across $\Gamma_\varepsilon$; that is the difference of the two traces of $u^{\varepsilon}$ on $\Gamma_\varepsilon$, $n$ is the outward unit normal, $\nu$ is the constant viscosity in $\Omega_\varepsilon \times (0,T)$, and $\mathbf{c}(z, \varsigma) = (c_{ij}(z, \varsigma))_{i,j=1,2,3}$; $z \in \mathbb{R}^3$ and $\varsigma \in \mathbb{R}^d$; $d$ being a positive integer, is a **measurable** $3 \times 3$ symmetric matrix corresponding to the viscosity in $\Sigma_\varepsilon \times (0,T)$. The time dependence of $\mathbf{c}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon})$ is governed by a stationary and ergodic random process $\xi_t$, which is defined on a probability space $(\Pi, \Upsilon, \mathbf{P})$ with values in $\mathbb{R}^d$, where $\Upsilon$ is a $\sigma$-algebra of subsets of $\Pi$ supplied with the probability measure $\mathbf{P}$. Let $(\mathcal{T}_x)_{x\in\mathbb{R}}$ be a dynamical system on $(\Pi, \Upsilon)$. That is, a family of operators satisfying for every $x, y \in \mathbb{R}$,

$$\begin{cases} \mathcal{T}_0 = \mathrm{Id}(\mathrm{Id} \text{ is the identical mapping}), \\ \mathcal{T}_{x+y} = \mathcal{T}_x \circ \mathcal{T}_y, \\ \mathbf{P}\big(\mathcal{T}_x^{-1} A\big) = \mathbf{P}(A), \quad \forall\, A \in \Upsilon \end{cases} \tag{2.8}$$

and such that, for every $A \in \Upsilon$, the set $\{(x,\omega) \in \mathbb{R} \times \Pi \,|\, \mathcal{T}_x\omega \in A\}$ is $dx \otimes d\mathbf{P}$-measurable. We suppose that $\mathcal{T}$ is ergodic (or metrically transitive) in the sense that for every set $A \in \Upsilon$ such that $\mathcal{T}_x A = A$, for every $x \in \mathbb{R}$, has a probability $\mathbf{P}(A)$ equal to 0 or 1. We suppose that $\xi_t$ is a stationary and ergodic process, that is, for every $n \in \mathbb{N}$, for every $t$, $t_1, \ldots, t_n \in \mathbb{R}$, and for every $B \in \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel $\sigma$-algebra on $\mathbb{R}^d$,

$$\mathbf{P}(\{\omega \,|\, \xi_{t+t_1}(\omega), \ldots, \xi_{t+t_n}(\omega) \in B\}) = \mathbf{P}(\{\omega \,|\, \xi_{t_1}(\mathcal{T}_t\omega), \ldots, \xi_{t_n}(\mathcal{T}_t\omega) \in B\}) \tag{2.9}$$

and (see for instance [6] and [13])

$$\lim_{L\to\infty} \frac{1}{L} \int_t^L \xi_t(\omega)dt = \mathbf{E}(\xi), \tag{2.10}$$

almost surely. Let $\mathcal{F}_{\le t}$ be the $\sigma$-algebra generated by $\xi_s$ for all $s \le t$ and $\mathcal{F}_{\ge t}$ be the $\sigma$-algebra generated by $\xi_s$ for all $s \ge t$. The *strong mixing coefficient* of the process $\xi_s$ is the function

$$\kappa(s) = \sup_{A\in\mathcal{F}_{\le t}, B\in\mathcal{F}_{\ge t+s}} |\mathbf{P}(A)\mathbf{P}(B) - \mathbf{P}(A\cap B)|. \tag{2.11}$$

The *uniform mixing coefficient* of the process $\xi_s$ is defined by

$$\phi(s) = \sup_{\substack{A\in\mathcal{F}_{\le t}, B\in\mathcal{F}_{\ge t+s} \\ P(B)\neq 0}} \left|\mathbf{P}(A) - \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(B)}\right| \tag{2.12}$$

and the *maximal correlation coefficient* of the process $\xi_s$ is defined through

$$\rho(s) = \sup_{\eta_1\in L^2(\Pi,\mathcal{F}_{\le t},P), \eta_2\in L^2(\Pi,\mathcal{F}_{\ge t+s},P)} \left|\frac{\mathbf{E}((\eta_1 - \mathbf{E}\eta_1)(\eta_2 - \mathbf{E}\eta_2))}{\sqrt{\mathbf{E}\big(\eta_1^2\big)\mathbf{E}\big(\eta_2^2\big)}}\right|. \tag{2.13}$$

We make the following assumptions:

(A1) There exists a positive constant $C$ such that for all $z \in \mathbb{R}^3$, $\varsigma \in \mathbb{R}^d$ and $\upsilon \in \mathbb{R}^3$,

$$|\mathbf{c}(z,\varsigma)| \le C, \quad \nu_{ij}(z,\varsigma)\upsilon_i\upsilon_j \ge C|\upsilon|^2, \tag{2.14}$$

where the summation convention with respect to repeated indices is used in the left-hand side of the second inequality.

(A2) At least one of the following relations holds:

$$\int_0^\infty \sqrt{\kappa(s)}ds < \infty, \quad \int_0^\infty \sqrt{\phi(s)}ds < \infty, \quad \int_0^\infty \sqrt{\rho(s)}ds < \infty. \tag{2.15}$$

Motivation for different scaling in $\Omega_\varepsilon$ and $\Sigma_\varepsilon$ allows for different values of the characteristic Fr (Froude), Eu (Euler), Re (Reynolds), and Sr (Strouhal) numbers. The assumption that the inertia effects are negligible in $\Sigma_\varepsilon$ leads to low Froude and Reynolds numbers and high Euler number (see for instance [22]) in $\Sigma_\varepsilon$. We assumed here that

$$\frac{\mathrm{Fr}}{\mathrm{Re}} = \varepsilon \quad \text{and} \quad \mathrm{Fr} \times \mathrm{Eu} = 1 \quad \text{in } \Sigma_\varepsilon. \tag{2.16}$$

On the other hand, according to [22],

$$\mathrm{Sr} + 1 = \mathrm{Eu} + \frac{1}{\mathrm{Re}} + \frac{1}{\mathrm{Fr}},$$

which implies, together with (2.16), that

$$\mathrm{FrSr} = O(1) \quad \text{in } \Sigma_\varepsilon.$$

Under the assumption (A1), for any initial condition $\mathbf{u} \in L^2(\Omega, \mathbb{R}^3)$ with $\operatorname{div} \mathbf{u} = \mathbf{0}$, and any $\varepsilon \in (0,1)$, problem (2.6)–(2.7) has **almost surely** a unique solution $(u^\varepsilon, p^\varepsilon)$ (see for example [26]) such that

$$\begin{cases} u^\varepsilon \in L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)) \cap L^\infty([0,T], L^2(\Omega)), \\ \dfrac{\partial u^\varepsilon}{\partial t} \in L^2(0,T,H^{-1}(\Omega,\mathbb{R}^3)), \\ p^\varepsilon \in L^2(0,T,L^2(\Omega,\mathbb{R})), \end{cases}$$

where

$$\mathbf{H}(\Omega,\mathbb{R}^3) = \{v \in H^1(\Omega,\mathbb{R}^3) \mid \operatorname{div}(v) = 0 \text{ in } \Omega, v = 0 \text{ on } \partial\Omega\backslash\Sigma, v_3 = 0 \text{ on } \Sigma\}. \tag{2.17}$$

We define for every $(x', x_3) \in \Omega$; $x' = (x_1, x_2)$ and every $(i,j) \in I_\varepsilon$,

$$T_\varepsilon(x) = \begin{cases} x & \text{if } x \in \Omega\backslash\Sigma_\varepsilon, \\ \left(x', (a(x') + \varepsilon)\dfrac{x_3}{a(x')} - \varepsilon\right) & \text{if } x \in \Omega \cap Z^\varepsilon_{ij}. \end{cases} \tag{2.18}$$

Observe that $T_\varepsilon$ is an invertible map from $\Omega$ into $\Omega_\varepsilon$, such that, for every $u \in H^1(\Omega_\varepsilon, \mathbb{R}^3)$,

$$\|u \circ T_\varepsilon\|_{H^1(\Omega;\mathbb{R}^3)} \leq C\|u\|_{H^1(\Omega_\varepsilon,\mathbb{R}^3)}, \tag{2.19}$$

where $C$ is a positive constant independent on $\varepsilon$. Our main result in this paper reads as follows:

**Theorem 2.1.** *Let* $(u^\varepsilon, p^\varepsilon)$ *be the solution of problem* (2.6)–(2.7). *Assume that assumptions* (*A*1)–(*A*2) *are fulfilled. Then, for* $\mathbf{P}$*-almost all* $\omega$,

$$u^\varepsilon \circ T^\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u^0 \quad \text{in } L^2(0,T,H^1(\Omega\backslash\Sigma,\mathbb{R}^3))\text{-weak},$$

$$\begin{aligned} u^\varepsilon &\underset{\varepsilon\to 0}{\to} u^0 \quad \textit{in } L^2(\Omega\times(0,T),\mathbb{R}^3)\textit{-strong},\\ \frac{\partial u^\varepsilon}{\partial t} &\underset{\varepsilon\to 0}{\rightharpoonup} \frac{\partial u^0}{\partial t} \quad \textit{in } L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)^*)\textit{-weak},\\ \overline{p^\varepsilon} &\underset{\varepsilon\to 0}{\to} p^0 \quad \textit{in } L^2(\Omega\times(0,T),\mathbb{R}^3)\textit{-strong}, \end{aligned} \tag{2.20}$$

