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The Lipkin-Meshkov-Glick Model and its Deformations Through Polynomial Algebras

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Abstract

We consider the many-particle Hamiltonian of Lipkin, Meshkov and Glick in the context of deformed polynomial algebras. The reducibility of the original model is proved according to the representations of these polynomial algebras. The LMG spectrum is recovered in such a way as well as supplementary eigenvalues associated to deformed LMG models.

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1 Introduction

Quantum mechanical equations whose solutions can be found analytically are rare. Only some interactions like the oscillator or the Coulomb ones enter this class of equations which are then called exactly solvable. Meanwhile one can weaken the condition of exact solvability by asking for the knowledge of a *finite* number of solutions only. This leads to what is referred to [1] as quasi-exact solvability (Q.E.S.). Quasi-exactly solvable models have been developed in the non-relativistic context essentially. They are characterized by the fact that, up to a change of variables as well as a transformation at the level of the wavefunctions, their Hamiltonians can be expressed as at most a quadratic function of the generators of a Lie algebra, namely $sl(2, R)$ in the one-dimensional case. These generators stabilize a finite-dimensional space and so do the Hamiltonians which can be easily diagonalized within this space.

One of these (physical) quasi-exactly solvable models is the Lipkin-Meshkov-Glick one [2] developed for treating many particle systems. Precisely, Lipkin, Meshkov and Glick (LMG) constructed a two N -fold degenerate level model where N is the number of fermions in the system. The two levels are separated by an energy ϵ . The simplified version of the LMG model whom we will be concerned with contains only terms which mix configurations. The corresponding Hamiltonian reads

$$H_{LMG} = \epsilon j_0 + \frac{\delta \epsilon}{2N} (j_+^2 + j_-^2) \quad (1)$$

where δ is the interaction strength while the $sl(2, R)$ generators j_0, j_{\pm} are realized as

$$j_0 = -\frac{N}{2} + \frac{1}{2} \sum_{m=1}^N (\alpha_m^\dagger \alpha_m + \beta_m^\dagger \beta_m), \quad (2)$$

$$j_+ = \sum_{m=1}^N \alpha_m^\dagger \beta_m^\dagger, \quad (3)$$

$$j_- = \sum_{m=1}^N \alpha_m \beta_m, \quad (4)$$

and satisfy

$$[j_0, j_{\pm}] = \pm j_{\pm}, [j_+, j_-] = 2j_0. \quad (5)$$

In the definitions (2)-(4), the fermion operators β_m^\dagger, β_m create and annihilate holes in the lower level while $\alpha_m^\dagger, \alpha_m$ create and annihilate particles in the upper level. These operators are such that

$$\{\alpha_m, \alpha_n^\dagger\} = \{\beta_m, \beta_n^\dagger\} = \delta_{mn}, \quad (6)$$

$$[\alpha_m, \beta_n] = [\alpha_m, \beta_n^\dagger] = [\beta_m, \alpha_n^\dagger] = [\alpha_m^\dagger, \beta_n^\dagger] = 0. \quad (7)$$

The Casimir operator of the $sl(2, R)$ algebra

$$C_1 = \frac{1}{2}\{j_+, j_-\} + j_0^2 \quad (8)$$

evidently commuting with the Hamiltonian (1), the later when realized in terms of matrices breaks up into submatrices each associated with a different value of j and of order $2j + 1$. Each state in a j multiplet has a different number of excited particle-hole pairs. The interaction in (1) mixes the states within the same j multiplet but cannot mix states having different eigenvalues of C_1 . It can only excite or de-excite two particle-hole pairs or in other words it can only change the eigenvalue of j_0 by two units. From the definition (2), it follows that the eigenvalues of j_0 are given by half the difference between the number of particles in the upper level and the number of particles in the lower level. Then the maximum eigenvalue of j_0 and of j is $\frac{N}{2}$. The largest matrix to be diagonalized in (1) is thus of dimension $N + 1$ ($=2j + 1$).

The main purpose of this paper is to revisit the LMG Hamiltonian given in (1) through the polynomial deformation point of view. In this context, we show that the largest matrix associated to a given N can be split into two submatrices of dimensions $\frac{N}{2} + 1$ and $\frac{N}{2}$ for N even and two submatrices, both of dimensions $\frac{N+1}{2}$ for N odd. This is due to the presence of an additional (with respect to (8)) invariant i.e. the Casimir operator of the deformed algebra. Moreover the polynomial deformation technique leads to new representations corresponding to new eigenvalues appropriate to a deformed LMG model.

