Closed-form response of a linear fractional visco-elastic oscillator under arbitrary stationary input

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Abstract

This paper studies the structural response of a single degree-of- freedom structure including a fractional derivative constitutive term. Unlike usual existing models for this kind of structure, the excitation is not necessarily a Markovian process but it is slowly varying in time, so that a timescale separation is used. Following the general formulation of the Multiple Timescale Spectral Analysis [1], the solution is developed as a sum of background and resonant components. Because of the specific shape of the frequency response function of a system equipped with a fractional viscoelastic device, the background component is not simply obtained as the variance of the loading divided by the stiffness of the system. On the contrary the resonant component is expressed as a simple extension of the existing formulation for a viscous system, at least at leading order. As a validation case, the proposed solution is shown to recover similar results (in the white noise excitation case) as former studies based on a stochastic averaging approach [2, 3, 4]. A better accuracy is however obtained in case of very small fractional exponent. Another example related to the buffeting analysis of a linear fractional viscoelastic system demonstrates the accuracy of the proposed formulation for colored excitation.

Keywords: Caputo fractional derivative, Riemann-Liouville fractional derivative, perturbation analysis, stochastic averaging, background component, resonant component

1. Fractional Calculus in Mechanics

Fractional calculus has attracted considerable attention over the last decades in the mechanical and structural engineering community. In some early works [5] dealing with the modeling of visco-elastic damping, a fractional derivative Maxwell model was used. An experimental demonstration [6] has confirmed the appropriateness of this model and further trigger scientific curiosity about fractional calculus in mechanics. This model is based on the Riemann—Liouville definition of the fractional derivative operator

$$\mathcal{D}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{y}(\tilde{t})}{(t-\tilde{t})^{\alpha}} d\tilde{t} \qquad (1)$$

where $\alpha \in [0,1]$ is the *fractional exponent*. Some other experimental investigations and identification techniques have shown that this model is able to accurately capture the dynamics of visco-elastic systems [7, 8, 9]. Also the linearity of this operator (the fractional derivative of a linear combination is the linear combination of the fractional derivatives) makes it rather attractive to combine viscoelastic devices with others existing features of linear stochastic dynamics [10, 11, 2, 12, 13]. The nonstationary solution of linear systems might be expressed in closed form [14], which constitutes another reason to use this type of model. As soon as nonlinear stochastic dynamical systems are considered, the exact solution is usually not obtained in closed form, even for wide-band (usually white noise) excitation. Approximations similar to or derived from the stochastic linearization and stochastic averaging methods [15, 16, 17, 18, 19], or those based on a Fokker-Planck equation of the process envelope [20] appear to be the most classical ways to deal with such problems. Narrow band excitations and complex dynamical interactions can be simplified with similar multiple scales approaches [21]. Other more realistic types of loadings [22] or even earthquake loadings of linear and nonlinear systems equipped with viscoelastic devices have also been considered in [23, 24].

Beside these approximations of the exact solution of the problem, other numerical techniques have been proposed to deal with the structural analysis of systems with fractional derivatives. In particular, Monte Carlo simulation methods are consistently used to validate approximations. Also there exist *ad hoc* simulation methods [14] which are computationally efficient, and methods based on Wiener path integral approaches [25, 26, 27, 28]. These techniques, together with exact assembling procedures of structural analysis [29], or finite element approaches [30, 4, 12],

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make it possible to study more realistic structures composed of several beams and columns, and several viscoelastic devices.