*where* $(u^0, p^0)$ *is the solution of the problem*

$$\begin{cases} \dfrac{\partial u^0}{\partial t} - \nu\Delta u^0 + \nabla p^0 = f & \textit{in } \Omega\times(0,T),\\ \operatorname{div} u^0 = 0 & \textit{in } \Omega,\\ u^0_\alpha = 0 & \textit{on } \partial\Omega\backslash\Sigma;\ \ \alpha = 1,2,\\ u^0_3 = 0 & \textit{on } \partial\Omega,\\ \nu\dfrac{\partial u^0_\beta}{\partial x_3}\bigg|_\Sigma = \overline{\langle\mathcal{C}\rangle} u^0_\beta(t) & \textit{on } \Sigma,\\ u^0(x,0) = \mathbf{u}(x) & \textit{for } x\in\Omega, \end{cases} \tag{2.21}$$

*where*

$$u^0_\beta(t) = u^0_\beta((x' + \widetilde{\sigma} W_t, 0, t)); \quad \beta = 1,2.$$

## 3. Conclusion

Biological fluid film in natural environments, such as the fluid films that line the epithelium of the eye, respiratory, gastrointestinal and urogenital tracts, are subject to considerable uncertainties and intrinsic spatial variability due to the biological and chemical heterogeneity and the random evolution of the mucus concentrations. In this paper, we considered a body lined by a thin heterogeneous biological film with a cellular microstructure and varying thickness $\varepsilon > 0$. The viscous fluid in the biological film is supposed to have Newtonian rheology but contains biological and chemical components with spatially periodic viscosity $\mathbf{c}(\frac{x}{\varepsilon}, \xi_{t/\varepsilon})$ regulated by a random time process $\xi_{t/\varepsilon}$, which can be related, for example, to the random structural evolution of the biofilm. Letting the thickness $\varepsilon$ tend to zero, we derived the following effective biological slip boundary condition on the wall of the body:

$$\nu\frac{\partial u^0_\beta}{\partial x_3}\bigg|_\Sigma = \overline{\langle\mathcal{C}\rangle} u^0_\beta(x' + \widetilde{\sigma} W_t, 0, t) \quad \text{on } \Sigma\times(0,T),$$

where the proportional coefficient $\overline{\langle\mathcal{C}\rangle}$ is defined in (A.34), $\widetilde{\sigma}$ is the matrix defined in $(A.26)_2$, and $W_t$ is the standard 2-dimensional Brownian motion. The tangential velocity $u^0_\beta(x' + \widetilde{\sigma} W_t, 0, t)$; $\beta = 1, 2$, is a function of the random trajectory $x' + \widetilde{\sigma} W_t$ of a finely concentrated particle on $\Sigma$ released initially from some point $x'$.

## Appendix A

### A.1. *Intermediate results*

Let $\zeta(z, s)$ be an ergodic stationary process in $s \in (0, \infty)$ with values in the space $L^2_{\text{per}}(Z)$ of $z$-periodic functions of $L^2(Z)$, such that

$$\|\zeta\|_{L^2(Z\times(0,1))} < \infty. \tag{A.1}$$

An example of such process is $\zeta(z, s) = a(z)\xi_s$, where $a \in L^2_{\text{per}}(Z)$ and $\xi$ is a Gaussian white noise with $\langle \xi_s \xi_{s'} \rangle = \delta(s - s')$; $\langle . \rangle$ being the average over different realizations of the random process $\xi$. We have the following result:

**Lemma A.1.** *Let* $\zeta^\varepsilon(x, s) = \zeta(\frac{x}{\varepsilon}, \frac{s}{\varepsilon})$. *Then*

(1) *For every* $\varphi \in C\big([0, T], C^1_0(\mathbb{R}^3)\big)$, *we have, with probability* 1,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \int_{\Sigma_\varepsilon} \varphi \zeta^\varepsilon(x, s) dx ds = \overline{\langle \zeta \rangle} \int_0^t \int_\Sigma \varphi(x', 0, s) dx' ds,$$

*where* $\overline{\langle \zeta \rangle} = \mathbf{E}(\int_Z \zeta(z, 0) dz)$.

(2) *For every sequence* $(v_\varepsilon)_\varepsilon \subset L^2(\Omega \times (0, T))$ *such that, for every* $t \in [0, T]$,

$$\sup_\varepsilon \frac{1}{\varepsilon} \int_0^t \int_{\Sigma_\varepsilon} (v_\varepsilon(x, s))^2 dx ds < +\infty,$$

*there exists a subsequence of* $(v_\varepsilon)_\varepsilon$, *still denoted in the same way, and* $v \in L^2(\Sigma \times (0, T))$, *such that, with probability* 1,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \int_{\Sigma_\varepsilon} v_\varepsilon \varphi \zeta^\varepsilon dx ds = \overline{\langle \zeta \rangle} \int_0^t \int_\Sigma v(x', s) \varphi(x', 0, s) dx' ds,$$

*for every* $\varphi \in C([0, T], C^1_0(\mathbb{R}^3))$.

**Proof.** (1) Let $\varphi \in C\big([0, T], C^1_0(\mathbb{R}^3)\big)$ and $t \in (0, T]$. Let $N(\varepsilon) = [1/\sqrt{\varepsilon}]$, where $[z]$ denotes the floor of $z$. Let $\pi = (t_k)^N_{k=0}$ be a partition of $[0, t]$, such that $t_k - t_{k-1} = \sqrt{\varepsilon}$, and $\phi = (s_k)^N_{k=1}$; $s_k \in [t_{k-1}, t_k]$ be a selection associated to $\pi$. Then

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \int_{\Sigma_\varepsilon} \varphi \zeta^\varepsilon(x, s) dx ds \\
&\quad = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \sum_{k=1}^{N(\varepsilon)} \int_{\Sigma_\varepsilon} \varphi(x, s_k) \int_{t_{k-1}}^{t_k} \zeta^\varepsilon(x, s) dx ds \\
&\quad = \lim_{\varepsilon\to 0} \sum_{k=1}^{N(\varepsilon)} \sum_{(i,j)\in I_\varepsilon} \varepsilon^2 \varphi(i\varepsilon, j\varepsilon, 0, s_k) \int_{t_{k-1}}^{t_k} \int_Z \zeta\left(z, \frac{s}{\varepsilon}\right) dz ds
\end{aligned}$$

$$= \lim_{\varepsilon\to 0}\sum_{k=1}^{N(\varepsilon)}\sum_{(i,j)\in I_\varepsilon}\left(\begin{aligned}&\varepsilon^2(t_k-t_{k-1})\varphi(i\varepsilon,j\varepsilon,0,s_k)\\&\qquad\times\int_0^1\int_Z\zeta\left(z,\frac{s(t_k-t_{k-1})}{\varepsilon}+\frac{t_{k-1}}{\varepsilon}\right)dzds\end{aligned}\right)$$

$$= \lim_{\varepsilon\to 0}\sum_{k=1}^{N(\varepsilon)}\sum_{(i,j)\in I_\varepsilon}\left(\begin{aligned}&\varepsilon^2\sqrt{\varepsilon}\varphi(i\varepsilon,j\varepsilon,0,s_k)\\&\qquad\times\int_0^1\int_Z\zeta\left(z,\frac{s}{\sqrt{\varepsilon}}+\frac{t_{k-1}}{\varepsilon}\right)dzds\end{aligned}\right), \tag{A.2}$$

where we have introduced the change of variables $s\mapsto s(t_k-t_{k-1})+t_{k-1}$, $z_1=\frac{x_1-i\varepsilon}{\varepsilon}$, $z_2=\frac{x_2-j\varepsilon}{\varepsilon}$, and $z_3=\frac{x_3}{\varepsilon}$. As $\zeta$ is a stationary ergodic process and, according to property $(2.8)_3$, $\mathcal{T}_x$ preserves the probability $\mathbf{P}$, we have almost surely, using the properties (2.9)–(2.10), that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_0^1\int_Z\zeta\left(z,\frac{s}{\sqrt{\varepsilon}}+\frac{t_{k-1}}{\varepsilon},\omega\right)dzds\\&\quad=\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_0^{1/\sqrt{\varepsilon}}\int_Z\zeta\left(z,s+\frac{t_{k-1}}{\varepsilon},\omega\right)dzds\\&\quad=\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_0^{1/\sqrt{\varepsilon}}\int_Z\zeta(z,s,\mathcal{T}_{t_{k-1}/\varepsilon}\omega)dzds\\&\quad=\lim_{\varepsilon\to 0}\sqrt{\varepsilon}\int_0^{1/\sqrt{\varepsilon}}\int_Z\zeta(z,s,\omega)dzds\\&\quad=\overline{\langle\zeta\rangle},\end{aligned}$$

which implies, together with (A.2), that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon}\varphi\zeta^\varepsilon(x,s)dxds=\overline{\langle\zeta\rangle}\int_0^t\int_\Sigma\varphi(x',0,s)dx'ds.$$

