LIÈGE université
Sciences Appliquées

Structural & Stochastic Dynamics
Urban & Environmental Engineering

Closed-form response of a linear fractional visco-elastic oscillator under arbitrary stationary input

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UEE
Urban & Environmental Engineering

Fractional derivatives

Classical differential operator :

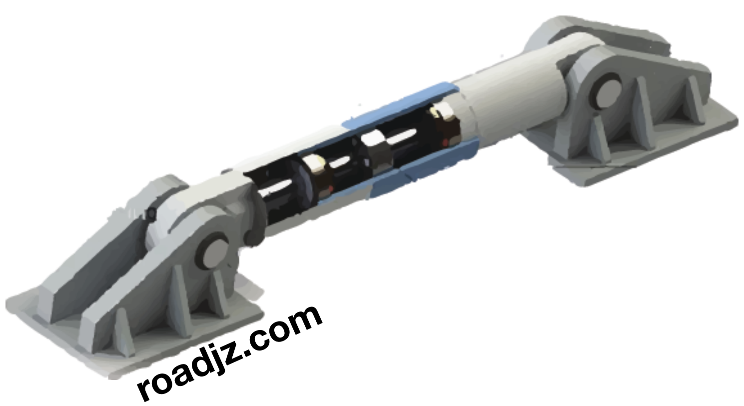
$$\mathcal{D}^0 y(t) = y(t) \qquad \mathcal{D}^1 y(t) = \dot{y}(t) \qquad \mathcal{D}^2 y(t) = \ddot{y}(t)$$

Fractional derivative operator (Riemann–Liouville) :

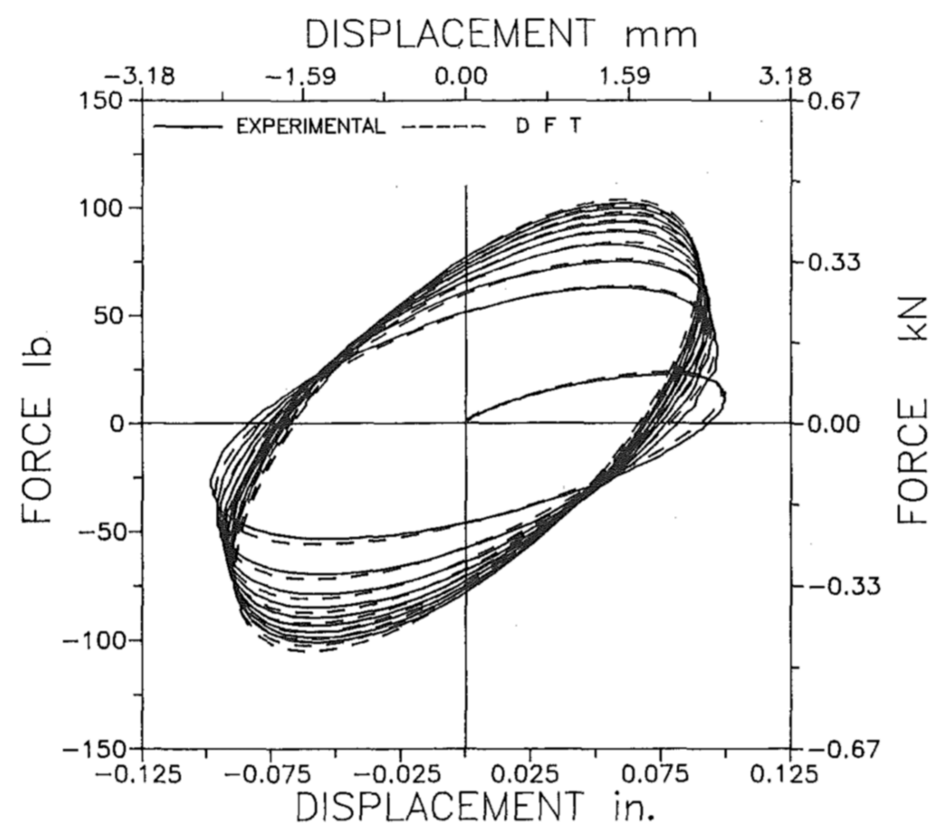
$$\mathcal{D}^\alpha y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{y}(\tilde{t})}{(t-\tilde{t})^\alpha} d\tilde{t}$$

(NB: linear operator)

Fractional derivatives



Constitutive law
of dissipative devices [1]

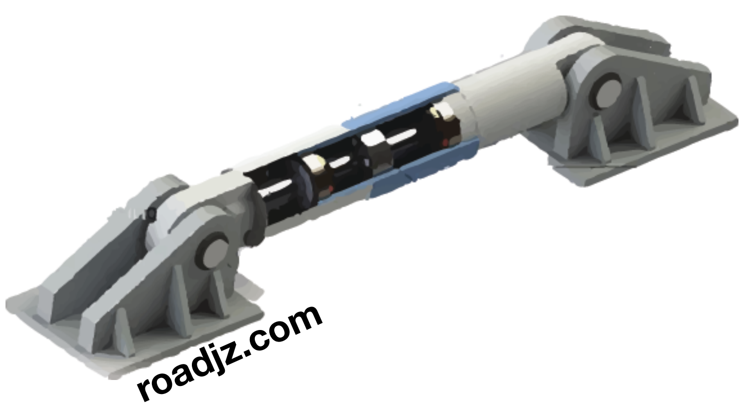


Fractional Maxwell model

$$F(t) + \lambda \mathcal{D}^{\alpha_1} F(t) = \mathcal{D}^{\alpha_2} y(t)$$

[1] Makris, Constantinou, 1991, Fractional-derivative maxwell model for viscous dampers

Fractional derivatives



Constitutive law of dissipative devices [1]

Fractional Order Controllers for path tracking[2]

Treatment of brain tumors [3]

[1] Makris, Constantinou, 1991, Fractional-derivative maxwell model for viscous dampers
[2] Suarez , Vinagre, Calderon, Monje and Chen, Using Fractional Calculus for Lateral and Longitudinal Control of Autonomous Vehicles
[3] Libertiaux, Pascon , Differential versus integral formulation of fractional hyperviscoelastic constitutive laws for brain tissue modelling