This review of the literature reveals two trends. On one side, there are simple dynamical systems with linear (or linearized) behaviour and white noise excitation, which possess closed form solutions. On the other side, there are numerical techniques to deal with more realistic loadings (sometimes nonstationary), more realistic structures and/or slight nonlinearities. The missing gap in-between is related to the understanding (via simple analytical solutions) of the behavior of complex structures subjected to more realistic loadings. As a first step towards this goal, we consider the stochastic analysis of a linear system subjected to a low frequency loading specified by its arbitrary power spectral density. More precisely, we consider the oscillator governed by

$$x''(\tau) + 2\xi \mathcal{D}^{\alpha} x(\tau) + x(\tau) = u(\tau) \qquad (2)$$

where ξ is the dimensionless *fractional coefficient* and $u(\tau)$ is a stationary random process with an arbitrary power spectral density $S_u(\Omega)$ with the restriction that its characteristic timescale is slow compared to the dimensionless natural frequency of the oscillator (which is equal to unity). This problem is a dimensionless version of a classical singe degree-of-freedom system with given mass, stiffness and damping, see [31] for the details of the scaling. This equation is for instance encountered in the analysis of wind-excited structures with visco-elastic devices, where the objective is to determine the statistics (the variance, mainly) of the structural response. The fact that the exogenous loading $u(\tau)$ could in principle be a non Markovian process is a specific contribution of this study.

This problem could be studied by means of the usual time domain multiple scales approach, or the stochastic averaging approach, but would require the consideration of three interacting timescales as well as fractional models —similar to those used to simulate realizations of wind fields [32],— for the augmented state coloring the excitation. This seems possible to solve the problem in this way, although no track evidences of this type of problem has been found in the literature. Instead, we take advantage of the linearity of the problem to derive a simplified solution in a frequency domain. It is based on the Multiple Timescale Spectral Analysis [1] which seeks the same objectives as the stochastic averaging, with slightly more versatility as shown next.

2. The Multiple Timescale Spectral Analysis

In 1961, Davenport proposed a method to compute the response of structures to the buffeting action of wind [33], which is known today as the Background/Resonant decomposition [34]. In this method, the total variance of the response, which corresponds to the area under the power spectral density of the response a sketched in Figure 3, is approached by the sum of two terms. The first one, the *background* component, corresponds to the quasi-static response of the structure and therefore depends on the stiffness of the system. The second one, the *resonant* component, corresponds to the resonance of the structure and is also governed by the damping in the system. It is possible to prove that this decomposition is valid in the necessary and sufficient condition that the two timescales associated with the problem of buffeting, i.e. the slow action of wind and the fast dynamics of the structure, are well separated. With the usual rules of perturbation theories, this translates into one order of magnitude at least [35].

In fact, it turns out that the decomposition proposed by Davenport in 1961 is just a particular case of the Multiple Timescales Spectral Analysis, which has been recently formalized by the author [1]. Under this general terminology, one can find an *ad hoc* version of the computation of integrals with small parameters, as typically depicted in Pertubation Methods textbooks, e.g. [35]. Indeed, in many applications of stochastic dynamics, the integrals to be computed (in the frequency domain) feature several small parameters. In particular, as soon as the ratio of two of the timescales of a problem is a small number, the problem is said to feature well separated timescales. Although this ratio is the most important small number, there are several additional small numbers in engineering applications: small damping, small nonlinearity, small stochasticity (see for instance [36] where four small numbers are identified and their smallness is exploited to construct very accurate approximate solutions of the problem).

The Multiple Timescale Spectral Analysis (MTSA) is a method to develop simple analytical solutions, or at least to reveal the influence of the different problem parameters to the various contributions to the stochastic response of a system. As stated by its name, the MTSA requires the existence of various well separated timescales in the problem. We notice that this does not really limit the possible field of application and corresponds anyways to usual assumptions formulated in deterministic and stochastic averaging [37]. The MTSA consists in recognizing the existence of these different timescales and consider the problem with all these timescales, successively, and provide local approximations in the frequency domain for the statistical moments associated with all of them. The method is rather general and is not limited to two timescales, nor to the estimation of the variance of the response. It has already been applied to the estimation of covariances of modal responses in case of multi-degree of freedom structures [38] (although this was before the general method has been fully formalized). In this context, the MTSA provides