(2) Let us now introduce the following measure:

$$\mu_\varepsilon^\pm=\frac{\mathbf{1}_{\Sigma_\varepsilon}(x)\mathbf{1}_{(0,t)}(s)\zeta^\pm\left(\frac{x}{\varepsilon},\frac{s}{\varepsilon}\right)dxds}{\varepsilon},$$

where $\zeta^+(z,s)=\max(\zeta(z,s),0)$, $\zeta^-(z,s)=\max(-\zeta(z,s),0)$, and $\mathbf{1}_A$ is the indicator function of the set $A$. We deduce from the above computations that $\left(\mu_\varepsilon^\pm\right)_\varepsilon$ converges in the weak sense of measures to the measure:

$$\mu^\pm=\overline{\langle\zeta^\pm\rangle}\mathbf{1}_\Sigma(x')\mathbf{1}_{(0,t)}(s)dx'ds,$$

when $\varepsilon$ goes to 0. Observing that

$$\begin{aligned}\int_{\mathbb{R}^4}|v_\varepsilon|d\mu_\varepsilon^\pm&\le\|\zeta\|_{L^2(Z\times(0,\infty))}\left(\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon}(v_\varepsilon(x,s))^2dxds\right)^{\frac{1}{2}}\\&\le C\left(\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon}(v_\varepsilon(x,s))^2dxds\right)^{\frac{1}{2}},\end{aligned}$$

we deduce, using the hypothesis on $(v_\varepsilon)_\varepsilon$, that the sequence $\left(v_\varepsilon\mu_\varepsilon^\pm\right)_\varepsilon$ converges, up to some subsequence, to some measure $\chi$, in the weak sense of measures. According to Fenchel's inequality, we have, for every $\varphi \in C_0^1(\mathbb{R}^4)$,

$$2\int_{\mathbb{R}^4} v_\varepsilon \varphi d\mu_\varepsilon^\pm - \int_{\mathbb{R}^4} (\varphi)^2 d\mu_\varepsilon^\pm \le \int_{\mathbb{R}^4} (v_\varepsilon)^2 d\mu_\varepsilon^\pm .$$

Then, passing to the limit, we get the following:

$$2\langle \chi, \varphi\rangle - \int_{\mathbb{R}^4} \varphi^2(x', 0, s) d\mu^\pm \le \liminf_{\varepsilon\to 0} \int_{\mathbb{R}^4} (v_\varepsilon)^2 d\mu_\varepsilon^\pm < +\infty.$$

This implies that

$$\sup\left\{ \langle \chi, \varphi\rangle \,|\, \varphi \in C_0^1(\mathbb{R}^4), \int_{\mathbb{R}^4} \varphi^2(x', 0, s) d\mu^\pm < +\infty \right\} < +\infty.$$

Thus, using Riesz' representation Theorem, we can identify $\chi$ with $v\mu^\pm$, for some $v \in L^2(\Sigma \times (0, T))$. As $\zeta = \zeta^+ - \zeta^-$, we obtain the result. □

### A.2. *Compactness results*

**Lemma A.2.** *Under assumption* (*A*1), *there exists almost surely a positive constant $C$ independent on $\varepsilon$, such that, for every $t \in [0, T]$, and every $\omega \in \Pi$,*

$$\begin{aligned} \int_0^t \int_{\Omega_\varepsilon} |\nabla u^\varepsilon|^2 dx ds + \varepsilon \int_0^t \int_{\Sigma_\varepsilon} |\nabla u^\varepsilon|^2 dx ds &\le C, \\ \int_\Omega |u^\varepsilon(x, t)|^2 dx + \int_0^t \int_{\Omega_\varepsilon} |u^\varepsilon|^2 dx ds &\le C, \\ \frac{1}{\varepsilon} \int_0^t \int_{\Sigma_\varepsilon} |u^\varepsilon|^2 dx ds &\le C. \\ \left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)^*)} &\le C, \end{aligned} \tag{A.3}$$

*where $\nabla$ is the gradient operator with respect to $x$ and $\mathbf{H}(\Omega, \mathbb{R}^3)^*$ is the topological dual of $\mathbf{H}(\Omega, \mathbb{R}^3)$.*

**Proof.** Let $t \in [0, T]$ and $s \in (0, t)$. Let $x' \in Y_{ij}^\varepsilon$; $(i, j) \in I_\varepsilon$, we write, for $x_3$ in the interval $(-\varepsilon, 0)$,

$$u^\varepsilon(x', x_3, s) = u^\varepsilon(x', -\varepsilon, s) + \int_{-\varepsilon}^{x_3} \frac{\partial u^\varepsilon}{\partial x_3}(x', z, s) dz. \tag{A.4}$$

Using Young's inequality, we derive

$$|u^\varepsilon(x', x_3, s)|^2 \le C\left( |u^\varepsilon(x', -\varepsilon, s)|^2 + \varepsilon \int_{-\varepsilon}^0 \left|\frac{\partial u^\varepsilon}{\partial x_3}\right|^2 dz \right), \tag{A.5}$$

from which we deduce that

$$\int_{Z_{ij}^{\varepsilon}} |u^{\varepsilon}(x,s)|^2 dx \leq C\left(\varepsilon \int_{Y_{ij}^{\varepsilon}} |u^{\varepsilon}(x',-\varepsilon,s)|^2 dx'\right) + C\left(\varepsilon^2 \int_{Z_{ij}^{\varepsilon}} |\nabla u^{\varepsilon}(x,s)|^2 dx\right) \tag{A.6}$$

and, summing over $I_\varepsilon$

$$\int_{\Sigma_\varepsilon} |u^{\varepsilon}(x,s)|^2 dx \leq C\left(\varepsilon \sum_{i,j} \int_{Y_{ij}^{\varepsilon}} |u^{\varepsilon}(x',-\varepsilon,s)|^2 dx'\right) + C\left(\varepsilon^2 \int_{\Sigma_\varepsilon} |\nabla u^{\varepsilon}(x,s)|^2 dx\right). \tag{A.7}$$

On the other hand, as $u^\varepsilon = 0$ on $\partial\Omega_\varepsilon \cap \partial\Omega$, there exists a positive constant $C$ independent on $\varepsilon$ such that

$$\sum_{i,j} \int_{Y_{ij}^{\varepsilon}} |u^{\varepsilon}(x',-\varepsilon,s)|^2 dx' \leq C \int_{\Omega_\varepsilon} |\nabla u^{\varepsilon}(x,s)|^2 dx. \tag{A.8}$$

We deduce from (A.7)–(A.8) that

$$\int_0^t \int_{\Sigma_\varepsilon} |u^{\varepsilon}(x,s)|^2 dxds \leq C\varepsilon \int_0^t \int_{\Omega_\varepsilon} |\nabla u^{\varepsilon}(x,s)|^2 dxds + C\varepsilon^2 \int_0^t \int_{\Sigma_\varepsilon} |\nabla u^{\varepsilon}(x,s)|^2 dxds. \tag{A.9}$$

Now, multiplying $(2.6)_{1,2,3}$ by $u^\varepsilon$, we get, using Green's formula, Young's inequalities, and the relation

$$\frac{\partial u^{\varepsilon}}{\partial t}.u^{\varepsilon} = \frac{1}{2}\frac{\partial |u^{\varepsilon}|^2}{\partial t},$$

that

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega} |u^{\varepsilon}|^2(x,t)dx + \nu \int_0^t \int_{\Omega_\varepsilon} |\nabla u^{\varepsilon}|^2 dxds + \varepsilon \int_0^t \int_{\Sigma_\varepsilon} \mathbf{c}\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon}\right) \nabla u^{\varepsilon}.\nabla u^{\varepsilon} dxds \\
&\quad = \int_0^t \int_{\Omega} f \cdot u^{\varepsilon} dxds + \frac{1}{2}\int_{\Omega} |\mathbf{u}|^2(x)dx \\
&\quad \leq C\left(\int_0^t \int_{\Omega_\varepsilon} |u^{\varepsilon}|^2 dxds\right)^{1/2} + C\left(\int_0^t \int_{\Sigma_\varepsilon} |u^{\varepsilon}|^2 dxds\right)^{1/2} + \frac{1}{2}\int_{\Omega} |\mathbf{u}|^2(x)dx.
\end{aligned} \tag{A.10}$$