Considered problem

$$x''(\tau) + 2\xi \mathcal{D}^{\alpha} x(\tau) + x(\tau) = u(\tau)$$

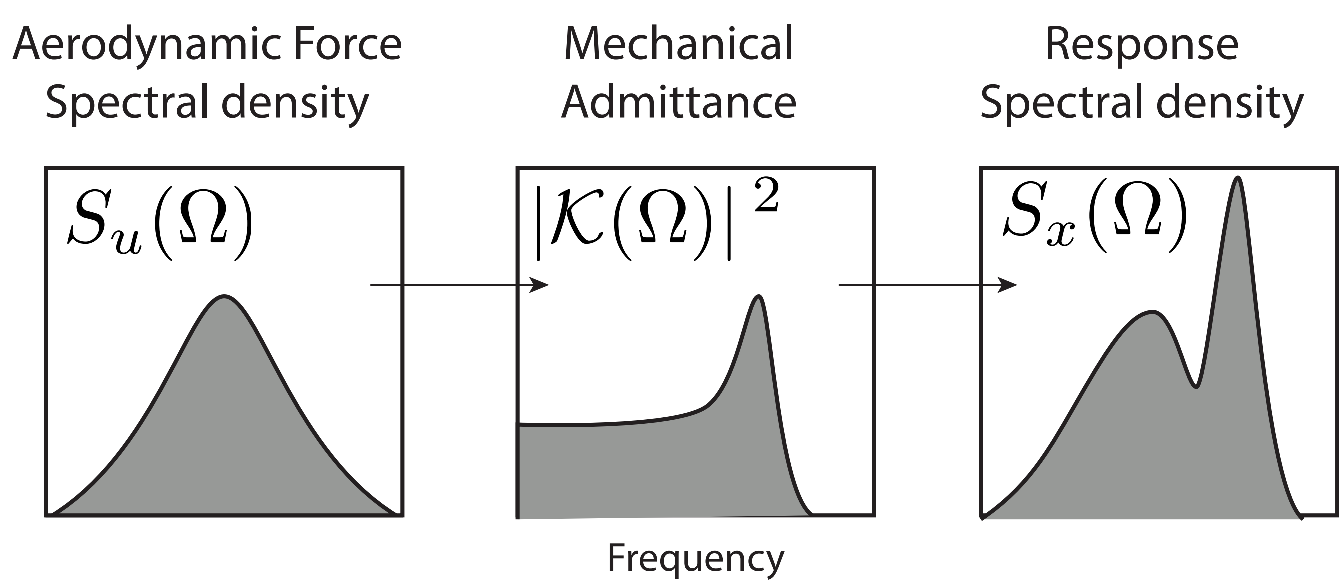
Linear equation — Solve in Frequency domain

$$\left(-\Omega^2 + 2\xi \mathrm{e}^{-\mathrm{i}\frac{\alpha\pi}{2}} \left|\Omega\right|^{\alpha} + 1\right) X(\Omega) = U(\Omega) \longrightarrow X(\Omega) = \mathcal{K}(\Omega) U(\Omega)$$

- Stationary loading
- Arbitrary loading PSD, $S_f(\omega)$
- Slow loading (turbulence)

→ Stead-state response ? $S_x(\omega), \sigma_x$

$$S_x(\Omega) = \left|\mathcal{K}(\Omega)\right|^2 S_u(\Omega)$$



Considered problem

$$x''(\tau) + 2\xi \mathcal{D}^{\alpha} x(\tau) + x(\tau) = u(\tau)$$

Existing solutions of similar problems

- Monte Carlo Simulations [5]

[5] Di Paola, Failla, Pirrotta. (2012). Stationary and non-stationary stochastic response of linear fractional viscoelastic systems

Considered problem

$$x''(\tau) + 2\xi \mathcal{D}^{\alpha} x(\tau) + x(\tau) = u(\tau)$$

Existing solutions of similar problems

- Monte Carlo Simulations [5]
- Stochastic Averaging (energy envelope) [6]

$$x(\tau) = A(\tau) \cos \Theta(\tau)$$
$$\dot{x}(\tau) = -A(\tau)\nu(A, \Theta) \sin \Theta(\tau)$$

$$x''(\tau) + 2\xi \mathcal{D}^{\alpha} x(\tau) + x(\tau) + \varepsilon z(x, x') = W(\tau)$$ [Yang et al.]

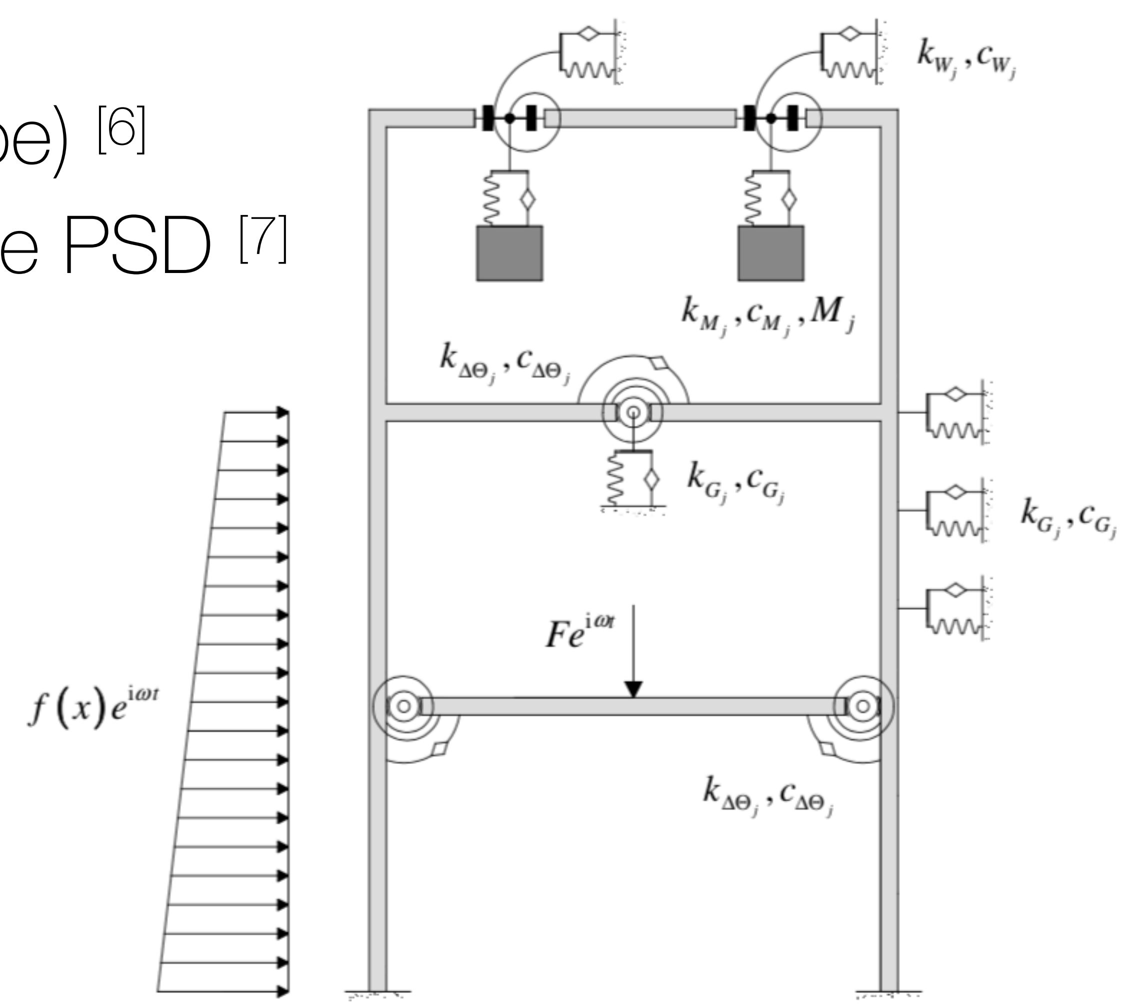
[6] Yang, Xu, Jia, Han (2015) Stationary response of nonlinear system with caputo-type fractional derivative damping under gaussian white noise excitation

