

Role of hidden color states in $2q-2\bar{q}$ systems

D. M. Brink

*University of Oxford, Theoretical Physics Department, 1 Keble Road, Oxford, OX1 3NP, United Kingdom
and Dipartimento di Fisica, Università degli Studi di Trento, I-38050 Povo, Trento, Italy*

Fl. Stancu

University of Liège, Institut de Physique B.5, Sart Tilman, B-4000 Liège 1, Belgium

(Received 31 August 1993)

We study a system of two quarks and two antiquarks of equal masses in the framework of a nonrelativistic potential model with color confinement and hyperfine spin-spin interaction. We propose a simple variational solution and conclude that the hyperfine interaction is needed to bind the system, in agreement with previous studies. Our results explicitly show the important role played by hidden color states in lowering the variational energy.

PACS number(s): 12.39.Pn, 12.39.Jh

I. INTRODUCTION

Multiquark systems formed of two quarks and two antiquarks have already been extensively studied in the framework of the bag [1–3], potential [4–6], and other models [7,8]. On the one hand, it is interesting to extend the available relativistic or nonrelativistic models beyond the hadron spectroscopy for which they proved successful. On the other hand, $2q-2\bar{q}$ systems are important for the study of hadronic molecules. Candidates for scalar meson molecules are the $f_0(975)$ and $a_0(980)$ resonances [1,5,9] and for vector mesons molecules the $G(1590)$ and $\Theta(1720)$ resonances [10,11]. These resonances have quantum numbers which are not consistent with a $q\bar{q}$ system. For a recent review, see, for example, Ref. [12]. The interpretation of these resonances as hadronic molecules raises the question of whether or not the $2q-2\bar{q}$ systems can form loosely bound states. This question has been extensively examined in the literature. In particular, in Refs. [5], [6], and [13] the role of unequal masses has also been discussed.

One difficulty is that the solution of the $2q-2\bar{q}$ Schrödinger equation including orbital, spin, and color degrees of freedom is not trivial. Several variational approaches have been proposed [4,6]. Our study is also based on a variational approach and is closely related to the work of Ref. [4]. Here we show that with an appropriate variational wave function with a few parameters one can obtain an attractive pocket in the energy surface if the spin-spin interaction is included. We also discuss the explicit role of the hidden color states. Our calculations have been carried out analytically to a large extent, and we present the basic analytic results as a guide for more elaborate treatments. The calculations here refer to the sector $T=0, S=0$ only, but they can be easily extended to other sectors.

II. HAMILTONIAN

In the present work, we choose the simplest Hamiltonian to describe the $2q-2\bar{q}$ system. It is nonrelativistic and

has the well-known form [4]

$$H = \sum_{i=1}^4 \left[m_i + \frac{p_i^2}{2m_i} \right] + \sum_{i<j} (V_{ij}^{\text{conf}} + V_{ij}^{\text{hyp}}), \quad (1)$$

with

$$V_{ij}^{\text{conf}} = - \left[e_0 + \frac{1}{2} k r_{ij}^2 \right] \frac{\hat{\lambda}_i \cdot \hat{\lambda}_j}{2} \quad (2)$$

and

$$V_{ij}^{\text{hyp}} = V_{ij}^{SS} = - \frac{8\pi\alpha_s}{3m_i m_j} f(r_{ij}) \mathbf{S}_i \cdot \mathbf{S}_j \frac{\hat{\lambda}_i \cdot \hat{\lambda}_j}{2}, \quad (3)$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ and m_i , \mathbf{r}_i , \mathbf{p}_i , \mathbf{S}_i , and $\hat{\lambda}_i/2$ are the mass, position, momentum, spin, and color operators of the i th particle. The coordinate space part of the spin-spin interaction has the regularized form

$$f(r_{ij}) = \frac{\sigma^3}{\pi^{3/2}} e^{-\sigma^2 r_{ij}^2}, \quad (4)$$

where σ is a parameter. The confinement potential (2) has a harmonic form for simplicity. In order to allow comparison with Ref. [4], we choose the same parameters as there. Our calculations are restricted to a single choice of parameters found in Table I of Ref. [4]. These are

$$\begin{aligned} m &= m_u = m_d = 330 \text{ MeV}, \\ \omega_0 &= 200 \text{ MeV}, \\ e_0 &= -352 \text{ MeV}, \\ \alpha_s &= 2.7, \\ \sigma &= 2 \text{ fm}^{-1}. \end{aligned} \quad (5)$$