Then, using Poincaré's inequality in $\Omega_\varepsilon$, assumption (A1), and (A.9), we obtain that

$$\begin{aligned}
&\int_0^t \int_{\Omega_\varepsilon} |\nabla u^{\varepsilon}|^2 dxds + \varepsilon \int_0^t \int_{\Sigma_\varepsilon} |\nabla u^{\varepsilon}|^2 dxds \\
&\quad \leq C\left(\left(\int_0^t \int_{\Omega_\varepsilon} |\nabla u^{\varepsilon}|^2 dxds\right)^{1/2} + \left(\varepsilon \int_0^t \int_{\Sigma_\varepsilon} |\nabla u^{\varepsilon}(x)|^2 dxds\right)^{1/2}\right).
\end{aligned} \tag{A.11}$$

from which we deduce $(A.3)_{1,2}$. On the other hand, combining (A.9), (A.10) and (A.11) we deduce $(A.3)_{3,4}$. Let $v \in C^1(0,T,\mathbf{H}(\Omega,\mathbb{R}^3))$ such that $v(x,0) = v(x,T) = 0$. Then, according to the above estimates, we have the following:

$$\left|\int_0^T \int_\Omega \frac{\partial u^\varepsilon}{\partial t}.v dx dt\right| = \left|\int_0^T \int_\Omega \frac{\partial v}{\partial t}.u^\varepsilon dx dt\right| \leq C\|\nabla v\|_{L^2(\Omega\times(0,T),\mathbb{R}^9)}.$$

This implies that, for every $\varepsilon \in (0,1)$,

$$\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|_{L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)^*)} \leq C. \qquad \square$$

In order to get estimates on the pressure $p^\varepsilon$, let us first define the zero mean value pressures:

$$\overline{p^\varepsilon} = p^\varepsilon - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} p^\varepsilon dx.$$

Then we have the following estimate:

**Lemma A.3.** *Under assumption* $(A1)$, *there exists a positive constant* $C$ *independent on* $\varepsilon$, *such that, for every* $\omega \in \Pi$,

$$\int_0^T \int_{\Omega_\varepsilon} (\overline{p^\varepsilon})^2 dx dt \leq C.$$

**Proof.** Let $t \in [0,T]$. Let $\psi_t^\varepsilon \in H_0^1(\Omega_\varepsilon,\mathbb{R}^3)$ be the solution of the following problem:

$$\begin{cases} \operatorname{div}\left(\psi_t^\varepsilon\right) = \overline{p^\varepsilon}(.,t) & \text{in } \Omega_\varepsilon, \\ \psi_t^\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

There exists a constant $C(\Omega)$ only depending on $\Omega$ and $T$ such that (see for instance [4]):

$$\left\|\nabla\psi_t^\varepsilon\right\|_{L^2(\Omega_\varepsilon,\mathbb{R}^9)} \leq C(\Omega)\|\overline{p^\varepsilon}(.,t)\|_{L^2(\Omega_\varepsilon)}.$$

Multiplying $(2.6)_{1,2}$ by $\psi_t^\varepsilon$ and using Green's formula, we obtain that

$$\nu\int_0^T \int_{\Omega_\varepsilon} \nabla u^\varepsilon \cdot \nabla\psi_t^\varepsilon dx dt$$
$$= \int_0^T \int_{\Omega_\varepsilon} f \cdot \psi_t^\varepsilon dx dt + \int_0^T \int_{\Omega_\varepsilon} (\overline{p^\varepsilon})^2 dx dt - \int_0^T \int_{\Omega_\varepsilon} \frac{\partial u^\varepsilon}{\partial t}.\psi_t^\varepsilon dx dt.$$

As

$$\left|\int_0^T \int_{\Omega_\varepsilon} f \cdot \psi_t^\varepsilon dx dt\right| \leq C\|\overline{p^\varepsilon}\|_{L^2(\Omega_\varepsilon\times(0,T))},$$

$$\left|\int_0^T \int_{\Omega_\varepsilon} \frac{\partial u^\varepsilon}{\partial t}.\psi_t^\varepsilon dxdt\right| \leq C \left\|\frac{\partial u^\varepsilon}{\partial t}\right\|_{L^2(0,T,H^{-1}(\Omega,\mathbb{R}^3))} \|\overline{p^\varepsilon}\|_{L^2(\Omega_\varepsilon\times(0,T))},$$

$$\left|\int_0^T \int_{\Omega_\varepsilon} \nabla u^\varepsilon \cdot \nabla\psi_t^\varepsilon dxdt\right| \leq C\|\overline{p^\varepsilon}\|_{L^2(\Omega_\varepsilon\times(0,T))} \int_0^T \|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon,\mathbb{R}^9)}dt$$

we get the desired estimate using Lemma A.2. □

We have the following compactness result:

**Proposition A.4.** *Let $(u^\varepsilon, p^\varepsilon)$ be a solution of problem* (2.6)–(2.7). *There exists a subsequence of $(u^\varepsilon, p^\varepsilon)_\varepsilon$, still denoted in the same way, such that, for* **P***-almost all $\omega$:*

(1) *The following convergences hold*

$$\begin{aligned}
&u^\varepsilon \circ T_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u^0 \quad \text{in } L^2(0,T,H^1(\Omega,\mathbb{R}^3))\text{-weak},\\
&u^\varepsilon \underset{\varepsilon\to 0}{\to} u^0 \quad \text{in } L^2(\Omega\times(0,T),\mathbb{R}^3)\text{-strong},\\
&\frac{\partial u^\varepsilon}{\partial t} \underset{\varepsilon\to 0}{\rightharpoonup} \frac{\partial u^0}{\partial t} \quad \text{in } L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)^*)\text{-weak},\\
&\overline{p^\varepsilon} \underset{\varepsilon\to 0}{\to} p^0 \quad \text{in } L^2(\Omega\times(0,T),\mathbb{R}^3)\text{-weak},
\end{aligned} \tag{A.12}$$

*with $u^0 \in L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)) \cap L^\infty([0,T],L^2(\Omega,\mathbb{R}^3))$ and $u^0(x,0) = \mathbf{u}(x)$ in $\Omega$.*

(2) *For every $t \in [0,T]$, every $\varphi \in C\big([0,T],C_0^1(\mathbb{R}^3)\big)$, and every $\omega \in \Pi$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \int_{\Sigma_\varepsilon} u^\varepsilon\varphi\zeta^\varepsilon dxds = \overline{\langle\zeta\rangle}\int_0^t \int_\Sigma u^*(x',s)\varphi(x',0,s)dx'ds,$$

*where $\zeta^\varepsilon(x,s) = \zeta(\frac{x}{\varepsilon},\frac{s}{\varepsilon})$; $\zeta(z,s)$ being an ergodic stationary process verifying* (A.1), *the constant $\overline{\langle\zeta\rangle}$ is defined in Lemma* A.$1_1$, *and*

$$u^*(x',s) = u^0(x',0,s),$$

*where $u^0(x',0,t)$ is, for almost every $t \in [0,T]$, the trace of $u^0 \in H^1(\Omega,\mathbb{R}^3)$ on $\Sigma$.*

**Proof.** (1) From estimates (A.3) and inequality (2.19), we deduce that the sequence $(u^\varepsilon \circ T_\varepsilon)_\varepsilon$ is bounded in $L^2(0,T,H^1(\Omega;\mathbb{R}^3))$. Then, up to some subsequence, the sequence $(u^\varepsilon \circ T_\varepsilon)_\varepsilon$ converges to some $u^0$ in $L^2(0,T,H^1(\Omega,\mathbb{R}^3))$-weak. From $(\text{A.3})_{1,3}$ it follows that $u^0$ belongs to $L^2(\Omega\times(0,T),\mathbb{R}^3)$ and, up to some subsequence,

$$\chi_\Omega\nabla(u^\varepsilon\circ T_\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \nabla u^0 \quad \text{in } L^2(0,T,L^2(\Omega,\mathbb{R}^9))\text{-weak},$$

where $\chi_\Omega$ is the characteristic function of $\Omega$. Let us write

$$\int_0^t\int_\Omega |u^\varepsilon - u^0|^2dxds = \int_0^t\int_{\Omega_\varepsilon{}^-} |u^\varepsilon - u^0|^2dxds + \int_0^t\int_{\Sigma_\varepsilon} |u^\varepsilon - u^0|^2dxds.$$

Using the above convergences, Sobolev embeddings and $(A.3)_{3,4}$, we prove that $(u^\varepsilon)_\varepsilon$ converges to $u^0$ in $L^2(\Omega \times (0,T), \mathbb{R}^3)$-strong. Using the trace Theorem, we have, up to some subsequence,

$$\lim_{\varepsilon\to 0}\int_0^t \int_{\Gamma_\varepsilon} \left|u^\varepsilon - u^0(.,0,.)\circ T_\varepsilon^{-1}\right| dx'ds$$
$$= \lim_{\varepsilon\to 0}\int_0^t \int_{\Sigma} |u^\varepsilon \circ T_\varepsilon - u^0(.,0,.)|\mathrm{Jac}(T_\varepsilon)dx'ds = 0. \tag{A.13}$$

