A way to extend the Pascal triangle to words
Joint work with Julien Leroy and Michel Rigo (ULiège)

Manon Stipulanti (ULiège) FRIA grantee

IRIF, Paris (France)
November 16, 2018
### Classical Pascal triangle

<table>
<thead>
<tr>
<th>( \binom{m}{k} )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>\cdots</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>5</td>
<td>10 10</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>6</td>
<td>15 20</td>
<td>15 6</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>7</td>
<td>21 35 35</td>
<td>21 7</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

**Usual binomial coefficients of integers:**

\[
\binom{m}{k} = \frac{m!}{(m-k)! \cdot k!}
\]

**Pascal’s rule:**

\[
\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}
\]

---

**Generalized Pascal triangles**

Manon Stipulanti (ULiège)
- Grid: first $2^n$ rows and columns

\[
\begin{array}{cccccc}
  & 0 & 1 & \cdots & 2^n-1 & 2^n \\
0 & \bullet & \bullet & \cdots & \bullet & \bullet \\
1 & \bullet & \bullet & \cdots & \bullet & \bullet \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2^n-1 & \bullet & \bullet & \cdots & \bullet & \bullet \\
2^n & \bullet & \bullet & \cdots & \bullet & \bullet \\
\end{array}
\]

\[
\begin{array}{cccc}
\begin{array}{c}
(0) \\
(1) \\
\vdots \\
(2^n-1)
\end{array} & \begin{array}{c}
(0) \\
(1) \\
\vdots \\
(2^n-1)
\end{array} & \cdots & \begin{array}{c}
0 \\
1 \\
\vdots \\
2^n-1
\end{array}
\end{array}
\]

$\mathbb{N}^2 \cap [0, 2^n]^2$
• Color the grid:
  Color the first $2^n$ rows and columns of the Pascal triangle
  \[
  \binom{m}{k} \mod 2 \quad 0 \leq m, k < 2^n
  \]
  in
  • white if $\binom{m}{k} \equiv 0 \mod 2$
  • black if $\binom{m}{k} \equiv 1 \mod 2$
• Color the grid:
  Color the first $2^n$ rows and columns of the Pascal triangle

$$\left( \binom{m}{k} \mod 2 \right)_{0 \leq m, k < 2^n}$$

in

- white if $\binom{m}{k} \equiv 0 \mod 2$
- black if $\binom{m}{k} \equiv 1 \mod 2$

• Normalize by a homothety of ratio $1/2^n$
  (bring into $[0, 1] \times [0, 1]$)
  $\leadsto$ sequence belonging to $[0, 1] \times [0, 1]$. 
The first six elements of the sequence

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The tenth element of the sequence
The Sierpiński gasket

Generalized Pascal triangles

Manon Stipulanti (ULiège)
Folklore fact

The latter sequence converges to the Sierpiński gasket when \( n \) tends to infinity (for the Hausdorff distance).
Folklore fact

The latter sequence converges to the Sierpiński gasket when $n$ tends to infinity (for the Hausdorff distance).

Definitions:

- **$\epsilon$-fattening** of a subset $S \subset \mathbb{R}^2$

  
  
  $$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff distance $d_h$

  $$d_h(S, S') = \min \{\epsilon \in \mathbb{R}_{\geq 0} \mid S \subset [S']_\epsilon \text{ and } S' \subset [S]_\epsilon\}$$
Replace usual binomial coefficients of integers by binomial coefficients of finite words
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet.*

**Example:** $101, 101001 \in \{0, 1\}^*$
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Example: 101, 101001 ∈ \{0, 1\}^*

Binomial coefficient of words

Let \( u, v \) be two finite words. The binomial coefficient \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

Example: \( u = 101001 \quad \quad v = 101 \)
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Example:** $101, 101001 \in \{0, 1\}^*$

---

**Binomial coefficient of words**

Let $u, v$ be two finite words. The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = 101001$ \hspace{1cm} $v = 101$ \hspace{1cm} 1 occurrence
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Example: 101, 101001 ∈ \{0, 1\}*

Binomial coefficient of words

Let \( u, v \) be two finite words. The binomial coefficient \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

Example: \( u = 101001 \) \( v = 101 \) 2 occurrences
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Example: 101, 101001 ∈ \( \{0, 1\}^* \)

Binomial coefficient of words

Let \( u, v \) be two finite words. The binomial coefficient \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

Example: \( u = \text{101001} \quad v = 101 \quad 3 \) occurrences
**Definition:** A finite word is a finite sequence of letters belonging to a finite set called alphabet.