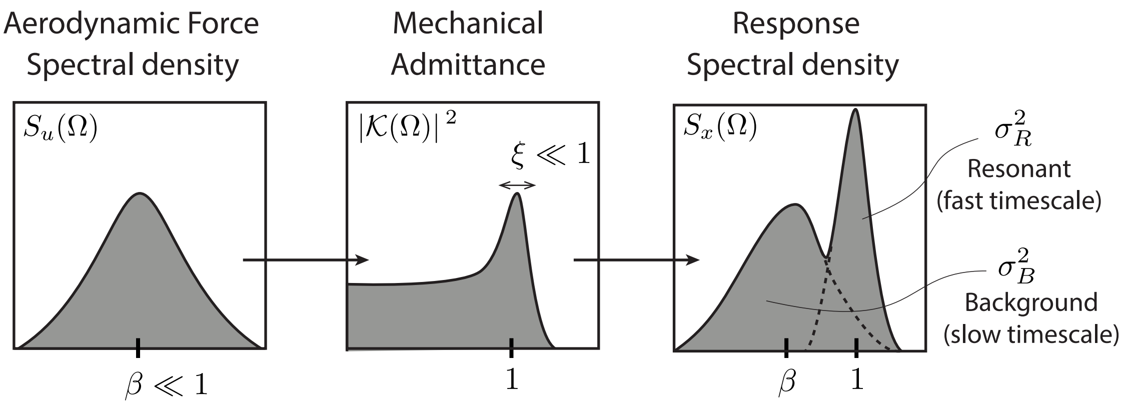

Figure 1: Sketch illustrating the decomposition of the response to buffeting into a slow background component and a fast resonant component [33].

a much simpler but very accurate solution for the estimation of the cross-correlation terms in a Complete Quadratic Combination, much simpler than the exact correlation coefficients [39], which are still widely used today despite their complexity. The MTSA also provides a very simple way to manage non proportional damping in a deterministic or stochastic context [31]. It has also given very concise and accurate results in the context of non Gaussian loading [40], nonstationary loading [41] slightly nonlinear systems [36], and oscillators featuring some visco-elastic components [31]. It is not limited to the variance of the response. Extensions to the estimation of the skewness [40] and excess [42] of the response have been developed. It is sometimes believed that averaging in the frequency domain simply consists in replacing the power spectral density of the load by a constant value, taken as the ordinate of the power spectral density of the load at natural frequency. This is fine for most of the problems dealing with the second-order moments, as confirmed again in this paper. However, this is not true for higher order statistics [40], and this is naturally and deductively shown with the Multiple Timescale Spectral Analysis. It therefore unifies under the same format, the necessary steps to deal with efficient and accurate solution of stochastic dynamics problems in the frequency domain.

3. Solution of The Problem

In this paper, the Multiple Timescales Spectral Analysis is applied to the considered problem, which demonstrates the ability of the method to deal with visco-elastic dampers, but also to provide an engineering insight into the different components of the structural response of a stochastically excited oscillator in the presence of dissipative devices modeled with fractional derivatives. To do so, let us first consider the Fourier transform of (2)

$$\left(1 + 2\xi\mathcal{C}\Omega|^{\alpha} - \Omega^2 - 2\xi i\mathcal{S}\Omega|^{\alpha}\right) X(\Omega) = U(\Omega) \quad (3)$$

where $\mathcal{C} = \cos\frac{\alpha\pi}{2}$ and $\mathcal{S} = \sin\frac{\alpha\pi}{2}$, so that the power spectral density of the response is given by

$$S_x(\Omega) = \frac{S_u(\Omega)}{\left(1 + 2\xi\mathcal{C}\,|\Omega|^{\alpha} - \Omega^2\right)^2 + \left(2\xi\mathcal{S}\,|\Omega|^{\alpha}\right)^2} := K(\Omega)S_u(\Omega). \quad (4)$$