Since $\operatorname{div} u^\varepsilon = 0$ in $\Omega$, we have, for every $\varphi \in \mathbf{C}_c^\infty(\Sigma)$,

$$\int_0^t \int_{\Sigma_\varepsilon} \operatorname{div}(u^\varepsilon)\varphi dxds = -\int_0^t \int_{\Sigma_\varepsilon} u_\tau^\varepsilon \cdot \nabla_\tau \varphi dxds - \int_0^t \int_{\Gamma_\varepsilon} u^\varepsilon \varphi n dx'ds = 0,$$

where $u_\tau^\varepsilon = \left(u_1^\varepsilon, u_2^\varepsilon\right)$, $n = e_3 = (0,0,1)^t$, and $\nabla_\tau \varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}\right)$. Then, passing to the limit using $(A.3)_3$ and (A.13), we deduce that $u_3^0 = 0$ on $\Sigma$. Now, since $\operatorname{div} u^\varepsilon = 0$ in $\Omega_\varepsilon$, we easily obtain that $\operatorname{div} u^0 = 0$ in $\Omega$. Thus, $u^0$ belongs to $\mathbf{H}(\Omega, \mathbb{R}^3)$. We deduce from Lemma A.2 and from the above computations that, up to some subsequence,

$$\frac{\partial u^\varepsilon}{\partial t} \underset{\varepsilon\to 0}{\rightharpoonup} \frac{\partial u^0}{\partial t} \quad \text{in } L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)^*)\text{-weak.}$$

Using Lemma A.2 and inequality (2.19), we deduce that, for almost every $t \in [0,T]$, there exists a subsequence $(u^{\varepsilon_l}(.,t))_{\varepsilon_l}$, such that

$$\int_\Omega |\nabla(u^{\varepsilon_l} \circ T_\varepsilon)(x,t)|^2 dx \le C_t < \infty.$$

This implies that, up to some subsequence, $(u^{\varepsilon_l}(.,t))_l$ converges to $u^0(.,t)$ in $\mathbf{H}(\Omega,\mathbb{R}^3)^*$-strong. Let $(t_n)_n$ be a dense sequence in $[0,T]\backslash\mathbb{N}$. We have

$$\begin{aligned}\|u^{\varepsilon_l}(.,t_n) - u^{\varepsilon_l}(.,t)\|_{\mathbf{H}(\Omega,\mathbb{R}^3)^*} &= \left\| \int_t^{t_n} \frac{\partial u^{\varepsilon_l}(.,s)}{\partial s} ds \right\|_{\mathbf{H}(\Omega,\mathbb{R}^3)^*} \\ &\le |t_n - t|^{\frac{1}{2}} \left( \int_t^{t_n} \left\| \frac{\partial u^{\varepsilon_l}}{\partial s} \right\|_{\mathbf{H}(\Omega,\mathbb{R}^3)^*} ds \right)^{\frac{1}{2}} \\ &\le C|t_n - t|^{\frac{1}{2}},\end{aligned}$$

from which we deduce that $(u^{\varepsilon_l}(.,t))_l$ converges to $u^0(.,t)$ in $\mathbf{H}(\Omega,\mathbb{R}^3)^*$-strong uniformly in $[0,T]$. Let $\delta > 0$. Then, according to [17, Chap. 1, Lemme 12.1], there exists a constant $C_\delta > 0$, such that

$$\begin{aligned}\int_\Omega |u^{\varepsilon_k}(.,t) - u^{\varepsilon_l}(.,t)|^2 dx &\le \delta \int_\Omega |\nabla u^{\varepsilon_l}(.,t)|^2 dx + \delta \int_\Omega |\nabla u^{\varepsilon_k}(.,t)|^2 dx \\ &\quad + C_\delta \|u^{\varepsilon_k}(.,t) - u^{\varepsilon_l}(.,t)\|^2_{\mathbf{H}(\Omega,\mathbb{R}^3)^*}.\end{aligned}$$

Thus, $(u^\varepsilon(.,t))_\varepsilon$ is a Cauchy sequence in $L^\infty([0,T], L^2(\Omega))$. We deduce from this that $u^0 \in L^\infty([0,T], L^2(\Omega))$ and $u^0(x,0) = \mathbf{u}(x)$ in $\Omega$. On the other hand, using inequality (2.19), we deduce that, up to some subsequence,

$$\overline{p^\varepsilon} \underset{\varepsilon\to 0}{\to} p^0 \quad \text{in } L^2(\Omega\times(0,T),\mathbb{R}^3)\text{-weak}.$$

(2) We deduce from Lemmas A.2$_4$ and A.1, that there exists $u^* \in L^2(\Sigma\times(0,T))$, such that, for every $\varphi \in C([0,T], C_0^1(\mathbb{R}^3))$, we have, up to some subsequence, and with probability 1,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^4} u^\varepsilon\varphi\zeta^\varepsilon d\mu_\varepsilon(x,s) = \overline{\langle\zeta\rangle}\int_0^t\int_\Sigma u^*(x',s)\varphi(x',0,s)dx'ds,$$

where

$$\mu_\varepsilon(x,s) = \frac{\mathbf{1}_{\Sigma_\varepsilon}(x)\mathbf{1}_{(0,t)}(s)dxds}{\varepsilon}.$$

Let us now define the measure $\mu_\varepsilon^0(x,s)$ through

$$\mu_\varepsilon^0(x,s) = \mathbf{1}_{\{x_3=-\varepsilon\}}\frac{\mathbf{1}_{\Sigma_\varepsilon}(x)\mathbf{1}_{(0,t)}(s)dxds}{\varepsilon}.$$

Then $\mu_\varepsilon^0(x,s)$ weakly converges, as $\varepsilon$ tends to zero, in the sense of measures to the measure $\mathbf{1}_\Sigma(x')\mathbf{1}_{(0,t)}(s)dx'ds$, thus, using the proof of Lemma A.1, one can see that, for every $\varphi \in C\big([0,T], C_0^1(\mathbb{R}^3)\big)$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^4} u^\varepsilon\varphi d\mu_\varepsilon^0(x,s) = \int_0^t\int_\Sigma u^*(x',s)\varphi(x',0,s)dx'ds. \tag{A.14}$$

Observing that

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\Gamma_\varepsilon} u^\varepsilon\varphi dx'ds = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^4} u^\varepsilon\varphi d\mu_\varepsilon^0(x,s),$$

we deduce, using (A.13) and (A.14), that

$$u^*(x',s) = u^0(x',0,s).$$

□

### A.3. *Local study*

Let us introduce the following result:

**Lemma A.5 ([6, Lemma 18.6]).** *Let $q_h$; $h \in \mathbb{N}^*$, be the solution of the following diffusion problem*:

$$\begin{cases} \dfrac{\partial q_h}{\partial s} + \operatorname{div}_z(\mathbf{c}(z,\xi_s)\nabla q_h) = 0 & \text{in } Z\times(-\infty,h);\ \ h\in\mathbb{N}^*, \\ q_h(z,h) = q_0 & \text{on } Z, \\ q_h(z,s) \quad \text{is 1-periodic in } z, & \end{cases} \tag{A.15}$$

*where $Z = Y\times(-1,0)$ and $q_0 \in L^2_{\text{per}}(Z)$, such that*

$$\int_Z q_0(z)dz = 0.$$

*Then, for every* $s \leq h - 1$,

$$|q_h(z,s)| \leq C\|q_0\|_{L^2(Z)} \exp(C(s-h)).$$

*If* $q_0 \in L^\infty(Z)$ *then, for every* $s \leq h$,

$$|q_h(z,s)| \leq C\|q_0\|_{L^\infty(Z)} \exp(C(s-h)).$$

Let us now consider the sequence of auxiliary equations:

$$\begin{cases} \dfrac{\partial \vartheta_h}{\partial s} + \operatorname*{div}_z(\mathbf{c}(z,\xi_s)\nabla\vartheta_h) = 0 & \text{in } Z \times (-\infty, h);\ \ h \in \mathbb{N}^*, \\ \vartheta_h|_{z_3=0} = 1 & \text{on } Y \times (-\infty, h), \\ \vartheta_h(z,h) = 1 & \text{on } Z, \\ \vartheta_h(z,s) \quad \text{is 1-periodic in } z. \end{cases} \tag{A.16}$$

We have the following convergence result.

**Proposition A.6.** *The sequence* $(\vartheta_h)_{h\in\mathbb{N}^*}$ *converge uniformly in compact sets of* $Z \times (-\infty,\infty)$, *as* $h \to \infty$, *to the positive solution* $\vartheta$ *of the problem*

$$\begin{cases} \dfrac{\partial \vartheta}{\partial s} + \operatorname*{div}_z(\mathbf{c}(z,\xi_s)\nabla\vartheta) = 0 & \textit{in } Z \times (-\infty, \infty), \\ \vartheta|_{z_3=0} = 1 & \textit{on } Y \times (-\infty, \infty), \\ \vartheta(z,s) \quad \textit{is 1-periodic in } z, \end{cases} \tag{A.17}$$

*with*

$$\int_Z \vartheta(z,s)dz = 1 \quad a.s.\ s \in (-\infty,\infty). \tag{A.18}$$

*Moreover* $\vartheta(z,s)$ *is continuous and satisfies the estimates*

$$\begin{cases} \|\vartheta(z,s)\|_{L^\infty(Z\times(-\infty,\infty))} \leq C, \\ \sup\limits_{k\in\mathbb{R}} \|\vartheta(z,s)\|_{L^2((k,k+1),H^1(Z))} \leq C, \\ \vartheta(z,s) \geq C_1, \\ \max\limits_{z\in Z, k\leq s\leq k+1} |\vartheta(z,s) - \vartheta_h(z,s)| \leq C\exp(C_2(k-h)), \end{cases} \tag{A.19}$$