2 The polynomial algebra point of view

We propose [3] to consider the following Hamiltonian

$$H = \epsilon(2J_0 + \delta(J_+ + J_-)) \quad (9)$$

instead of (1). In (9) the operators J_0, J_{\pm} satisfy the following polynomial algebra (compare with (5))

$$[J_0, J_{\pm}] = \pm J_{\pm}, \quad (10)$$

$$[J_+, J_-] = -\frac{16}{N^2}J_0^3 + \frac{2}{N^2}(2j^2 + 2j - 1)J_0 \quad (11)$$

where j is an eigenvalue of $C_1=(8)$. Such a choice is justified by the fact that a particular realization of the algebra (10)-(11) is

$$J_0 = \frac{1}{2}j_0, \quad J_{\pm} = \frac{1}{2N}j_{\pm}^2, \quad (12)$$

the Hamiltonian (9) then coinciding with the LMG one in (1). However other realizations with respect to (12) will be available in general, leading in such a way to new eigenvalues.

Indeed the Casimir operator of the polynomial algebra (10)-(11) is

$$C_2 = J_+J_- - \frac{4}{N^2}J_0^4 + \frac{8}{N^2}J_0^3 + \frac{2j^2 + 2j - 5}{N^2}J_0^2 - \frac{2j^2 + 2j - 1}{N^2}J_0 \quad (13)$$

and two types of finite-dimensional representations arise. The first ones are defined according to

$$\begin{aligned} J_0 |J, M\rangle &= (M + c) |J, M\rangle, \\ J_+ |J, M\rangle &= f(M) |J, M + 1\rangle, \\ J_- |J, M\rangle &= g(M) |J, M - 1\rangle, \end{aligned} \quad (14)$$

with $M = -J, -J + 1, \dots, J - 1, J$, $J = 0, \frac{1}{2}, 1, \dots$ and

$$\begin{aligned} f(M - 1)g(M) &= \frac{1}{N^2}(J - M + 1)(J + M) \\ &\quad (2j^2 + 2j - 1 - 4J^2 - 4J - 4M^2 + 4M + 8(1 - 2M)c - 24c^2). \end{aligned} \quad (15)$$

The real number c can take three distinct values [4] given by

$$c = 0 \quad (16)$$

and

$$c = \pm \sqrt{\frac{1}{4}j(j + 1) - \frac{1}{8} - J(J + 1)}. \quad (17)$$

The second representations are characterized by the following equations

$$\begin{aligned} J_0 | J', M' \rangle &= \left(\frac{M'}{2}\right) | J', M' \rangle, \\ J_+ | J', M' \rangle &= f'(M') | J', M' + 2 \rangle, \\ J_- | J', M' \rangle &= g'(M') | J', M' - 2 \rangle, \end{aligned} \quad (18)$$

where $J' = 0, 1, 2, \dots$ and

$$f'(M' - 2)g'(M') = \frac{1}{4N^2}(J' - M' + 2)(J' + M')(2j^2 + 2j - 1 - J'^2 - 2J' - M'^2 + 2M') \quad (19)$$

if $M' = -J', -J' + 2, \dots, J' - 2, J'$ and

$$f'(M' - 2)g'(M') = \frac{1}{4N^2}(J' - M' + 1)(J' + M' - 1)(2j^2 + 2j - J'^2 - M'^2 + 2M') \quad (20)$$

if $M' = -J' + 1, -J' + 3, \dots, J' - 3, J' - 1$. In the cases where $J' = \frac{1}{2}, \frac{3}{2}, \dots$, J' must be equal to j (M' to m) and

$$f'(m - 2)g'(m) = \frac{1}{4N^2}(j + m)(j + m - 1)(j - m + 1)(j - m + 2). \quad (21)$$

In the following we will drop these second representations due to the fact that they are reducible. Indeed evaluating the eigenvalues of the Casimir operator $C_2=(13)$ both within the representations (14) and (18), we can easily be convinced that

$$(J' = n)_{(18)} = (J = \frac{n}{2}, c = 0)_{(14)} \oplus (J = \frac{n-1}{2}, c = 0)_{(14)} \quad (22)$$

and

$$(J' = j = n + \frac{1}{2})_{(18)} = (J = \frac{n}{2}, c = \frac{1}{4})_{(14)} \oplus (J = \frac{n}{2}, c = -\frac{1}{4})_{(14)} \quad (23)$$

for any integer n . Moreover the original LMG model defined in (1) or equivalently in (9) with the realization (12) being clearly connected to the representations (18) with $J' = j$ (J' being an integer or a half integer), we can conclude that this LMG model is in fact a reducible one. More precisely, following Eq. (22) (resp. (23)), a LMG matrix of dimension $2n + 1$ (resp.