Considered problem

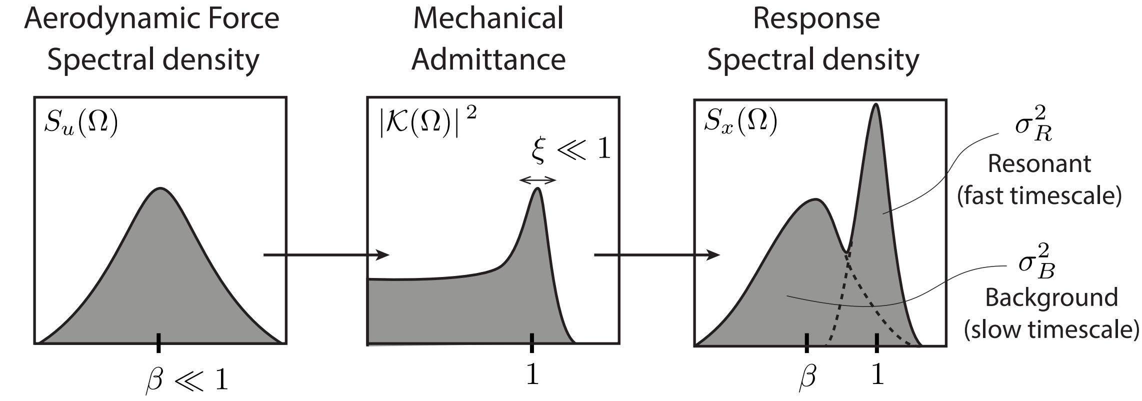
$$x''(\tau) + 2\xi \mathcal{D}^{\alpha} x(\tau) + x(\tau) = u(\tau)$$

Existing solutions of similar problems

- Monte Carlo Simulations [5]
- Stochastic Averaging (energy envelope) [6]
- (Numerical) Integration of the response PSD [7]



[7] Failla (2017). Stationary response of beams and frames with fractional dampers through exact frequency response functions.

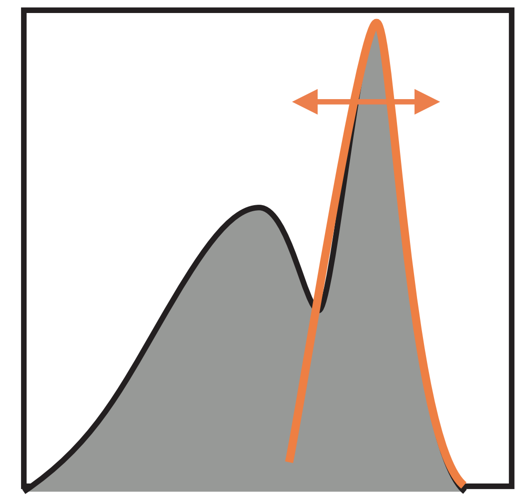


Aerodynamic Force
Spectral density
$S_u(\Omega)$
$\beta \ll 1$
Mechanical
Admittance
$|\mathcal{K}(\Omega)|^2$
$\xi \ll 1$
1
Response
Spectral density
$S_x(\Omega)$
σ_R^2
Resonant
(fast timescale)
σ_B^2
Background
(slow timescale)
β
1

The Multiple Timescale Spectral Analysis [8]

« Use smallness of contributing components to separate them »

Response Spectral density

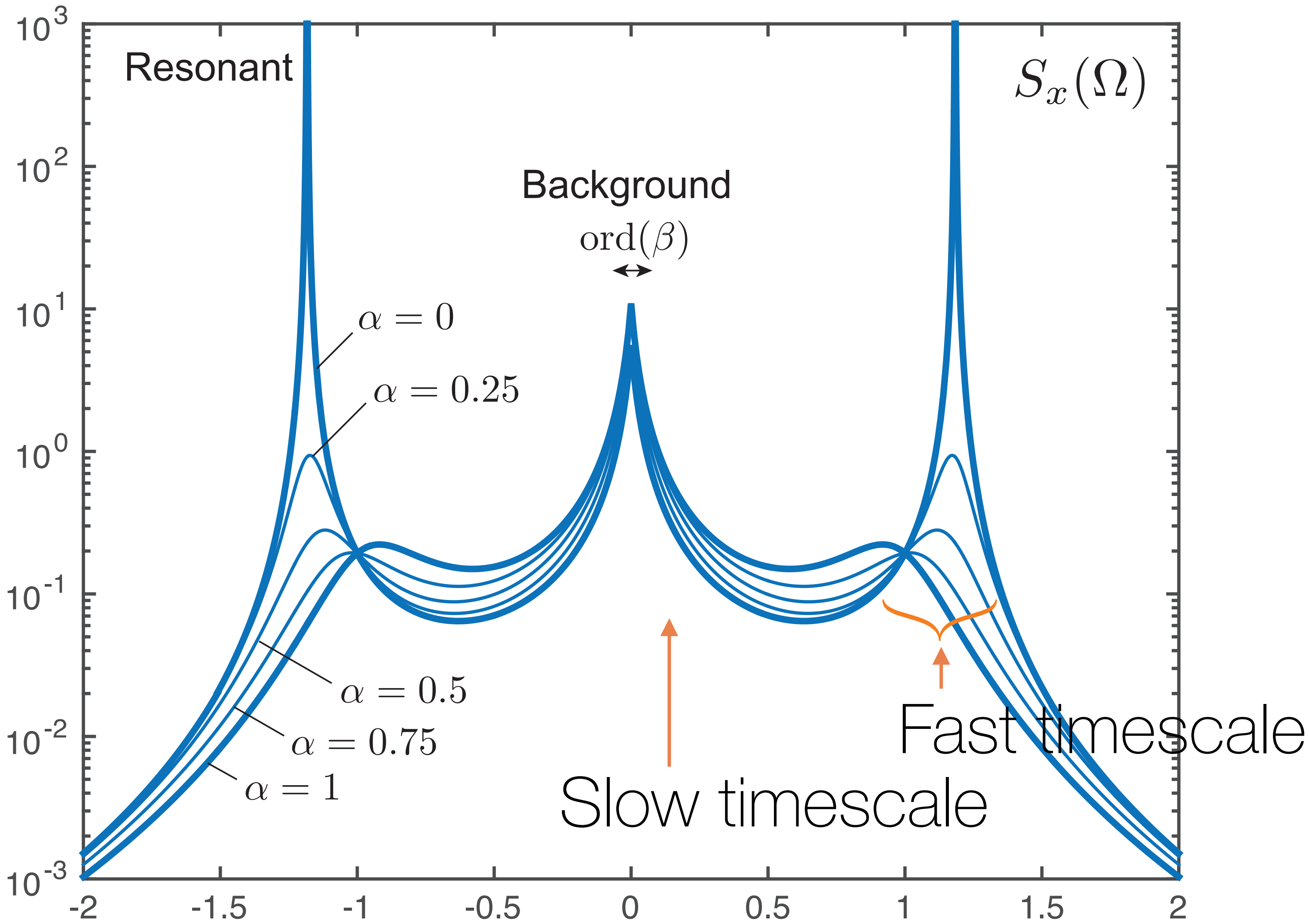


- Identify the components to the integral
- Loop on components
  - Find local approximation
  - Use stretched coordinate to make domain of order 1
  - Subtract off approximation
  - Construct residual
  - Iterate