This equation defines the kernel $K(\Omega)$. The only assumption required for the method to be applicable is that the central frequency β in the loading $S_u(\Omega)$ be much smaller that unity, $\beta \ll 1$ i.e. that the stochastic loading is slower that the dynamics of the system. This kernel is illustrated in Figure 2 for several values of α. It has some peculiarities: (i) the resonance peak located near $\Omega = \pm 1$ in the viscous case ($\alpha = 1$) regularly moves to higher frequencies as $\alpha \to 0$, i.e. as the fractional derivative term tends to correspond to a stiffness term. In the limiting case $\alpha = 0$, the fractional derivative corresponds to a usual stiffness term and the peak is located at abscissa $\Omega_p = \sqrt{1 + 2\xi} \simeq 1 + \xi$; (ii) the frequency response function passes trough a common *crossing point*, at abscissa $\Omega = 1$, no matter the fractional exponent α; (iii) the intercept is $K(0) = 1$ provided $\alpha \neq 0$. As $\alpha \to 0$, a short boundary layer, whose extent is of order α, develops in the neighborhood of the origin and creates the transition from the upper bound $K(0) = 1$ to the lower bound $K(\Omega) \simeq \frac{1}{(1+2\xi)^2}$. For $\alpha \to 0$, the size of this transition zone tends to zero; for $\alpha = 0$, there is no transition anymore and $K(0) = \frac{1}{(1+2\xi)^2}$.

As a result of the fractional powers of Ω appearing in $K(\Omega)$, the response of the system at second order, its variance, defined as

$$\sigma_x^2 = \int_{-\infty}^{+\infty} S_x(\Omega) d\Omega, \quad (5)$$

is unfortunately not available in a simple closed form, even for simple forms of $S_u(\Omega)$. This is all the more valid for complex expressions of $S_u(\Omega)$ corresponding non Markovian processes.

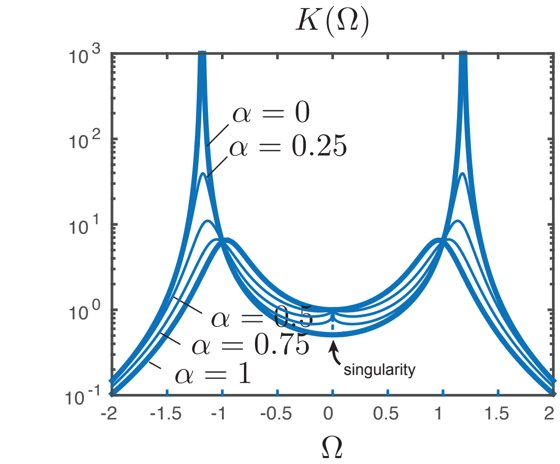

Figure 2: Representation of the kernel $K(\Omega)$ for various values of the fractional exponent α. Other parameters: $\xi = 0.2$.

Figure 3 shows some examples of the power spectral density of the structural response given by (4). This function features two distinct types of peaks: one in the low-frequency range around $\Omega \simeq 0$ and over a domain whose extent is of order β (the background component) and the other in the order-one frequency range (the resonant component). They are well distinct because $\beta \ll 1$.

Application of the Multiple Timescale Spectral Analysis consists in successively focusing on the different contributions to the response. In this application, there are only 2 components.

First, the background component is evaluated by rescaling the frequency axis Ω with the stretched coordinate ζ defined as $\Omega = \beta\zeta$. With this scaling, the background component develops over a domain of order 1 and, considering the separation of timescales $\beta \ll 1$, the kernel $K\left[\Omega\left(\zeta\right)\right]$can be approximated by

$$\hat{K}\left[\Omega\left(\zeta\right)\right] = \frac{1}{1 + 4\xi\mathcal{C}\beta^{\alpha}\left|\zeta\right|^{\alpha} + 4\xi^{2}\beta^{2\alpha}\left|\zeta\right|^{2\alpha}}. \tag{6}$$