*where* $C$, $C_1$ *and* $C_2$ *are nonrandom positive constants.*

**Proof.** By the maximum principle (see for instance [14]), the solution $\vartheta_h(z,s)$ of problem (A.16) is positive. Moreover, multiplynig equation $(\text{A.16})_1$ by 1 and integrating by parts over the set $Z \times (s,h)$, we get, for all $s \leq h$,

$$\int_Z \vartheta_h(z,s)dz = 1. \tag{A.20}$$

Then, using the Harnack inequality, we obtain that

$$0 < C_1 \le \vartheta_h(z,s) \le C < \infty, \tag{A.21}$$

where $C$ and $C_1$ are nonrandom constants independent on $h$. Observe now that $\vartheta_{h+k} - \vartheta_h$ is the solution of the following problem in $Z \times (-\infty, h)$:

$$\begin{cases} \dfrac{\partial(\vartheta_{h+k} - \vartheta_h)}{\partial s} + \operatorname*{div}_z(\mathbf{c}(z,\xi_s)\nabla(\vartheta_{h+k} - \vartheta_h)) = 0, \\ (\vartheta_{h+k} - \vartheta_h)|_{z_3=0} = 0, \end{cases}$$

with

$$\int_Z (\vartheta_{h+k} - \vartheta_h) dz = 0, \tag{A.22}$$

from which we deduce, using Lemma A.5 and the estimate (A.21), that

$$\|\vartheta_{h+k} - \vartheta_h\|_{C(Z\times[s,s+1])} \le C_1 \exp(C(s-h)), \tag{A.23}$$

which implies that $(\vartheta_h)_h$ is a Cauchy sequence which uniformly converges on compact sets of $Z \times (-\infty, \infty)$, as $h \to \infty$, to a continuous function $\vartheta$. Passing to the limit in (A.21)–(A.23), as $k \to \infty$, we obtain the estimates $(A.19)_{1,3,4}$, and passing to the limit in (A.22) as $k \to \infty$, taking into account the equality (A.20), we obtain the equality (A.18). On the other hand, multiplying Eq. $(A.16)_1$ by $\vartheta_h$ and integrating by parts over the set $Z \times (s,h)$, we deduce that

$$\int_s^h \int_Z \mathbf{c}(z,\xi_s)\nabla\vartheta_h . \nabla\vartheta_h dz ds \le C,$$

where $C$ is a nonrandom constant independent on $h$. Thus, using hypothesis (A1), we have

$$\int_{Z\times(s,s+1)} |\nabla\vartheta_h|^2 dz ds \le C,$$

uniformly with respect to $\omega \in \Pi$ and $s < h$. We deduce that the sequence $(\nabla\vartheta_h)_h$ weakly converges to $\nabla\vartheta$ in the space $L^2_{\text{loc}}(Z \times (-\infty,\infty))$, and, as a consequence, the estimate $(A.19)_2$. Passing to the limit in the integral identity corresponding to problem (A.16) and using the continuity of $\vartheta(z,s)$, we deduce that $\vartheta(z,s)$ is the solution of problem (A.17). □

We define the random process $\eta = (\eta_i)_{i=1,2,3}$ through

$$\eta_i(s) = \int_Z \frac{\partial c_{ij}(z,\xi_s)}{\partial z_j} \vartheta(z,s) dz. \tag{A.24}$$

where $(c_{ij}(z,\xi_s))_{i,j=1,2,3} = \mathbf{c}(z,\xi_s)$. We introduce the matrix $\sigma^2 = (\sigma^2_{ij})_{i,j=1,2,3}$

$$\sigma^2_{ij} = \int_0^\infty \mathbf{E}(\eta_i(s)\eta_j(0) + \eta_j(s)\eta_i(0)) ds. \tag{A.25}$$

We also define the following matrices:

$$\overline{\sigma} = (\sigma_{\alpha\beta})_{\alpha,\beta=1,2}, \quad \widetilde{\sigma} = (\sigma_{\alpha j})_{\alpha=1,2,j=1,2,3}, \tag{A.26}$$

$\sigma$ being the square root of the matrix $\sigma^2$ defined in (A.25). The following result is crucial for the description of the limit problem (2.21):

**Proposition A.7.** *The process $\eta(s)$ satisfies the functional central limit Theorem with covariance matrix $\sigma^2$:*

$$\sqrt{\varepsilon}\int_0^{t/\varepsilon}\eta(s)ds \underset{\varepsilon\to 0}{\overset{\mathcal{L}}{\to}} \sigma W_t \quad \text{in } C([0,T],\mathbb{R}^3).$$

**Proof.** The proof follows the lines of the proof of [12, Lemma 3]. Let us consider the following two auxiliary problems:

$$\begin{cases} \dfrac{\partial \vartheta^1}{\partial s} + \operatorname*{div}_z(\mathbf{c}(z,\xi_s)\nabla\vartheta^1) = 0 & \text{in } Z\times(-\infty,T/2), \\ \vartheta^1|_{z_3=0} = 1/2 & \text{on } Y\times(-\infty,T/2), \\ \vartheta^1(z,T/2) = 1 & \text{on } Z, \\ \vartheta^1(z,s) \quad \text{is 1-periodic in } z & \end{cases} \tag{A.27}$$

and

$$\begin{cases} \dfrac{\partial \vartheta^2}{\partial s} + \operatorname*{div}_z(\mathbf{c}(z,\xi_s)\nabla\vartheta^2) = 0 & \text{in } Z\times(-\infty,T/2), \\ \vartheta^2|_{z_3=0} = 1/2 & \text{on } Y\times(-\infty,T/2), \\ \vartheta^2(z,T/2) = \vartheta(z,T/2)-1 & \text{on } Z, \\ \vartheta^2(z,s) \quad \text{is 1-periodic in } z. & \end{cases} \tag{A.28}$$

Then $\vartheta(z,s) = \vartheta^1(z,s)+\vartheta^2(z,s)$ on $(0,T/2)$ and $\eta(0) = \eta^1(0)+\eta^2(0)$, where

$$\eta_i^m(0) = \int_Z \frac{\partial c_{ij}}{\partial z_j}(z,\xi_0)\vartheta^m(z,0)dz; \quad m=1,2.$$

Since $\vartheta^m(z,0)$ and $\eta^1(0)$ are $\mathcal{F}_{\le\frac{T}{2}}$-measurable, we have, using one of the mixing conditions (A2) (for example the last one), that

$$\|\mathbf{E}(\eta^1(0)/\mathcal{F}_{\ge T})\|_{L^2(\Pi)} \le \rho(T/2)\|\eta^1(0)\|_{L^2(\Pi)} \le C\rho(T/2). \tag{A.29}$$

On the other hand, using Lemma A.5, one has

$$|\vartheta^2(z,0)| \le C\exp(-CT/2),$$

from which we deduce that

$$\|\mathbf{E}(\eta^2(0)/\mathcal{F}_{\ge T})\|_{L^2(\Pi)} \le \|\eta^2(0)\|_{L^2(\Pi)} \le C\exp(-CT/2). \tag{A.30}$$

Combining (A.29) and (A.30), we get

$$\|\mathbf{E}(\eta(0)/\mathcal{F}_{\ge T})\|_{L^2(\Pi)} \le C(\exp(-CT)+\rho(T/2)).$$

Then, under assumption (A2), we obtain, using the Prokhorov Theorem (see for instance [2, Chap. 1, Sec. 5]) and the functional central limit Theorem (see for instance [18, Chap. 9]), that

$$\sqrt{\varepsilon}\int_0^{t/\varepsilon}\eta(s)ds \underset{\varepsilon\to 0}{\overset{\mathcal{L}}{\to}} \sigma W_t \quad \text{in } C([0,T],\mathbb{R}^3). \qquad \square$$

### A.4. *Proof of the main result*

In this subsection, we give the proof of our main result. According to Proposition A.4, there exists a subsequence of $(u^\varepsilon, p^\varepsilon)_\varepsilon$, still denoted in the same way, which verifies the convergences 2.20 for **P**-almost all $\omega$, with $u^0 \in L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)) \cap L^\infty([0,T],L^2(\Omega,\mathbb{R}^3))$ and $u^0(x,0) = \mathbf{u}(x)$ in $\Omega$. Let us define the sequence $\left(u^0_\varepsilon(x,t)\right)_\varepsilon$ of test-functions through

$$u^0_\varepsilon(x,t) = u^0(x,s) + u^0\big(X_1^\varepsilon, X_2^\varepsilon, 0, t\big)(\vartheta^\varepsilon(x,t)-1), \tag{A.31}$$

where

$$\begin{cases} X_1^\varepsilon = x_1 + \dfrac{1}{\sqrt{\varepsilon}}\displaystyle\int_0^t \eta_1\left(\frac{s}{\varepsilon}\right)ds, \\ X_2^\varepsilon = x_2 + \dfrac{1}{\sqrt{\varepsilon}}\displaystyle\int_0^t \eta_2\left(\frac{s}{\varepsilon}\right)ds, \\ \vartheta^\varepsilon(x,t) = \vartheta\left(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon}\right), \end{cases} \tag{A.32}$$