[8] Denoël, 2015. Multiple timescale spectral analysis

The Multiple Timescale Spectral Analysis [9]

$$x''(\tau) + 2\xi \mathcal{D}^{\alpha} x(\tau) + x(\tau) = u(\tau)$$



$\xi = 0.2$

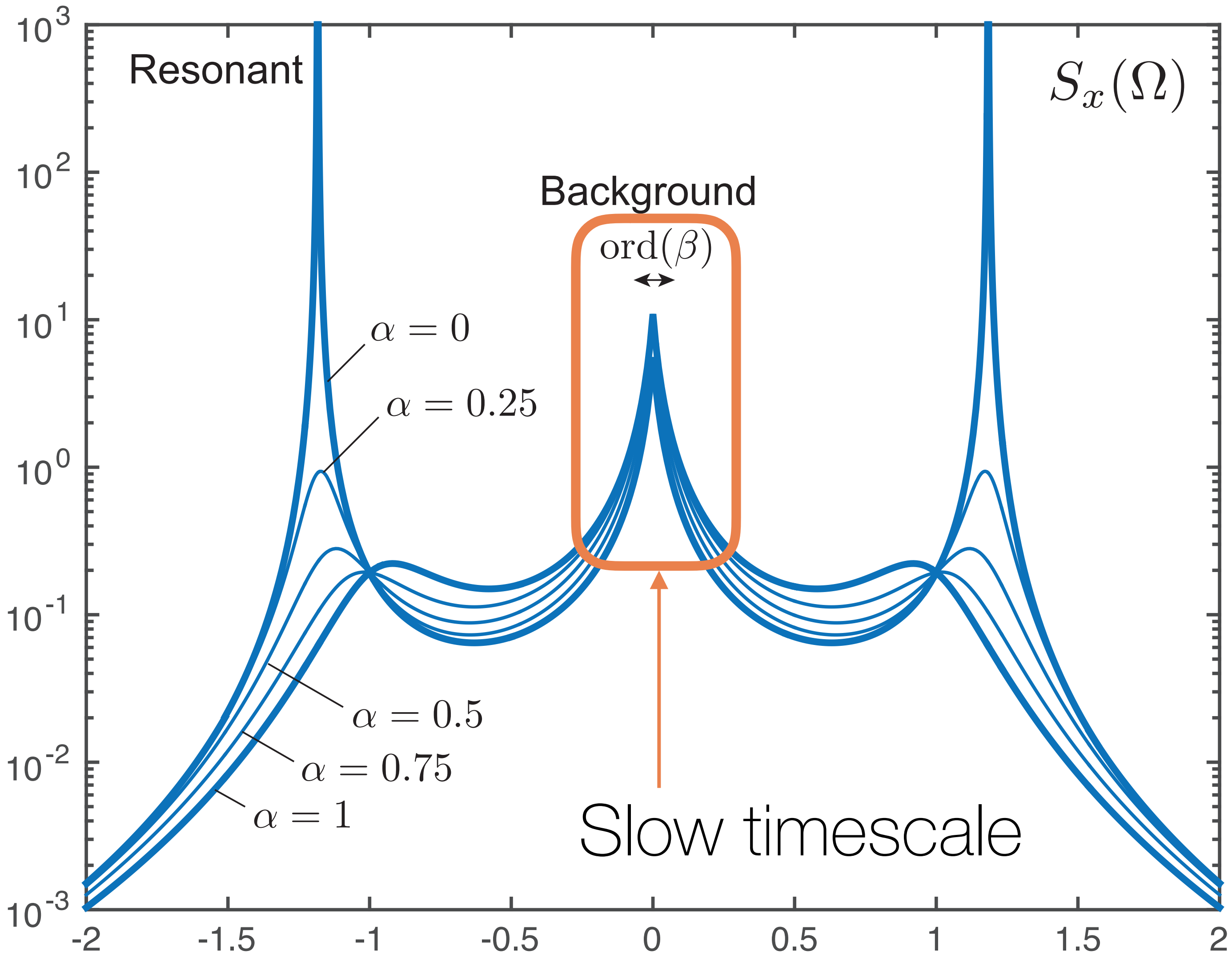
Location of the resonant peak:

$$\begin{array}{ll} \sim 1 & \text{if } \alpha \simeq 1 \\ \sim \sqrt{1+2\xi} & \text{if } \alpha \simeq 0 \end{array}$$

[9] Denoël, 2018. Multiple timescale spectral analysis of a linear fractional viscoelastic system under colored excitation

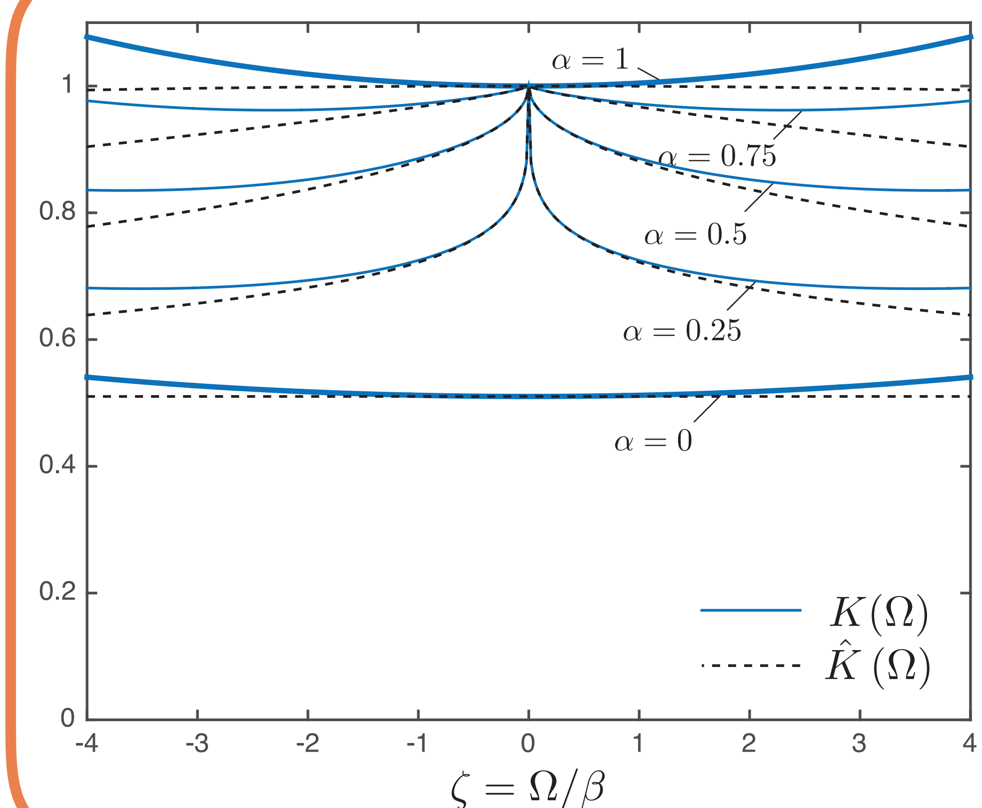
The Multiple Timescale Spectral Analysis [9]