This is the frequency response function of a lowpass fractional filter [43]. The expression $S_u(\Omega)\hat{K}\left(\Omega\right)$ is therefore a local approximation of $S_x(\Omega)$ in the neighborhood of the origin, for $\Omega \sim \beta \ll 1$. This approximation is represented by dotted lines in Figure 3, for $\alpha = 0.25$ and $\xi = 0.2$. Using this approximation, the background component of the response is expressed as

$$\begin{aligned} \sigma_{x,b}^{2} &= \int_{-\infty}^{+\infty} S_u(\Omega)\hat{K}\left(\Omega\right)d\Omega \\ &= \int_{-\infty}^{+\infty} \frac{S_u(\Omega)}{1 + 4\xi\mathcal{C}\left|\Omega\right|^{\alpha} + 4\xi^{2}\left|\Omega\right|^{2\alpha}} d\Omega \end{aligned} \tag{7}$$

which is the lowpass fractional filtered energy in the loading. For $\alpha \simeq 1$ and $\xi \ll 1$, the frequency response function of this filter tends to unity and

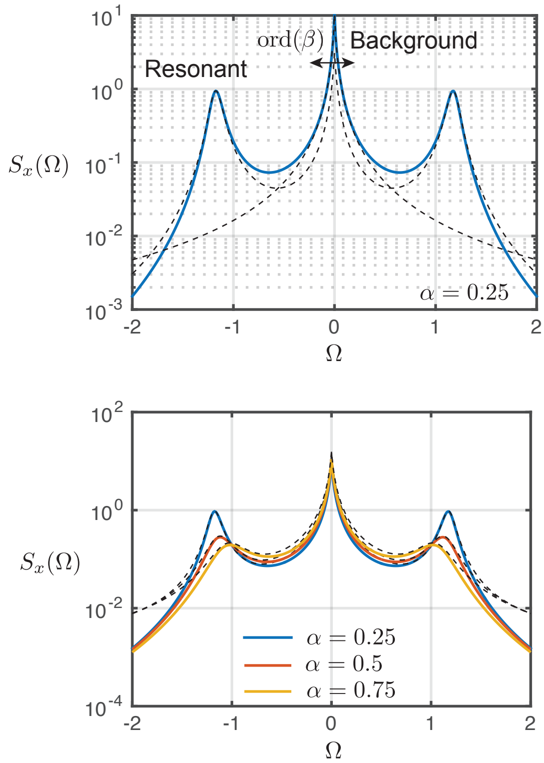

Figure 3: Examples of the power spectral density of the structural response for various values of the fractional exponent α. Other parameters: $\xi = 0.2$, $\beta = 0.05$.

$$\lim_{\alpha \to 1, \xi \to 0} \sigma_{x,b}^{2} = 1 \tag{8}$$

which is the well-known result from linear stochastic dynamics.

Second, the resonant component needs to be developed. To do so a remainder is constructed by subtracting this first approximation $\hat{K}\left(\Omega\right)$ from the original function to integrate, that is

$$\begin{aligned} r_1 &= \int_{-\infty}^{+\infty} S_x(\Omega) - S_u(\Omega)\hat{K}\left(\Omega\right)d\Omega \\ &= \int_{-\infty}^{+\infty} S_u(\Omega)\left(K\left(\Omega\right) - \hat{K}\left(\Omega\right)\right)d\Omega. \end{aligned} \tag{9}$$

The function to be integrated features two symmetrical peaks which will equally contribute the resonant part of the response. So we only only focus on the positive peak, then multiply by two. It is possible to prove [31] that the peaks (in absolute value) are located close to abscissa

$$\Omega_p = 1 + \mathcal{C}\xi - \left[\alpha + \left(\frac{1}{2} - \alpha\right)\mathcal{C}^2\right]\xi^2 + \mathcal{O}(\xi^2). \tag{10}$$

The position of the peak is a perturbation of 1 (the dimensionless natural frequency) and

- for $\alpha \simeq 1$, the fractional derivative resembles a viscous effect, $\mathcal{C} = \cos \frac{\alpha\pi}{2} \ll 1$ and the position of the peak is very close to $1 - \xi^2$, the peak position of the viscously damped system.
- for $\alpha \simeq 0$, $\mathcal{C} \simeq 1$ and the position of the peak is located close to abscissa $1 + \xi$; this is consistent with existing results obtained with a stochastic averaging approach, at least at leading order in ξ [18].