**Example:** $101, 101001 \in \{0, 1\}^*$

---

**Binomial coefficient of words**

Let $u, v$ be two finite words.

The binomial coefficient $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = 101001$ $v = 101$ $4$ occurrences
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Example:** 101, 101001 ∈ \{0, 1\}*

**Binomial coefficient of words**

Let \(u, v\) be two finite words. The *binomial coefficient* \(\binom{u}{v}\) of \(u\) and \(v\) is the number of times \(v\) occurs as a subsequence of \(u\) (meaning as a “scattered” subword).

**Example:** \(u = 10\overline{1001}\) \(v = 101\) 5 occurrences
**Definition**: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Example**: $101, 101001 \in \{0, 1\}^*$

---

**Binomial coefficient of words**

Let $u, v$ be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example**: $u = 101001$  \hspace{1cm} $v = 101$ \hspace{1cm} 6 occurrences
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Example: 101, 101001 ∈ \{0, 1\}*

Binomial coefficient of words

Let $u, v$ be two finite words. The binomial coefficient $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

Example: $u = 101001$ \quad $v = 101$

\[
\Rightarrow \binom{101001}{101} = 6
\]
Remark:
Natural generalization of binomial coefficients of integers

With a one-letter alphabet \( \{a\} \)

\[
\binom{a^m}{a^k} = \binom{\underbrace{a \cdots a}_{m \text{ times}}}{\underbrace{a \cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall m, k \in \mathbb{N}
\]
Definitions:

- \( \text{rep}_2(n) \) greedy base-2 expansion of \( n \in \mathbb{N}_{>0} \) starting with 1
- \( \text{rep}_2(0) = \varepsilon \) where \( \varepsilon \) is the empty word

\[
\begin{array}{c|c|c}
 n & \text{rep}_2(n) \\
\hline
0 & \varepsilon \\
1 & 1 \times 2^0 \\
2 & 1 \times 2^1 + 0 \times 2^0 \\
3 & 1 \times 2^1 + 1 \times 2^0 \\
4 & 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 \\
5 & 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
6 & 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\
\vdots & \vdots \\
\hline
\end{array}
\]

\( \{\varepsilon\} \cup 1\{0, 1\}^* \)
Generalized Pascal triangle $P_2$ in base 2

<table>
<thead>
<tr>
<th>$(\text{rep}_2(m))$</th>
<th>$\epsilon$</th>
<th>1</th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>11</td>
<td>1    $\textbf{2}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$\text{rep}_2(m)$</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>101</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>110</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>111</td>
<td>1 $\textbf{3}$</td>
<td>0</td>
<td>3</td>
<td>$\textbf{1}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

Binomial coefficient of finite words:

$(\text{rep}_2(m)) 
\binom{(\text{rep}_2(k))}{(\text{rep}_2(m))}$

Rule (not local):

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b} \binom{u}{v}$$
The classical Pascal triangle

<table>
<thead>
<tr>
<th>( \text{rep}_2(m) )</th>
<th>( \varepsilon )</th>
<th>1</th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 1 )</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 10 )</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 11 )</td>
</tr>
<tr>
<td>( \text{rep}_2(k) )</td>
<td>100</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( 100 )</td>
</tr>
<tr>
<td>101</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( 101 )</td>
</tr>
<tr>
<td>110</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( 110 )</td>
</tr>
<tr>
<td>111</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( 111 )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

Generalized Pascal triangles
Questions:

- After coloring and normalization can we expect the convergence to an analogue of the Sierpiński gasket?
- Could we describe this limit object?
**Grid:** first $2^n$ rows and columns of $P_2$

$$\begin{array}{cccccc}
0 & 1 & \cdots & 2^n - 1 & 2^n \\
0 & & & & & \\
1 & & & & & \\
\vdots & & & & & \\
2^n - 1 & & & & & \\
2^n & & & & & \\
\end{array}$$

$\mathbb{N}^2 \cap [0, 2^n]^2$
• Color the grid:
  Color the first $2^n$ rows and columns of the generalized Pascal triangle $P_2$

  $\left(\left(\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \mod 2\right)_{0 \leq m, k < 2^n}\right)$

  in

  • white if $\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 0 \mod 2$
  • black if $\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 1 \mod 2$

• Normalize by a homothety of ratio $1/2^n$
  (bring into $[0, 1] \times [0, 1]$)

  $\leadsto$ sequence $(U_n)_{n \geq 0}$ belonging to $[0, 1] \times [0, 1]$

  $U_n = \frac{1}{2^n} \bigcup_{u, v \in \{\varepsilon\} \cup 1\{0, 1\}^* \text{ s.t. } |u|, |v| \leq n, \binom{u}{v} \equiv 1 \mod 2} \{(\text{val}_2(v), \text{val}_2(u)) + Q\}$

  $Q = [0, 1] \times [0, 1]$
The elements $U_0, \ldots, U_5$
The element $U_9$

Lines of different slopes: $1, 2, 4, 8, 16, \ldots$

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The ($\star$) condition

\begin{align*}
(u, v) \text{ satisfies } (\star) \iff \begin{cases} 
    u, v \neq \varepsilon \\
    \binom{u}{v} \equiv 1 \bmod 2 \\
    \binom{u}{v_0} = 0 = \binom{u}{v_1}
\end{cases}
\end{align*}

Example: $(u, v) = (101, 11)$ satisfies $(\star)$

\[
\begin{align*}
\binom{101}{11} &= 1 \\
\binom{101}{110} &= 0 \\
\binom{101}{111} &= 0
\end{align*}
\]
Lemma: Completion

\((u, v)\) satisfies \((\ast)\) \(\Rightarrow\) \((u_0, v_0), (u_1, v_1)\) satisfy \((\ast)\)

**Proof:** Since \((u, v)\) satisfies \((\ast)\)

\[
\binom{u}{v} \equiv 1 \mod 2, \quad \binom{u}{v_0} = 0 = \binom{u}{v_1}
\]

Proof for \((u_0, v_0)\):

\[
\binom{u_0}{v_0} = \underbrace{\binom{u}{v_0}}_{= 0 \text{ since } (\ast)} + \underbrace{\binom{u}{v}}_{\equiv 1 \mod 2} \equiv 1 \mod 2
\]

If \(\binom{u_0}{v_{00}} > 0\) or \(\binom{u_0}{v_{01}} > 0\), then \(v_0\) is a subsequence of \(u\).
This contradicts \((\ast)\).

Same proof for \((u_1, v_1)\).
Example: \((u, v) = (101, 11)\) satisfies \((*) \Rightarrow \binom{u}{v} \equiv 1 \mod 2\)

\[
\begin{array}{c}
U_3 \\
U_4 \\
U_5
\end{array}
\]

\(\leadsto\) Creation of segments of slope 1
Endpoint \((3/8, 5/8) = (\text{val}_2(11)/2^3, \text{val}_2(101)/2^3)\)
Length \(\sqrt{2} \cdot 2^{-3}\)

\(S_{u,v} \subset [0, 1] \times [1/2, 1]\) endpoint \((\text{val}_2(v)/2^{|u|}, \text{val}_2(u)/2^{|u|})\)
length \(\sqrt{2} \cdot 2^{-|u|}\)
Definition: Set of segments of slope 1

\[ \mathcal{A}_0 = \bigcup_{(u,v)} S_{u,v} \subset [0, 1] \times [1/2, 1] \]

satisfying (⋆)