Background component (slow)



$\xi = 0.2$

Local approximation



$$\sigma_{x,b}^2 = 1 - 4\xi \cos\frac{\alpha\pi}{2} m_{u,\alpha}$$

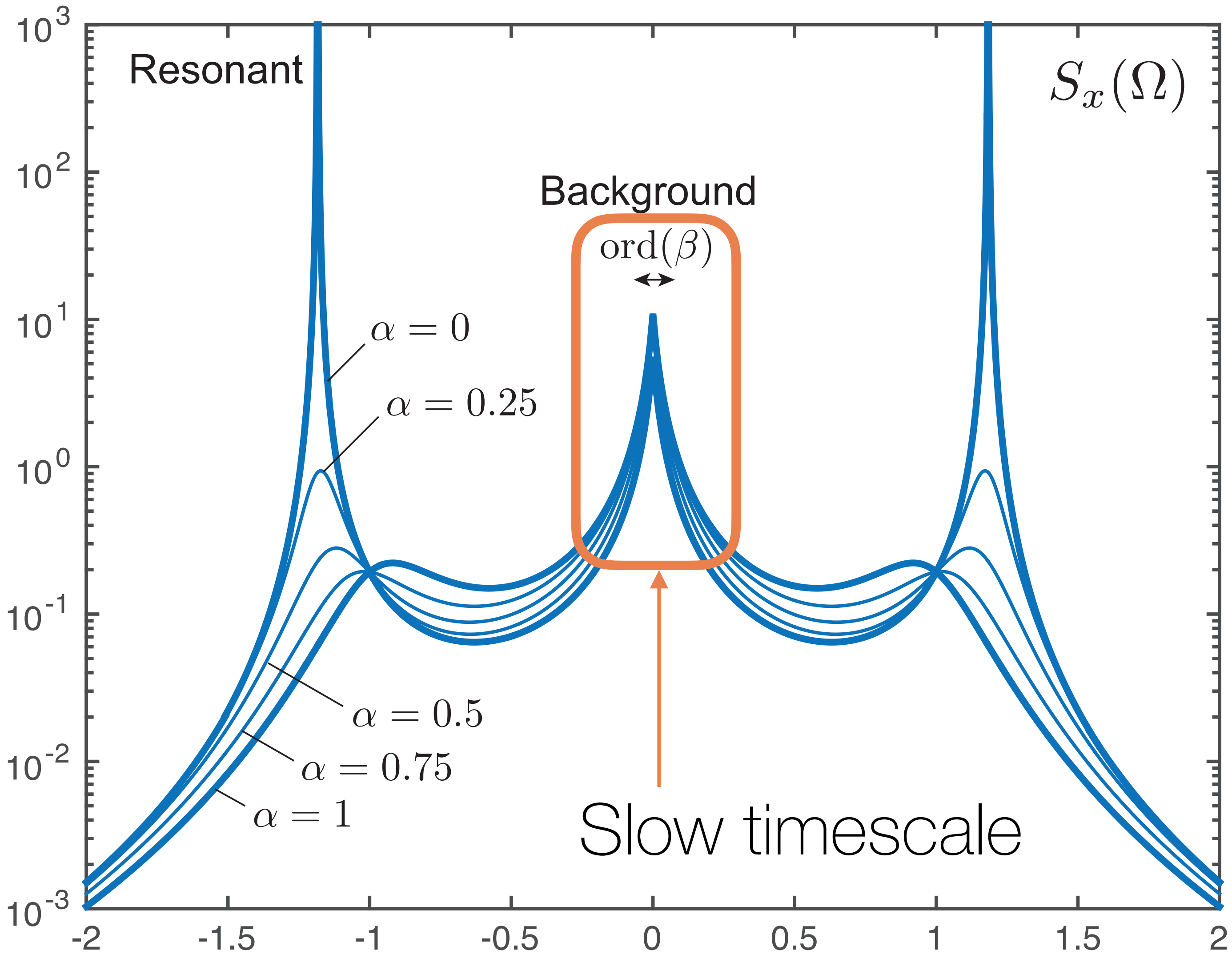
Fractional spectral moment

Scalar (no integration)

[9] Denoël, 2018. Multiple timescale spectral analysis of a linear fractional viscoelastic system under colored excitation

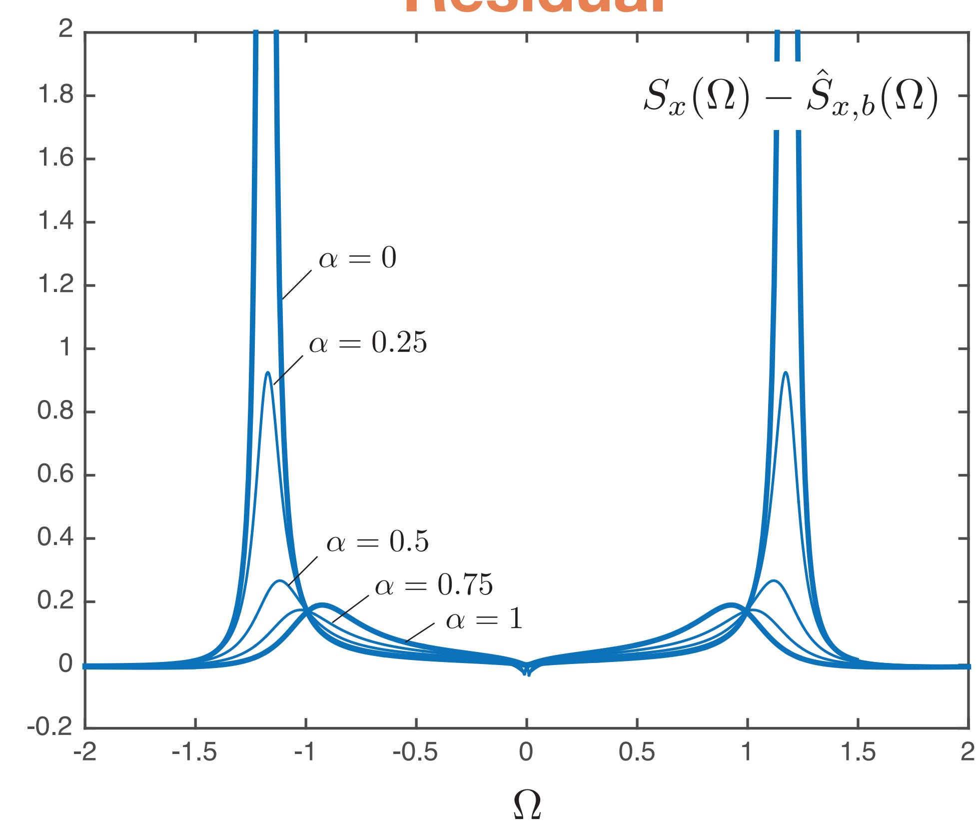
The Multiple Timescale Spectral Analysis [9]