The natural stretched coordinate to focus on the peak in the positive region is therefore η defined as

$$\Omega = 1 + \mathcal{C}\xi + \xi\eta_2 \quad \Longleftrightarrow \quad \eta = \frac{\Omega - 1 - \mathcal{C}\xi}{\xi}. \tag{11}$$

Using this stretching, the integrand in (9) becomes, after some simplifications [31]

$$S_u(\Omega)\left(K(\Omega) - \hat{K}(\Omega)\right) \simeq \hat{\mathcal{K}}(\eta) := \frac{S_u(1+\mathcal{C}\xi)}{4\xi^2} \frac{1}{c_2\eta^2 + c_1\eta + c_0} \tag{12}$$

where the coefficients $c_0 = \mathcal{S}^2\left(1 + 2\xi\alpha\mathcal{C}\right)$, $c_1 = \frac{1}{2}\xi(1 + (1-4\alpha)\left(1 - 2\mathcal{S}^2\right))$ and $c_2 = 1 + 2\xi(1-\alpha)\mathcal{C}$. This approximation is also represented by dotted lines in Figure 3, for $\alpha = 0.25$ and $\xi = 0.2$. The approximation of the remainder r_1, multiplied by two is the resonant contribution to the response. It reads

$$\sigma_{x,r}^2 = 2\int_{-\infty}^{+\infty} \hat{\mathcal{K}}(\eta)\xi \mathrm{d}\eta = \frac{\pi S_u\left(1 + \mathcal{C}\xi\right)}{2\rho\xi} \tag{13}$$

where $\rho = \sqrt{4c_0c_2 - c_1^2} = [4\left(1 + 2\xi\mathcal{C}\right)\mathcal{S}^2 + \mathrm{ord}\left(\xi^2\right)]^{1/2}$. Truncating ρ to its leading order terms for consistency with the previous orders of approximations, the resonant contribution to the response finally finally simplifies into

$$\sigma_{x,r}^2 = \frac{\pi S_u\left(1 + \mathcal{C}\xi\right)}{2\mathcal{S}\xi\sqrt{1 + 2\xi\mathcal{C}}}. \tag{14}$$

To summarize, the background/resonant decomposition of the variance of a linear oscillator with fractional derivatives is given by

$$\sigma_x^2 = \sigma_{x,b}^2 + \sigma_{x,r}^2 = \tag{15}$$

$$\int_{-\infty}^{+\infty} \frac{S_u(\Omega)}{1 + 4\xi\mathcal{C}\left|\Omega\right|^{\alpha} + 4\xi^2\left|\Omega\right|^{2\alpha}} d\Omega + \frac{\pi S_u\left(1 + \mathcal{C}\xi\right)}{2\mathcal{S}\xi\sqrt{1 + 2\xi\mathcal{C}}}. \tag{16}$$

This expression regularly extends the well-known background resonant decomposition in case of integer derivative ($\alpha = 1$, $\mathcal{C} = 0$, $\mathcal{S} = 1$), which is

$$\lim_{\alpha \to 1} \sigma_x^2 = 1 + \frac{\pi S_u(1)}{2\xi}.$$

It also shows that, at leading order, the response of a fractionally damped oscillator depends on the power spectral density of the loading, computed for a unique value of the frequency: $\Omega = 1 + \mathcal{C}\xi$. This is the only way the response depends on the power spectral density of the loading. With this approximation, we show that the Markovianity of the input of this system is secondary; in other words, the proposed solution is valid no matter the shape of the power spectral density (rational fractions of Ω or not).