Generalized Pascal triangles

Manon Stipulanti (ULiège)
Example: (1, 1) satisfies (∗)
Segment $S_{1,1}$ endpoint $(1/2, 1/2)$ length $\sqrt{2} \cdot 2^{-1}$
$c : (x, y) \mapsto (x/2, y/2) \quad h : (x, y) \mapsto (x, 2y)$
Example: \((1, 1)\) satisfies \((\ast)\)

Segment \(S_{1,1}\) endpoint \((1/2, 1/2)\) length \(\sqrt{2} \cdot 2^{-1}\)

\(c: (x, y) \mapsto (x/2, y/2)\)  \(h: (x, y) \mapsto (x, 2y)\)
Example: $(1, 1)$ satisfies $(*)$
Segment $S_{1,1}$ endpoint $(1/2, 1/2)$ length $\sqrt{2} \cdot 2^{-1}$
$c : (x, y) \mapsto (x/2, y/2) \quad h : (x, y) \mapsto (x, 2y)$
Example: \((1, 1)\) satisfies \((\star)\)

Segment \(S_{1,1}\) endpoint \((1/2, 1/2)\) length \(\sqrt{2} \cdot 2^{-1}\)

\(c : (x, y) \mapsto (x/2, y/2)\) \hspace{1cm} \(h : (x, y) \mapsto (x, 2y)\)
Example: \((1, 1)\) satisfies \((\star)\)

Segment \(S_{1,1}\) endpoint \((1/2, 1/2)\) length \(\sqrt{2} \cdot 2^{-1}\)

\(c: (x, y) \mapsto (x/2, y/2)\) \hspace{1cm} \(h: (x, y) \mapsto (x, 2y)\)
**Definition:** Set of segments of different slopes
\[ c : (x, y) \mapsto \left( \frac{x}{2}, \frac{y}{2} \right) \]
\[ h : (x, y) \mapsto (x, 2y) \]

\[ \mathcal{A}_n = \bigcup_{0 \leq j \leq i \leq n} h^j(c^i(A_0)) \]
**Definition:** Set of segments of different slopes

\[ c : (x, y) \mapsto (x/2, y/2) \]
\[ h : (x, y) \mapsto (x, 2y) \]

\[ A_n = \bigcup_{0 \leq j \leq i \leq n} h^j(c^i(A_0)) \]

**Lemma:** \((A_n)_{n \geq 0}\) is a Cauchy sequence
**Definition:** Set of segments of different slopes

\[ c : (x, y) \mapsto (x/2, y/2) \]

\[ h : (x, y) \mapsto (x, 2y) \]

\[ \mathcal{A}_n = \bigcup_{0 \leq j \leq i \leq n} h^j(c^i(\mathcal{A}_0)) \]

**Lemma:** \((\mathcal{A}_n)_{n \geq 0}\) is a Cauchy sequence

**Definition:** Limit object \(\mathcal{L}\)
A key result

Theorem (Leroy, Rigo, S., 2016)

The sequence \((U_n)_{n \geq 0}\) of compact sets converges to the compact set \(L\) when \(n\) tends to infinity (for the Hausdorff distance).

“Simple” characterization of \(L\): (⋆) condition
First step: coloring the cells of the grids regarding the parity

**Extension using Lucas’ theorem**

Everything still holds for binomial coefficients $\equiv r \mod p$ with

- base-2 expansions of integers
- $p$ a prime
- $r \in \{1, \ldots, p-1\}$

**Theorem (Lucas, 1878)**

Let $p$ be a prime number. If $m = m_k p^k + \cdots + m_1 p + m_0$ and $n = n_k p^k + \cdots + n_1 p + n_0$ then

$$
\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \mod p.
$$
Example with $p = 3$, $r = 2$

Left: binomial coefficients $\equiv 2 \mod 3$
Right: estimate of the corresponding limit object
Extension to any integer base

Everything still holds for binomial coefficients \( \equiv r \mod p \) with
- base-\( b \) expansions of integers with \( b \geq 2 \)
- \( p \) a prime
- \( r \in \{1, \ldots, p - 1\} \)