Background component (slow)



$\xi = 0.2$

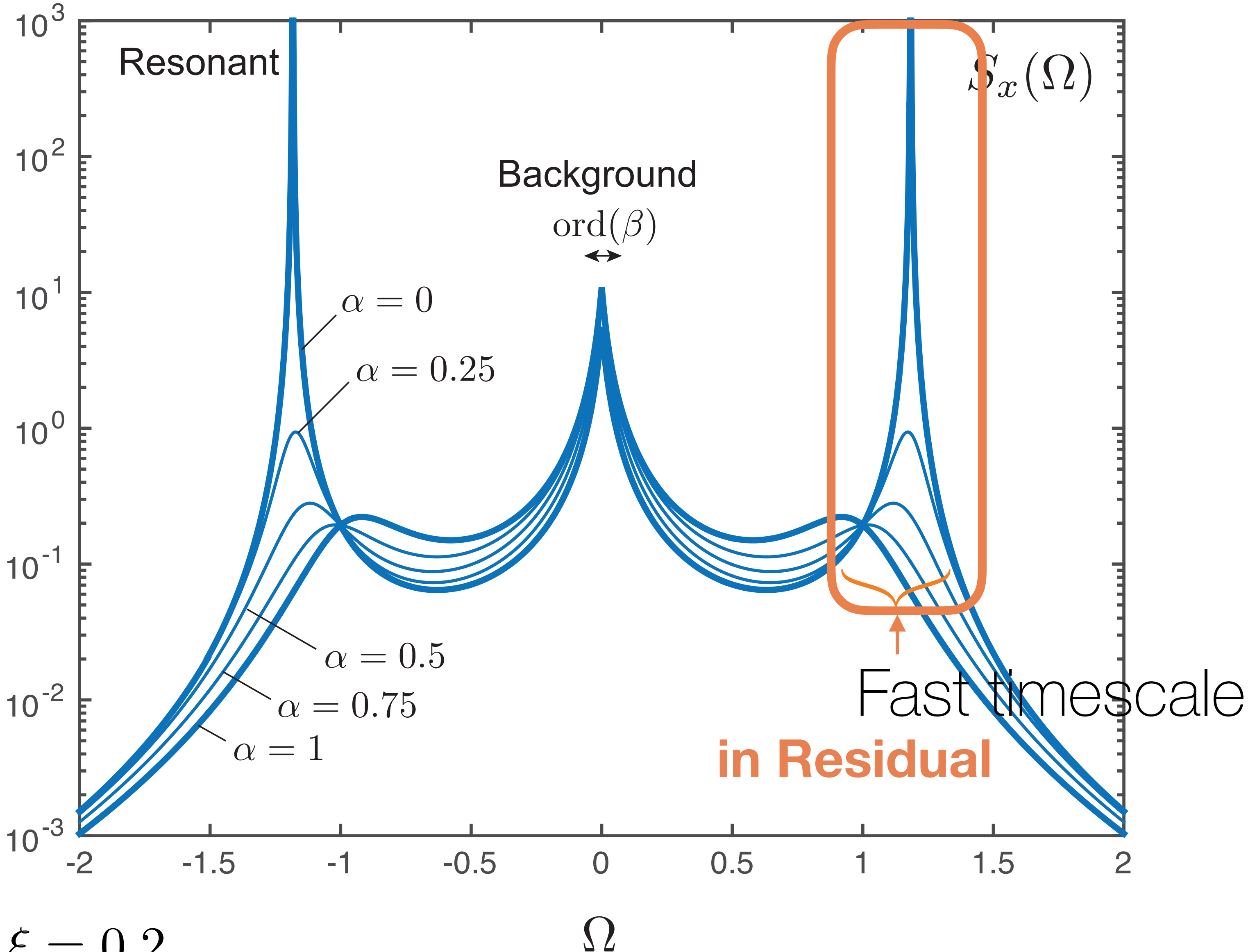
Residual



[9] Denoël, 2018. Multiple timescale spectral analysis of a linear fractional viscoelastic system under colored excitation

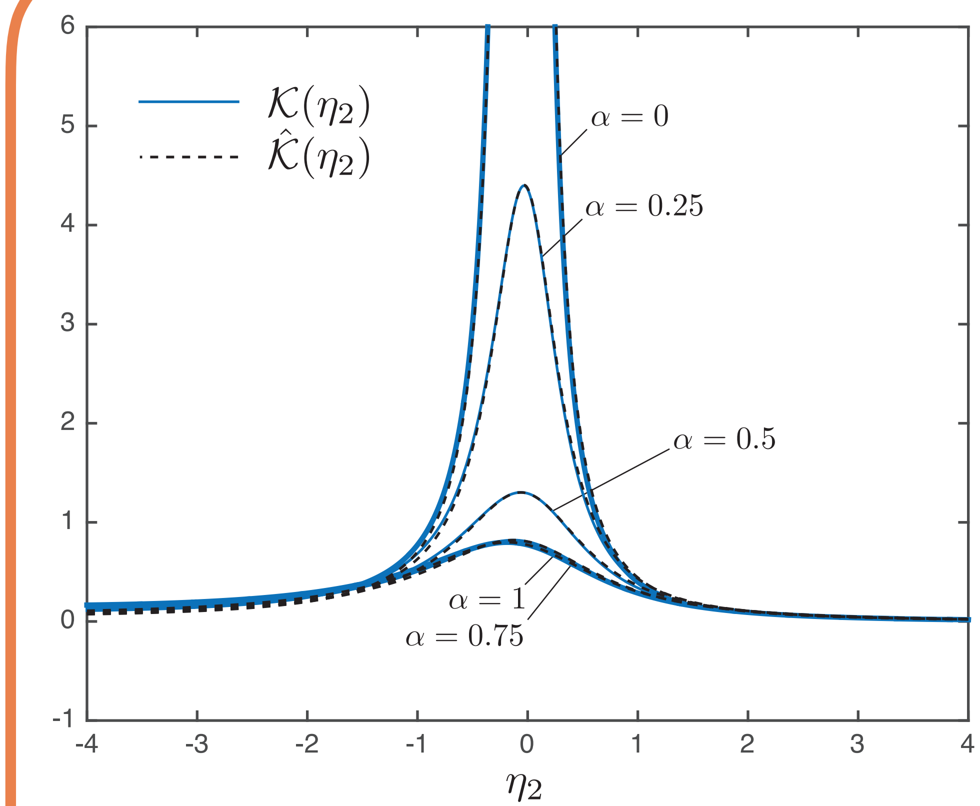
The Multiple Timescale Spectral Analysis [9]