$2n+2$) can be split into a direct sum of two submatrices of dimensions $n+1$ (resp. $n+1$) and n (resp. $n+1$) if we concentrate on the Hamiltonian (9) coming from the polynomial algebra point of view. We thus obtain the results we have mentioned in the Introduction with $N = 2n$ (resp. $N = 2n+1$), such results being significant for a large number of particles. We are going to illustrate these results on specific examples but before doing so, let us notice that due to the irreducibility of the representations (14), we can conclude that searching the eigenvalues of the Hamiltonian $H=(9)$ is equivalent to the diagonalization of the matrix $\langle H \rangle$ given by

$$\begin{pmatrix} 2J+2c & \delta f(J-1) & 0 & 0 & \cdot & \cdot & 0 \\ \delta g(J) & 2J-2+2c & \delta f(J-2) & 0 & \cdot & \cdot & 0 \\ 0 & \delta g(J-1) & 2J-4+2c & \delta f(J-3) & \cdot & \cdot & 0 \\ 0 & 0 & \delta g(J-2) & 2J-6+2c & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & -2J+2+2c & \delta f(-j) \\ 0 & 0 & 0 & 0 & \cdot & \delta g(-J+1) & -2J+2c \end{pmatrix}. \quad (24)$$

3 Examples

3.1 The $N = 2$ -case

We first limit ourselves to this simplest case in order to illustrate easily our statements. The complete LMG matrix is of dimension 4, corresponding to the four possibilities of two particles for two levels (the two particles can be on the lower level, or on the upper one, or one particle can be on the lower level while the other one can be on the upper one or vice-versa). Following the original LMG Hamiltonian (1), this dimension 4 decomposes into 3 and 1 while the Hamiltonian (9) together with the representations (14) is such that 4 splits into $2+1+1$ (corresponding to $J = \frac{1}{2}$ and $J = 0$ twice). The eigenvalues E (with $\epsilon = 1$) can be obtained through the diagonalization of the three matrices (24) of respective dimensions 2, 1 and 1. The results are summarized in the following table

j	J	E
0	0	0
1	0	0
	$\frac{1}{2}$	$\pm\sqrt{1 + \frac{1}{4}\delta^2}$

3.2 The $N = 8$ -case

For $N = 8$, there are $2^8 = 256$ states. The largest original LMG matrix corresponds to $j = \frac{N}{2} = 4$, the other ones being associated to $j = 3$ (7 times), $j = 2$ (20 times), $j = 1$ (28 times) and $j = 0$ (14 times). Following the decompositions (22)-(23) and the Hamiltonian (9), the polynomial algebra point of view leads to another decomposition: $J = 2$ (1 time), $J = \frac{3}{2}$ (8 times), $J = 1$ (27 times), $J = \frac{1}{2}$ (48 times) and $J = 0$ (42 times). The corresponding eigenvalues come from the diagonalization of the matrices (24) and are given in unit ϵ in the following table

j	J	E
0	0	0
1	0	0
	$\frac{1}{2}$	$\pm\sqrt{1 + \frac{1}{64}\delta^2}$
2	$\frac{1}{2}$	$\pm\sqrt{1 + \frac{9}{64}\delta^2}$
	1	$0, \pm\sqrt{4 + \frac{3}{16}\delta^2}$
3	1	$0, \pm\sqrt{4 + \frac{15}{16}\delta^2}$
	$\frac{3}{2}$	$\pm\sqrt{5 + \frac{33}{64}\delta^2 \pm \sqrt{16 + \frac{3}{2}\delta^2 + \frac{27}{128}\delta^4}}$
4	$\frac{3}{2}$	$\pm\sqrt{5 + \frac{113}{64}\delta^2 \pm \sqrt{16 + \frac{19}{2}\delta^2 + \frac{275}{128}\delta^4}}$
	2	$0, \pm\sqrt{10 + \frac{59}{32}\delta^2 \pm \sqrt{36 - \frac{9}{8}\delta^2 + \frac{2025}{1024}\delta^4}}$

4 Supplementary eigenvalues

The tables in the previous Section take account of the LMG eigenvalues only but by using the representations of the polynomial algebra subtending such a quasi-exact model. However this polynomial algebra is richer than the $sl(2, R)$ algebra usually used inside the LMG context. Its representations have three labels (J, c, j) instead of one (j) for $sl(2, R)$. Thus the number

of representations is larger. This is particularly clear from the second table corresponding to $N = 8$. Indeed when $j = 2$ we can see that the LMG context is recovered when $J = \frac{1}{2}$ and $J = 1$ while the case $J = 0$ is missing and must correspond to another model. The same situation holds for $j = 3$, $J = 0$, $J = \frac{1}{2}$ and $j = 4$, $J = 0$, $J = \frac{1}{2}$, $J = 1$. These new possibilities (excluded by the Hamiltonian (1) but not by the ones given in (9)) lead to supplementary eigenvalues as summarized in the following table

j	J	E
2	0	0
3	0	0
4	$\frac{1}{2}$	$\pm\sqrt{1 + \frac{21}{64}\delta^2}$
	0	0
	$\frac{1}{2}$	$\pm\sqrt{1 + \frac{37}{64}\delta^2}$
	1	$0, \pm\sqrt{4 + \frac{31}{16}\delta^2}$