4. Validation, illustrations and discussion

In order to validate the proposed solution, we determine the response of the oscillator subjected to a loading specified by

$$S_u(\Omega) = \frac{0.546}{\beta\left(1 + 1.64\frac{|\Omega|}{\beta}\right)^{5/3}} \tag{17}$$

with $\beta \ll 1$ is a small dimensionless characteristic frequency. In wind engineering applications, it is related to the slow turbulence, compared to the natural frequency of the structure. The accuracy of the proposed formulation is assessed by comparison with a reference result obtained by accurate numerical integration of the exact power spectral density of the response. Integration is performed with the adaptive algorithm proposed in Wolfram Mathematica [44], with default integration parameters of Version 11.0.1.0.

Figures 4 and 5 show the variance of the response obtained with the proposed formulation (MTSA) and by numerical integration of the exact analytical formulation (Exact). The variance is represented as a function of α for given values of ξ, and as a function of ξ for given values of α. It is also given for two values of β, namely $\beta = 0.01$ and $\beta = 0.1$. These two small numbers correspond to typical values of this parameter in buffeting applications.

In all cases, the proposed formulation (MTSA) provides very accurate results, when compared to the reference solution (Exact). The smaller β, the more accurate. This is consistent with the methodology to develop the approximate solution and with the inherent assumptions in the Multiple Timescale Spectral Analysis. The same observation also holds for ξ which also needs to be a relatively small number. In fact, this relative smallness can be discussed with this example. Indeed, the comparison shows that the proposed method is very accurate for values of ξ which are as large as 1.

In both figures, the background component $\sigma_{x,b}^2$ is shown with dashed lines. Where the total variance is similar to background component $\sigma_{x,b}^2$, the resonant counterpart to the response is negligible and the response is actually quasi-static. This happens for large values of the damping (see Figure 4) or, for given ξ,

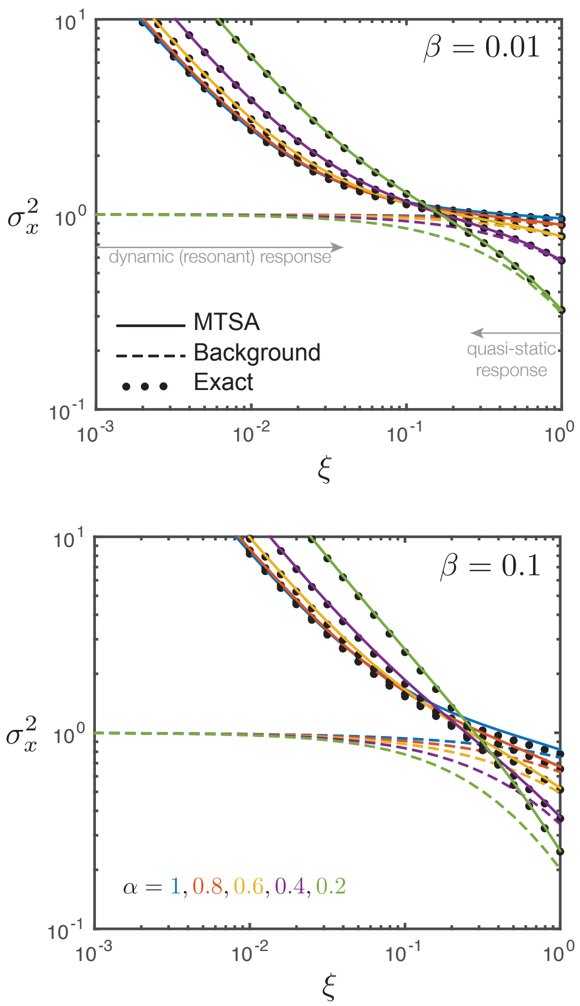

Figure 4: Variances of the response of the system subjected to the buffeting type excitation, for $\beta = 0.01$ (top) and $\beta = 0.1$ (bottom). Represented as a function of the fractional coefficient ξ. and for various values of the fractional exponent α (Please see online version for colors)