Resonant component (fast)



$\xi = 0.2$

Local approximation



$$\eta_2 = \frac{\Omega - (1 + \mathcal{C}\xi)}{\xi}$$

$$\sigma_{x,r}^2 = \frac{\pi S_u (1 + \mathcal{C}\xi)}{2\mathcal{S}\xi\sqrt{1 + 2\xi\mathcal{C}}}$$

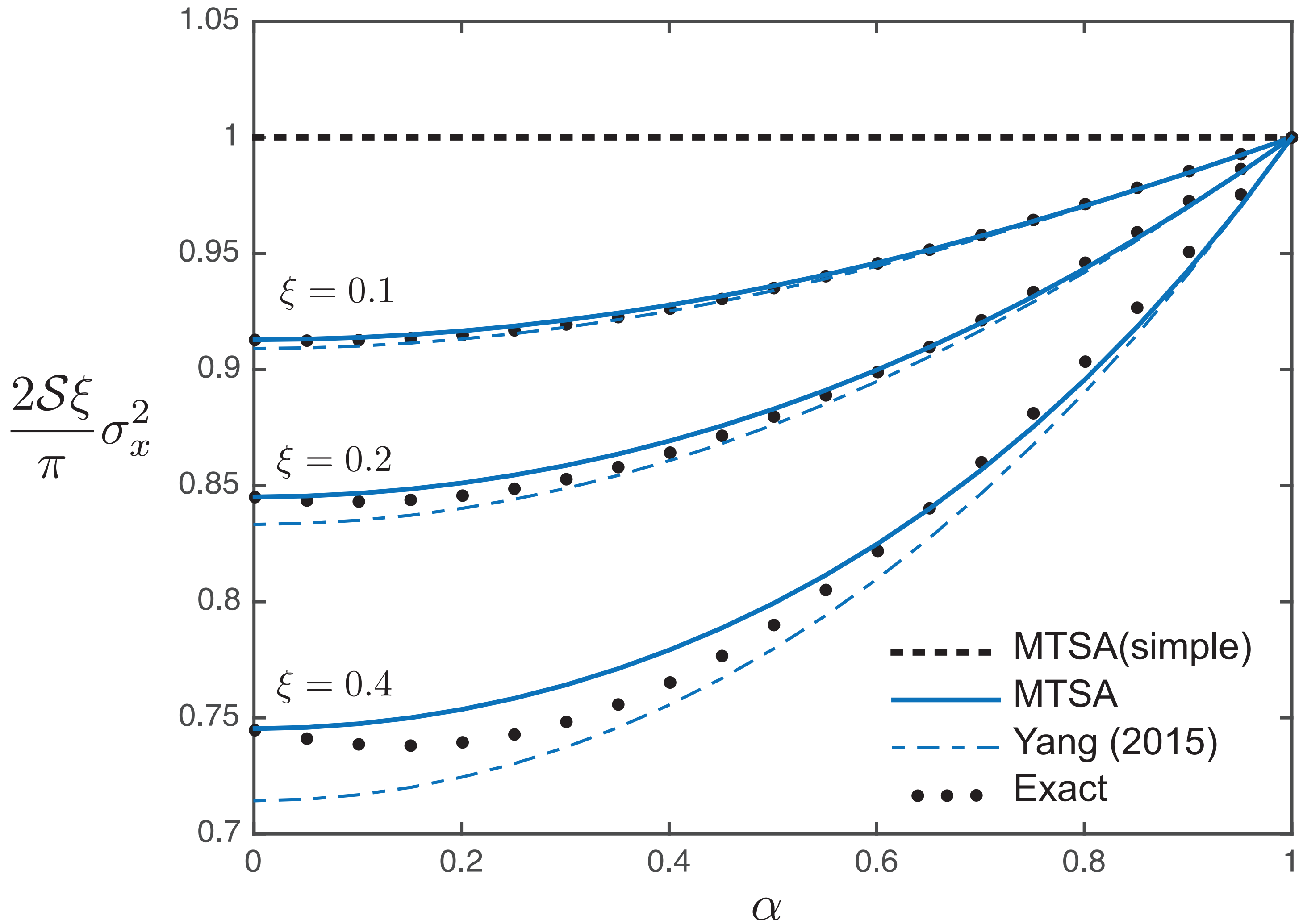
$$\mathcal{C} = \cos\frac{\alpha\pi}{2} \qquad \mathcal{S} = \sin\frac{\alpha\pi}{2}$$

[9] Denoël, 2018. Multiple timescale spectral analysis of a linear fractional viscoelastic system under colored excitation

Validation

In summary: $\sigma_x^2 = \sigma_{x,b}^2 + \sigma_{x,r}^2 = 1 - 4\xi\mathcal{C}m_{u,\alpha} + \dfrac{\pi S_u\,(1+\mathcal{C}\xi)}{2\mathcal{S}\xi\sqrt{1+2\xi\mathcal{C}}}$

1. Delta-correlated excitation



Multiple timescale spectral analysis

$$\sigma_x^2 = \frac{\pi}{2\mathcal{S}\xi}\frac{1}{\sqrt{1+2\xi\mathcal{C}}}$$

Stochastic Averaging [Yang, 2015]