$\vartheta$ being the solution of the cell problem (A.17) in Proposition A.6 of Appendix A.3, and $\eta = (\eta_i)_{i=1,2,3}$ is the random process defined in (A.24). The new variables $X_m^\varepsilon$; $m = 1,2$, which verify the stochastic differential equations:

$$\begin{cases} dX_m^\varepsilon(t) = \dfrac{1}{\sqrt{\varepsilon}}\eta_m\left(\dfrac{t}{\varepsilon}\right)dt; \quad m = 1,2, \\ X_m^\varepsilon(0) = x_m, \end{cases} \tag{A.33}$$

describe the random perturbations of the trajectory around the horizontal position $(x_1,x_2)$ of a particle fluid in the biofilm. The function $u^0_\varepsilon(x,t)$ is thus the sum of $u^0$ on $\Sigma\times[0,T]$ and a local random perturbation described by the quantity $u^0(X_1^\varepsilon, X_2^\varepsilon, 0, t)(\vartheta^\varepsilon(x,t)-1)$. Let us define the mean

$$\overline{\langle\mathcal{C}\rangle} = \mathbf{E}\left(\int_Z \mathbf{c}(z,\xi_s)\nabla_z\vartheta.\nabla_z\vartheta(z,s)dz\right). \tag{A.34}$$

To complete our proof, we need the following result:

**Lemma A.8.** *Let $u^\varepsilon$ be the (velocity) solution of problem* (2.6)–(2.7)*. Assume that assumptions (A1)–(A2) are fulfilled. Then*

(1) *For every $\varphi \in C\big([0,T], C_0^1(\mathbb{R}^3)\big)$, every $\omega \in \Pi$, and every $t \in [0,T]$,*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon} u^0_\varepsilon\varphi\zeta^\varepsilon dxds = \overline{\langle\zeta\rangle}\int_0^t\int_\Sigma u^0(x',0,s)\varphi(x',0,s)dx'ds,$$

*where* $\zeta^\varepsilon(x,s) = \zeta\big(\frac{x}{\varepsilon}, \frac{s}{\varepsilon}\big)$; $\zeta(z,s)$ *being an ergodic stationary process verifying* (A.1).

(2) *There exists a subsequence of* $(u^\varepsilon)_\varepsilon$, *still denoted in the same way, such that, for every* $\omega \in \Pi$ *and every* $\psi \in C([0,T], C^1(\Sigma\times\mathbb{R}))$ *with* $\psi = 0$ *on* $\partial\Sigma\times\mathbb{R}$, *we have*

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\big(\nabla u^\varepsilon - \nabla u^0_\varepsilon\big).\nabla(\psi\vartheta^\varepsilon)dxds = 0.$$

(3) *For every* $\omega\in\Pi$ *and every* $\psi\in C([0,T], C^1(\Sigma\times[0,\infty),\mathbb{R}^3))$ *such that* $\psi = 0$ *on* $\partial\Sigma\times[0,\infty)$, *we have*

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla u^0_\varepsilon.\nabla(\psi\vartheta^\varepsilon)dxds$$
$$= \overline{\langle\mathcal{C}\rangle}\int_0^t\int_\Sigma u^0(x' + \widetilde{\sigma}W_t, 0, s).\psi(x',0,s)ds,$$

*where* $\widetilde{\sigma}$ *is the matrix defined in* $(\text{A.26})_2$.

**Proof.** (1) Using the construction of $\big(u^0_\varepsilon\big)_\varepsilon$, Proposition A.4 and Lemma A.1, one can see that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon}u^0_\varepsilon\varphi\zeta^\varepsilon dxds = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon}u^0(x',-\varepsilon,s)\varphi\zeta^\varepsilon dxds$$
$$= \overline{\langle\zeta\rangle}\int_0^t\int_\Sigma u^0(x',0,s)\varphi(x',0,s)dx'ds.$$

(2) Since $\mathbf{c}\big(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\big)$ is symmetric, we have the following:

$$\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\big(\nabla u^\varepsilon - \nabla u^0_\varepsilon\big).\nabla(\psi\vartheta^\varepsilon)dxds$$
$$= \varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla(\psi\vartheta^\varepsilon).\big(\nabla u^\varepsilon - \nabla u^0_\varepsilon\big)dxds. \tag{A.35}$$

Then, using the Green formula, we have the following:

$$\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla(\psi\vartheta^\varepsilon).\big(\nabla u^\varepsilon - \nabla u^0_\varepsilon\big)dxds$$
$$= -\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon}\operatorname*{div}_z\left(\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon\right)\psi.\big(u^\varepsilon - u^0_\varepsilon\big)dxds$$
$$- \int_0^t\int_{\Sigma_\varepsilon}\operatorname*{div}_z\left(\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_x\psi\right).\vartheta^\varepsilon\big(u^\varepsilon - u^0_\varepsilon\big)dxds$$
$$+ \int_0^t\int_{\partial\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon.n\psi.\big(u^\varepsilon - u^0_\varepsilon\big)d\sigma(x)ds$$
$$+ \varepsilon\int_0^t\int_{\partial\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_x\psi n.\vartheta^\varepsilon\big(u^\varepsilon - u^0_\varepsilon\big)d\sigma(x)ds, \tag{A.36}$$

where $\operatorname{div}_z$ is the divergence with respect to $z \in Z$, $\nabla_z$ is the gradient with respect to $z$, $n$ is the outward unit normal on $\partial\Sigma_\varepsilon$, and $d\sigma(x)$ is the surfacic Lebesgue measure on $\partial\Sigma_\varepsilon$. Using assumption (A1), formula $(\text{A.32})_3$, estimate $(\text{A.3})_4$ in Lemma A.2, and Lemma A.1, we have, up to some subsequence,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon}\operatorname*{div}_z\left(\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon\right)\psi.u^\varepsilon dxds$$
$$=\overline{\left\langle\operatorname*{div}_z(\mathbf{c}(z,\xi_s)\nabla_z\vartheta)\right\rangle}\int_0^t\int_\Sigma u^0(x',0,s).\psi(x',0,s)dx'ds,$$

where, according to Lemma A.1, $\overline{\langle\operatorname{div}_z(\mathbf{c}(z,\xi_s)\nabla_z\vartheta)\rangle}$ is the mean of the process $\operatorname{div}_z(\mathbf{c}(z,\xi_s)\nabla_z\vartheta)$.

On the other hand, using the construction of the sequence $\left(u_\varepsilon^0\right)_\varepsilon$ and Lemma A.1, we obtain that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon(\omega)}\operatorname*{div}_z\left(\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon\right)\psi.u_\varepsilon^0 dxds$$
$$=\overline{\left\langle\operatorname*{div}_z(\mathbf{c}(z,\xi_s)\nabla_z\vartheta)\right\rangle}\int_0^t\int_\Sigma u^0(x',0,s).\psi(x',0,s)dx'ds.$$

Hence,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon(\omega)}\operatorname*{div}_z\left(\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon\right)\psi.\big(u^\varepsilon-u_\varepsilon^0\big)dxds=0 \tag{A.37}$$

and

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\Sigma_\varepsilon}\operatorname*{div}_z\left(\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_x\psi\right).\vartheta^\varepsilon\big(u^\varepsilon-u_\varepsilon^0\big)dxds=0. \tag{A.38}$$

Besides, we have the following:

$$\int_0^t\int_{\partial\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon.\nu\psi.(u^\varepsilon-u_\varepsilon^0)d\sigma(x)ds$$
$$=\int_0^t\int_{\Sigma^-}\mathbf{c}\left(\frac{x'}{\varepsilon},-\frac{\varepsilon}{2},\xi_{\frac{s}{\varepsilon}}\right)\nabla_z\vartheta^\varepsilon.n\psi.\big(u^\varepsilon-u_\varepsilon^0\big)dx'ds$$
$$-\int_0^t\int_{\Gamma_\varepsilon}\mathbf{c}\left(\frac{x'}{\varepsilon},\frac{\varepsilon}{2},\xi_{\frac{s}{\varepsilon}}\right)\nabla_z\vartheta^\varepsilon.n\psi.\big(u^\varepsilon-u_\varepsilon^0\big)dx'ds, \tag{A.39}$$

where $n=e_3$. Thus, using assumption (A1), Proposition $\text{A.4}_2$ and formula $(\text{A.32})_3$ of $\vartheta^\varepsilon$, we get

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\partial\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon.n\psi.\big(u^\varepsilon-u_\varepsilon^0\big)d\sigma(x)ds$$
$$=\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\partial\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_x\psi n.\vartheta^\varepsilon\big(u^\varepsilon-u_\varepsilon^0\big)d\sigma(x)ds=0. \tag{A.40}$$

Hence, combining (A.35)–(A.40), we obtain that

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\left(\nabla u^\varepsilon-\nabla u_\varepsilon^0\right).\nabla(\psi\vartheta^\varepsilon)dxds=0.$$