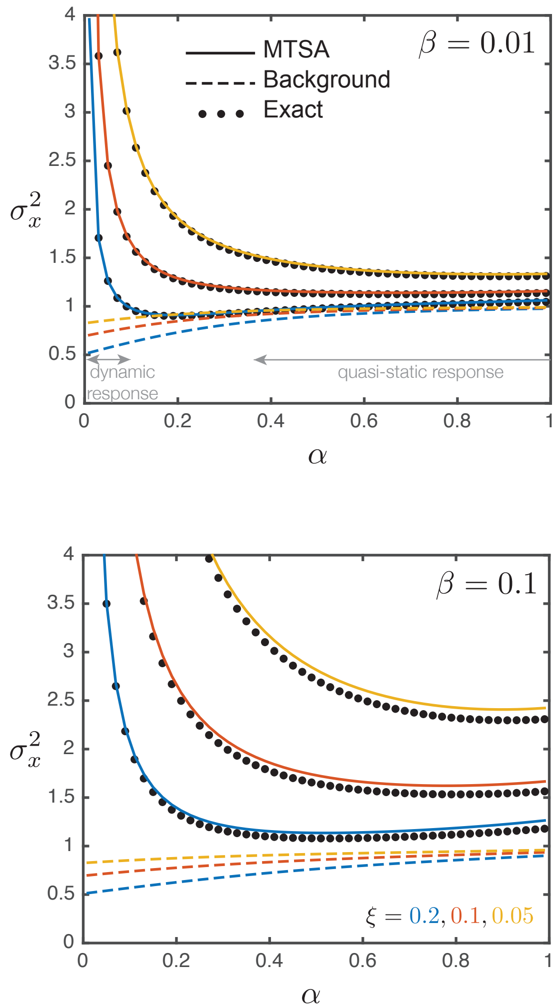

Figure 5: Variances of the response of the system subjected to the buffeting type excitation, for $\beta = 0.01$ (top) and $\beta = 0.1$ (bottom). Represented as a function of the fractional exponent α and for various values of the fractional coefficient ξ. (Please see online version for colors)

for large values of α, i.e. as the behaviour of the fractional damper tends to viscosity (see Figure 5).

5. Conclusions

In this paper, we have applied the Multiple Timescale Spectral Analysis to the structural analysis of a linear system equipped with a viscoelastic device. The proposed formulation extends the well known background/resonant decomposition which is usually applied in the field of wind engineering. It shows that

- the background component can be determined without any prior knowledge of the dynamics of the system; it consists in a fractional filtered version of the input;
- the power spectral density of the loading just enters in the solution of this problem through the value of the power spectral density at a frequency equal to $\Omega = 1 + \cos\frac{\alpha\pi}{2}\xi$;
- the resonant component of the response $\sigma_{x,r}^2$ is proportional to the power spectral density;
- the resonant component of the response $\sigma_{x,r}^2$ does not scale with the inverse of the damping ratio anymore (as what would be obtained in the integer derivative case). It rather scales with the inverse of $\xi\sqrt{1+2\xi\mathcal{C}}$. In the limit case $\alpha \to 1$ (viscous damping), $\mathcal{C} \to 0$, the usual scaling is recovered
- the resonant component scales with $\mathcal{S}^{-1} = \csc\frac{\alpha\pi}{2}$, which tends to infinity as $\alpha \to 0$. This results from the fact that there is no damping anymore in this limit case, and the dynamic response is unbounded.

These preliminary results are very promising. Future works should combine the developments summarized in this paper with other contexts of application of the Multiple Timescale Spectral Analysis and consider in this way slightly nonlinear systems (with the help of a Volterra model) or to multiple degree-of-freedom structures equipped with fractional derivative dissipative devices.

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