Limiting for example ourselves to the maximal value of j , i.e. $j = 4$ we can see that these supplementary eigenvalues are surprisingly close to some of the original LMG ones. Indeed when $\delta = 1$, we have

j	J	E
4	0	0
	$\frac{1}{2}$	± 1.256
	1	$0, \pm 2.437$
	$\frac{3}{2}$	$\pm 1.228, \pm 3.467$
	2	$0, \pm 2.402, \pm 4.232$

The same kind of results hold for any number of particles. In order to fix the ideas, for an even number $N = 2n$ of particles, the largest matrix corresponds to $j = n$, the values $J = \frac{n-1}{2}, \frac{n}{2}$ give rise to the LMG eigenvalues while the cases $J = 0, \frac{1}{2}, 1, \dots, \frac{n}{2} - 1$ lead to the supplementary ones, close (and bigger than) the LMG ones. Moreover the closeness is better for δ smaller.

A natural question then arises: to what kind of model do correspond these supplementary eigenvalues? In order to answer this question, let us once again concentrate on the case of $N = 8$ particles and, this time, on the representations (18). We have five different values in what concerns J' ($=0, 1, 2, 3, 4$). Realizing the matrix associated to J_+ , for example, according to (18)-(20) leads to

$$J_+ = \frac{1}{16}M(J')j_+^2 \quad (25)$$

where $M(J')$ is a diagonal matrix of dimension $2J' + 1$ being differently fixed with the different values of J' . It is interesting to note that this diagonal matrix reduces to the identity one for $J' = J'_{max} = \frac{N}{2} = 4$ only, in agreement with (12). The Hamiltonian (9) is thus

$$H = \epsilon j_0 + \frac{\delta\epsilon}{2N} (M(J') j_+^2 + j_-^2 M(J')) \quad (26)$$

with $J' = 0, 2, 4, \dots, \frac{N}{2}$ and $M(\frac{N}{2}) = I$. Moreover in general the operator (25) and its adjoint can also be written as

$$J_+ = \frac{1}{2N} M(J') j_+^2 \equiv \frac{1}{2N} (j'_+)^2, \quad (27)$$

$$J_- = \frac{1}{2N} j_-^2 M(J') \equiv \frac{1}{2N} (j'_-)^2 \quad (28)$$

with

$$[j_0, j'_\pm] = \pm j'_\pm, \quad (29)$$

$$[j'_+, j'_-] = \sum_{k=0}^{J'-1} c_k j_0^{2k+1} \quad (30)$$

where c_k are coefficients being fixed according to N and J' . The relations (28)-(29) are those of a polynomial deformation of $sl(2, R)$ except when $J' = 1$ and $J' = \frac{N}{2}$ where it is equivalent to $sl(2, R)$ ($J' = 0$ leading to trivial results). We can then conclude by saying that our model (9) or equivalently (26) is made of a usual LMG model (corresponding to $J' = \frac{N}{2}$) and $\frac{N}{2}$ deformed (due to $M(J') \neq I$) LMG models (corresponding to $J' = 0, 1, \dots, \frac{N}{2} - 1$) giving rise to these supplementary eigenvalues we have mentioned.

5 Summary

We have presented a calculation of the whole spectrum of the Lipkin-Meshkov-Glick Hamiltonian considered in the context of a deformed polynomial algebra. For any given number N of particles the spectrum first divides into j multiplets of the $sl(2, R)$ algebra. The eigenvalues associated with the largest j are non degenerate except for $E = 0$. We have shown that the Hamiltonian

matrix of each j further splits into two submatrices corresponding to two distinct irreducible representations of the deformed polynomial algebra. In order to illustrate the method we have derived explicit analytic expressions for the eigenvalues of the LMG Hamiltonian for $N = 2$ and 8. Our method can evidently be extended to any N .

Furthermore we have shown that the deformed polynomial algebra related to the LMG model implies a larger spectrum than that of the model itself. Some of the new eigenvalues present characteristics similar to those of the LMG model and actually correspond to a superposition of specific deformed LMG models where, once again, deformed polynomial algebras play a prominent role.

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