(3) We have, using the expression (A.31), that

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}&\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla u_\varepsilon^0.\nabla(\psi\vartheta^\varepsilon)dxds\\ &=\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla\vartheta^\varepsilon.\nabla\vartheta^\varepsilon u^0\big(X_1^\varepsilon,X_2^\varepsilon,0,t\big).\psi(x,s)dxds\\ &\quad+\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla\vartheta^\varepsilon.\nabla\psi u^0\big(X_1^\varepsilon,X_2^\varepsilon,0,t\big)\vartheta^\varepsilon dxds\\ &\quad+\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_{x'}u^0\big(X_1^\varepsilon,X_2^\varepsilon,0,t\big).\nabla\psi(\vartheta^\varepsilon-1)\vartheta^\varepsilon dxds\\ &\quad+\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla u^0(x',0,s).\nabla(\psi\vartheta^\varepsilon)dxds.\end{aligned}$$

One has, using assumption (A1), that

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}&\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla\vartheta^\varepsilon.\nabla\psi\vartheta^\varepsilon u^0\big(X_1^\varepsilon,X_2^\varepsilon,0,s\big)dxds\\ &=\lim_{\varepsilon\to 0}\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon.\nabla\psi\vartheta^\varepsilon u^0\big(X_1^\varepsilon,X_2^\varepsilon,0,s\big)dxds=0\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}&\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_{x'}u^0\big(X_1^\varepsilon,X_2^\varepsilon,0,t\big).\nabla\psi(\vartheta^\varepsilon-1)\vartheta^\varepsilon dxds\\ &=\lim_{\varepsilon\to 0}\varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla u^0(x',0,s).\nabla(\psi\vartheta^\varepsilon)dxds=0.\end{aligned}$$

Observe now that, for every $\varphi\in C([0,T],\mathbb{R}^2)$,

$$\begin{aligned}\int_\Sigma&|u^0(x'+\varphi(s),0,s)|^2dx'ds\\ &\le C\left(\int_\Omega|u^0(x'+\varphi(s),x_3,s)|^2dxds+|\varphi(s)|\int_\Omega|\nabla u^0(x,s)|^2dxds\right)\\ &\le C\|\varphi\|_\infty\int_\Omega|\nabla u^0(x,s)|^2dxds.\end{aligned}$$

This implies that

$$\int_0^T\int_\Sigma|u^0(x'+\varphi(s),0,s)|^2dx'ds\le C\|\varphi\|_\infty.$$

Consequently, the mapping $\Phi : C([0,T],\mathbb{R}^2) \to L^2(\Sigma \times (0,T))$ defined by

$$\Phi(\varphi)(x', s) = u^0(x' + \varphi(s), 0, s),$$

is continuous. Hence, taking into account assumption (A2), Lemma A.7, and Lemma A.1, we obtain that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_0^t\int_{\Sigma_\varepsilon} \varepsilon\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla u^0_\varepsilon.\nabla(\psi\vartheta^\varepsilon)dxds\\
&\quad=\lim_{\varepsilon\to 0}\int_0^t\int_{\Sigma_\varepsilon} \varepsilon\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla\vartheta^\varepsilon.(\nabla\vartheta^\varepsilon)u^0(X_1^\varepsilon,X_2^\varepsilon,0,t).\psi(x,s)dxds\\
&\quad=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\int_{\Sigma_\varepsilon} \mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla_z\vartheta^\varepsilon.(\nabla_z\vartheta^\varepsilon)^\varepsilon u^0(X_1^\varepsilon,X_2^\varepsilon,0,t).\psi(x,s)dxds\\
&\quad=\overline{\langle\mathcal{C}\rangle}\int_0^t\int_\Sigma u^0(x'+\widetilde{\sigma}W_s,0,s).\psi(x',0,s)ds. \qquad \square
\end{aligned}$$

Now, we continue the proof of Theorem 2.1. Let $v \in L^2(0,T,\mathbf{H}(\Omega,\mathbb{R}^3)) \cap L^\infty([0,T];L^2(\Omega,\mathbb{R}^3))$ and $u^0(x,0) = \mathbf{u}(x)$ in $\Omega$. Since $\vartheta^\varepsilon|_{\Gamma_\varepsilon\times[0,T]} = 1$, we can define its extension $\widetilde{\vartheta}^\varepsilon$ to the whole $\Omega$ by

$$\widetilde{\vartheta}^\varepsilon = \begin{cases}\vartheta^\varepsilon & \text{in } \Sigma_\varepsilon\times[0,T],\\ 1 & \text{in } (\Omega\backslash\Sigma_\varepsilon)\times[0,T].\end{cases}$$

Then, multiplying equations (2.6) by $v\widetilde{\vartheta}^\varepsilon$ and integrating by parts, using (2.7), we get

$$\begin{aligned}
&\int_0^t\int_\Omega \frac{\partial u^\varepsilon}{\partial t}.vdxds + \int_0^t\int_{\Omega_\varepsilon}\nu\nabla u^\varepsilon.\nabla vdxds + \varepsilon\int_0^t\int_{\Sigma_\varepsilon}\mathbf{c}\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon}\right)\nabla u^\varepsilon.\nabla(v\vartheta^\varepsilon)dxds\\
&\quad= \int_0^t\int_\Omega f.v\vartheta^\varepsilon dxds. \qquad \text{(A.41)}
\end{aligned}$$

Then, applying Proposition A.4 and Lemma A.8 for some subsequence of $(u^\varepsilon)_\varepsilon$ still denoted in the same way, we get, using $(\text{A.19})_1$ for $\vartheta^\varepsilon$ in the fourth integral,

$$\begin{aligned}
&\int_0^t\int_\Omega\frac{\partial u^0}{\partial t}.vdxds + \int_0^t\int_\Omega\nu\nabla u^0.\nabla vdxds\\
&\quad+\overline{\langle\mathcal{C}\rangle}\int_0^t\int_\Sigma(u^0(x'+\widetilde{\sigma}W_t,0,s))_\beta.v_\beta(x',0,s)dx'ds\\
&\quad=\int_0^t\int_\Omega f.vdxds
\end{aligned}$$

and, using Green's formula,

$$\int_0^t \int_\Omega \frac{\partial u^0}{\partial t} v dx ds - \int_0^t \int_{\Omega^+} \nu \Delta u^0 . v dx ds - \nu \int_\Sigma \frac{\partial u^0_\alpha}{\partial x_3} v_\alpha(x', 0) dx' ds$$
$$+ \int_0^t \int_\Omega \nabla p^0 . v dx ds + \overline{\langle \mathcal{C} \rangle} \int_0^t \int_\Sigma (u^0(x' + \widetilde{\sigma} W_t, 0, s))_\beta . v_\beta(x', 0, s) dx' ds$$
$$= \int_0^t \int_\Omega f . v dx ds,$$

from which we deduce that $(u^0, p^0)$ is the unique solution of problem (2.21). The **uniqueness** of $(u^0, p^0)$ implies that the whole sequence $(u_\varepsilon)_\varepsilon$ verifies the convergence (2.20).

**Remark A.9.** (The case where $\xi_t$ is a diffusion process) Let us suppose that $(\xi_t)_{t \geq 0}$ be a stationary ergodic diffusion process, with values in $\mathbb{R}^d$, given by

$$d\xi_t = \mathbf{b}(\xi_t) dt + \lambda(\xi_t) dW_t,$$

with generator

$$L = q_{kl} \frac{\partial^2}{\partial y_k \partial y_l} + \mathbf{b}(y) . \nabla_y; \quad (q_{kl})_{k,l=1,\dots,d} = \lambda \lambda^* / 2.$$

Let us introduce the time reverse process $\zeta_t = \xi_{-t}$ with generator

$$\widetilde{L} = \widetilde{q}_{kl} \frac{\partial^2}{\partial y_k \partial y_l} + \widetilde{\mathbf{b}}(y) . \nabla_y$$

and suppose that the following condition holds instead of (A2):

(A2′) The diffusion coefficients $\widetilde{q}_{kl}$ and their first-order derivatives are uniformly bounded,

$$|\widetilde{q}_{kl}(y)| + |\nabla_y \widetilde{q}_{kl}(y)| \leq C$$

and the operator $\widetilde{L}$ is uniformly elliptic, for every $v \in \mathbb{R}^d$,

$$\widetilde{q}_{kl} v_k v_l \geq C |v|^2.$$

The vector $\widetilde{\mathbf{b}}(y)$ admits the polynomial estimate

$$|\widetilde{\mathbf{b}}(y)| + |\nabla_y \widetilde{\mathbf{b}}(y)| \leq C(1 + |y|^\kappa),$$

for some $\kappa > 0$, and there exist numbers $\mu > -1$ and $R > 0$, such that, for all $y \in \{y : |y| \geq R\}$,

$$\frac{\widetilde{\mathbf{b}}(y) . y}{|y|} \leq -C |y|^\mu.$$

Then (see [12, Paragraph 5]), we have

$$\sqrt{\varepsilon} \int_0^{t/\varepsilon} \eta(-s) ds \underset{\varepsilon \to 0}{\overset{\mathcal{L}}{\to}} \sigma W_t \quad \text{in } C([0, T], \mathbb{R}^3)$$

and, using the Prokhorov Theorem (see for instance [2, Chap. 1, Sec. 5]), the convergence of the sequence $(\sqrt{\varepsilon}\int_0^{t/\varepsilon}\eta(-s)ds)_\varepsilon$ is equivalent to the convergence of the sequence $(\sqrt{\varepsilon}\int_0^{t/\varepsilon}\eta(s)ds)_\varepsilon$.

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