Example: base 3, \( \equiv 1 \mod 2 \)
Fibonacci numeration system

Definitions:

- Fibonacci numbers \((F(n))_{n \geq 0}\):
  \[ F(0) = 1, \ F(1) = 2, \ F(n + 2) = F(n + 1) + F(n) \quad \forall n \geq 0 \]
  \(1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots \)

- \(\text{rep}_F(n)\) greedy Fibonacci representation of \(n \in \mathbb{N}_{>0}\) starting with 1

- \(\text{rep}_F(0) = \varepsilon\) where \(\varepsilon\) is the empty word

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\text{rep}_F(n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(\varepsilon)</td>
</tr>
<tr>
<td>1</td>
<td>(1 \times F(0))</td>
</tr>
<tr>
<td>2</td>
<td>(1 \times F(1) + 0 \times F(0))</td>
</tr>
<tr>
<td>3</td>
<td>(1 \times F(2) + 0 \times F(1) + 0 \times F(0))</td>
</tr>
<tr>
<td>4</td>
<td>(1 \times F(2) + 0 \times F(1) + 1 \times F(0))</td>
</tr>
<tr>
<td>5</td>
<td>(1 \times F(3) + 0 \times F(2) + 0 \times F(1) + 0 \times F(0))</td>
</tr>
<tr>
<td>6</td>
<td>(1 \times F(3) + 0 \times F(2) + 0 \times F(1) + 1 \times F(0))</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>
Generalized Pascal triangle $P_F$ in Fibonacci base

<table>
<thead>
<tr>
<th>$(\text{rep}_F(m))$</th>
<th>$\varepsilon$</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>101</th>
<th>1000</th>
<th>1001</th>
<th>1010</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>100</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$\text{rep}_F(m)$</td>
<td>101</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>1000</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>1001</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>1010</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>...</td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\ddots$</td>
</tr>
</tbody>
</table>

Binomial coefficient of finite words: $(\text{rep}_F(m))$ $\binom{\text{rep}_F(k)}{\varepsilon}$

Rule (not local):

$$\left(\begin{array}{c}
u a \\vb\end{array}\right) = \left(\begin{array}{c}u \\vb\end{array}\right) + \delta_{a,b}\left(\begin{array}{c}u \\v\end{array}\right)$$
The first six elements of the sequence \((U'_n)_{n \geq 0}\)

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The tenth element

Lines of different slopes:

Golden Ratio
The tenth element

Lines of different slopes: $\varphi^n$, $n \geq 0$, with $\varphi = \frac{1+\sqrt{5}}{2}$ Golden Ratio
Similar convergence result

**Theorem (S., 2018)**

The sequence \((U'_n)_{n \geq 0}\) of compact sets converges to a limit compact set \(L'\) when \(n\) tends to infinity (for the Hausdorff distance).

“Simple” characterization of \(L'\): \((\star')\) condition
\[ \beta \in \mathbb{R}_{>1} \quad A_\beta = \{0, 1, \ldots, \lceil \beta \rceil - 1\} \]

\[ x \in [0, 1) \rightsquigarrow x = \sum_{j=1}^{+\infty} c_j \beta^{-j}, \quad c_j \in A_\beta \]

Greedy way: \( d_\beta(x) = c_1 c_2 c_3 \cdots \)

\[ d_\beta(1) = \begin{cases} 
(\beta - 1)^\omega & \text{if } \beta \in \mathbb{N} \\
(\lceil \beta \rceil - 1)d_\beta(1 - (\lceil \beta \rceil - 1)/\beta) & \text{if } \beta \notin \mathbb{N}
\end{cases} \]

**Definition**

\( \beta \in \mathbb{R}_{>1} \) is a *Parry number* if \( d_\beta(1) \) is ultimately periodic.