$$\sigma_x^2 = \frac{\pi}{2\mathcal{S}\xi}\frac{1}{1+\xi\mathcal{C}}$$

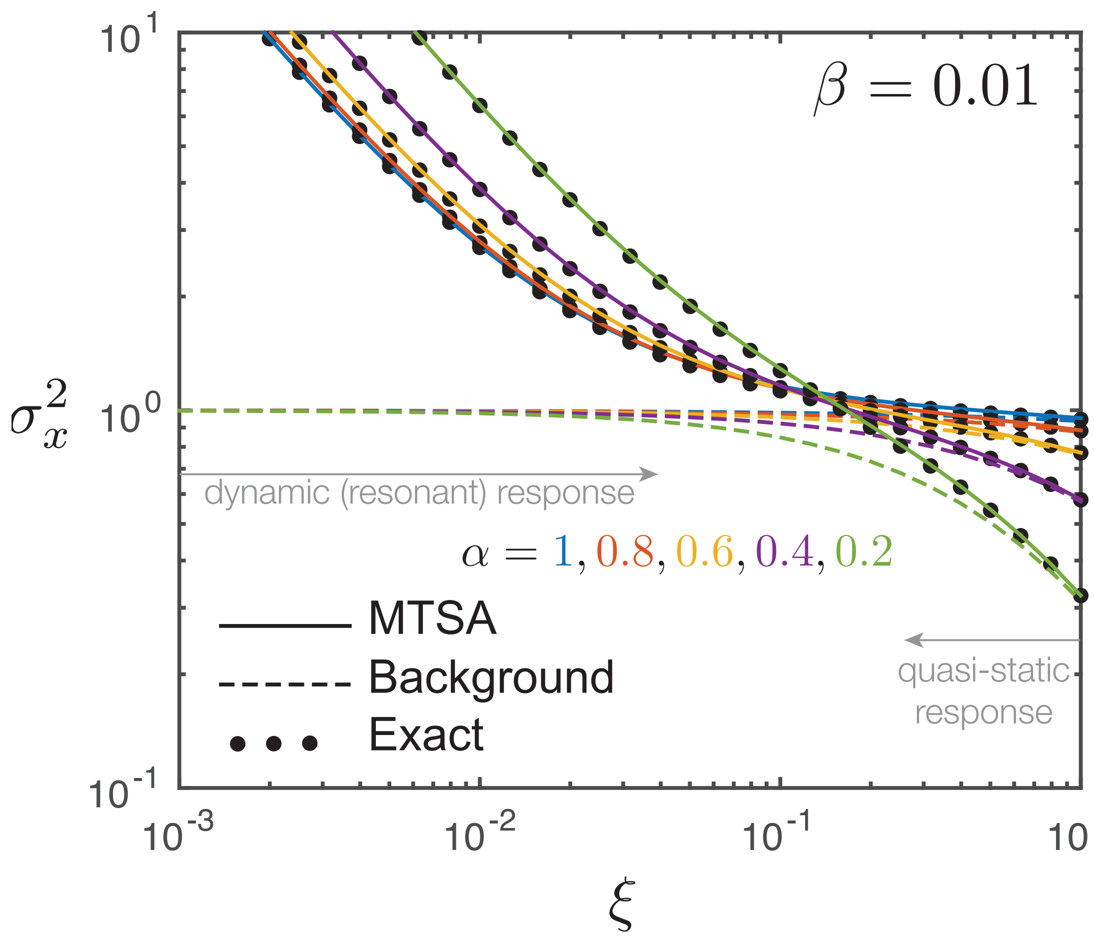
$$\mathcal{C} = \cos\frac{\alpha\pi}{2} \qquad \mathcal{S} = \sin\frac{\alpha\pi}{2}$$

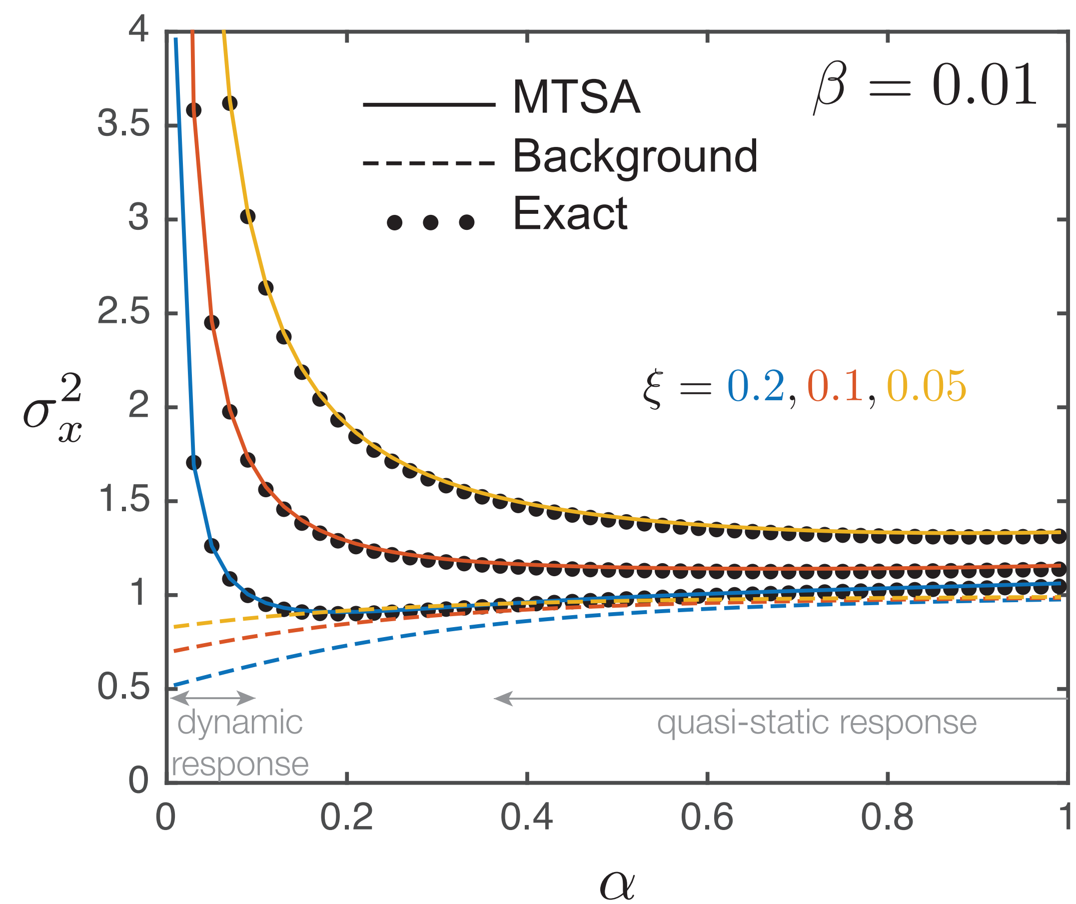
[9] Denoël, 2018. Multiple timescale spectral analysis of a linear fractional viscoelastic system under colored excitation

Validation

2. Wind spectrum excitation

$$S_f(\omega;\beta) = \sigma_f \frac{\frac{0.546}{\beta}}{\left(1+\frac{1.64}{\beta}|\omega|\right)^{5/3}}$$





[9] Denoël, 2018. Multiple timescale spectral analysis of a linear fractional viscoelastic system under colored excitation

Conclusion

Multiple Timescale Spectral Analysis

- Assumptions are made in the frequency domain
- Shares many features of time domain multiple scales method (secular terms, freedom in developing approximations)
- Already applied in many other contexts: MDOF, slight nonlinearities, higher statistical moments (skewness, kurtosis), modal coupling, wave spectrum analysis.

- A simple expression for the response of a SDOF fractional visco-elastic system. Variance is obtained by a simple algebraic equation

$$\sigma_x^2 = \sigma_{x,b}^2 + \sigma_{x,r}^2 = 1 - 4\xi\mathcal{C}m_{u,\alpha} + \frac{\pi S_u\left(1+\mathcal{C}\xi\right)}{2\mathcal{S}\xi\sqrt{1+2\xi\mathcal{C}}}$$

- Simple way to identify the regime in which the structure responds

Thank you !

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