**Example:** \( b \in \mathbb{N}_{>1} \): \( d_b(1) = (b - 1)^\omega \)

\( \varphi \): \( d_\varphi(1) = 110^\omega \)

Parry number \rightsquigarrow \text{algebraic integer whose conjugates have modulus less than } \beta \text{ (Perron number)}
A *linear numeration system* is a sequence $(U(n))_{n \geq 0}$ such that

- $U$ increasing
- $U(0) = 1$
- $\sup_{n \geq 0} \frac{U(n+1)}{U(n)}$ bounded by a constant
- $U$ linear recurrence relation
  \[ \exists k \geq 1, \exists a_0, \ldots, a_{k-1} \in \mathbb{Z} \text{ such that} \]
  \[ U(n + k) = a_{k-1} U(n + k - 1) + \cdots + a_0 U(n) \quad \forall n \geq 0 \]

Greedy representation in $(U(n))_{n \geq 0}$:

\[ n = \sum_{j=0}^{\ell} c_j U(j) \quad \text{rep}_U(n) = c_\ell \cdots c_0 \in L_U = \text{rep}_U(\mathbb{N}) \]

Example: integer base $(b^n)_{n \geq 0}$ with $b \in \mathbb{N}_{>1}$,
Fibonacci numeration system $(F(n))_{n \geq 0}$
Parry number $\beta \in \mathbb{R}_{>1} \leadsto$ linear numeration system $(U_\beta(n))_{n \geq 0}$

- $d_\beta(1) = t_1 \cdots t_m 0^\omega$

  $U_\beta(0) = 1$
  $U_\beta(i) = t_1 U_\beta(i - 1) + \cdots + t_i U_\beta(0) + 1 \quad \forall 1 \leq i \leq m - 1$
  $U_\beta(n) = t_1 U_\beta(n - 1) + \cdots + t_m U_\beta(n - m) \quad \forall n \geq m$

- $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$

  $U_\beta(0) = 1$
  $U_\beta(i) = t_1 U_\beta(i - 1) + \cdots + t_i U_\beta(0) + 1 \quad \forall 1 \leq i \leq m + k - 1$
  $U_\beta(n) = t_1 U_\beta(n - 1) + \cdots + t_{m+k} U_\beta(n - m - k) \quad \forall n \geq m + k$
  $+ U_\beta(n - k) - t_1 U_\beta(n - k - 1) - \cdots$
  $- t_m U_\beta(n - m - k)$

Examples:

- $b \in \mathbb{N}_{>1} \leadsto (b^n)_{n \geq 0}$ base $b$
- $\varphi \leadsto (F(n))_{n \geq 0}$ Fibonacci numeration system
• $\beta \in \mathbb{R}_{>1}$ Parry number

• $\left( U_\beta(n) \right)_{n \geq 0}$ Parry numeration system

• Generalized Pascal triangle $P_\beta$ in $\left( U_\beta(n) \right)_{n \geq 0}$ indexed by words of $L_{U_\beta}$

• Sequence of compact sets extracted from $P_\beta$ (first $U_\beta(n)$ rows and columns of $P_\beta$)

• Convergence to a limit object (same technique)
  • Lines of different slopes: $\beta^n$, $n \geq 0$
  • (⋆') condition

• Works modulo any prime number
Example 1

\( \varphi^2 \)
$\beta_1 \approx 2.47098$ dominant root of $P(X) = X^4 - 2X^3 - X^2 - 1$
Example 3

$\beta_2 \approx 1.38028$ dominant root of $P(X) = X^4 - X^3 - 1$
Example 4

\[ \beta_3 \approx 2.80399 \text{ dominant root of } P(X) = X^4 - 2X^3 - 2X^2 - 2 \]
Example 5

$\beta_4 \approx 1.32472$ dominant root of polynomial $P(X) = X^5 - X^4 - 1$
In this talk:

<table>
<thead>
<tr>
<th></th>
<th>Generalized Pascal triangle</th>
<th>Convergence mod $p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>base 2</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>integer base</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Fibonacci</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Parry</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>
Other related work

- Regularity of the sequence counting subword occurrences: result for any integer base \( b \) and the Fibonacci numeration system
- Behavior of the summatory function: result for any integer base \( b \) (exact behavior) and the Fibonacci numeration system (asymptotics)