The parameter ω_0 is related to the harmonic force constant k , as usual, i.e., $\omega_0 = (k/m)^{1/2}$, which leads to

$$k = 339 \text{ MeV fm}^{-2}. \quad (6)$$

We deal with u and d quarks only. The parameters (5) were fitted in Ref. [4] to the $q\bar{q}$ problem. With a wave function of type

$$\phi_0 = \frac{a^{3/2}}{\pi^{3/4}} \exp \left[-\frac{a^2}{2} r_{ij}^2 \right], \quad (7)$$

the expectation value of the two-body Hamiltonian is minimized with respect to a to give

$$\begin{aligned} m_\pi &= 172.7 \text{ MeV at } a = 2 \text{ fm}^{-1}, \\ m_\rho &= 770 \text{ MeV at } a = 1.1 \text{ fm}^{-1}, \end{aligned} \quad (8)$$

which imply

$$\begin{aligned} \langle r^2 \rangle_\pi^{1/2} &= 0.30 \text{ fm}, \\ \langle r^2 \rangle_\rho^{1/2} &= 0.56 \text{ fm}. \end{aligned} \quad (9)$$

III. WAVE FUNCTION

For describing the $2q-2\bar{q}$ system, one can introduce three alternative coordinate systems as shown in Fig. 1. They can be used at various stages in order to simplify calculations. Suppose that particles 1 and 2 are quarks and 3 and 4 are antiquarks. The three possibilities are

$$\rho = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_3), \quad \rho' = \frac{1}{\sqrt{2}}(\mathbf{r}_2 - \mathbf{r}_4), \quad (10)$$

$$\mathbf{x} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_3 - \mathbf{r}_2 - \mathbf{r}_4),$$

$$\alpha = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_4), \quad \alpha' = \frac{1}{\sqrt{2}}(\mathbf{r}_2 - \mathbf{r}_3), \quad (11)$$

$$\mathbf{y} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_4 - \mathbf{r}_2 - \mathbf{r}_3),$$

$$\sigma = \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2), \quad \sigma' = \frac{1}{\sqrt{2}}(\mathbf{r}_3 - \mathbf{r}_4), \quad (12)$$

$$\lambda = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 - \mathbf{r}_4).$$

Note that (10) and (11) correspond to meson-meson channels, and we shall call them direct and exchange channels. With the above notation, one can also write

$$\rho = \frac{1}{\sqrt{2}}(\lambda + \mathbf{y}), \quad \rho' = \frac{1}{\sqrt{2}}(\lambda - \mathbf{y}), \quad (13)$$

$$\alpha = \frac{1}{\sqrt{2}}(\lambda + \mathbf{x}), \quad \alpha' = \frac{1}{\sqrt{2}}(\lambda - \mathbf{x}), \quad (14)$$

$$\sigma = \frac{1}{\sqrt{2}}(\mathbf{x} + \mathbf{y}), \quad \sigma' = \frac{1}{\sqrt{2}}(\mathbf{x} - \mathbf{y}). \quad (15)$$

These relations will be used later. Note also that permutations of particles i and j lead to

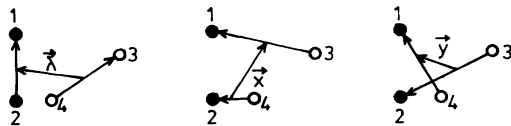


FIG. 1. Three possible ways to define relative coordinates for a $2q-2\bar{q}$ system. Solid and open circles represent quarks and antiquarks, respectively.

$$(12)\rho = \alpha, \quad (12)\rho' = \alpha', \quad (34)\rho = \alpha', \quad (34)\rho' = \alpha,$$

$$(12)\mathbf{x} = -\mathbf{y}, \quad (34)\mathbf{x} = \mathbf{y}, \quad (23)\mathbf{x} = \lambda, \quad (14)\mathbf{x} = -\lambda, \quad (16)$$

$$(12)\lambda = \lambda, \quad (34)\lambda = \lambda, \quad (23)\mathbf{y} = \mathbf{y}, \quad (14)\mathbf{y} = \mathbf{y}.$$

The most general orbital wave function with $L=0$ is a function of six scalar quantities. They can be chosen in many ways. A choice appropriate for the asymptotic meson-meson channel with particles 1 and 3 in the first meson and 2 and 4 in the second would be

$$R(x) = R(\rho^2, \rho'^2, \rho \cdot \rho', x^2, \rho \cdot \mathbf{x}, \rho' \cdot \mathbf{x}). \quad (17a)$$

The corresponding wave function in the exchange channel is

$$R(y) = R(\alpha^2, \alpha'^2, \alpha \cdot \alpha', y^2, \alpha \cdot \mathbf{y}, \alpha' \cdot \mathbf{y}). \quad (17b)$$

We have $R(y) = (12)R(x) = (34)R(x)$. In the asymptotic region when the two mesons are well separated, the wave function $R(x)$ would be a function only of ρ^2, ρ'^2 , and x^2 and would not depend on the angles. Weinstein and Isgur choose orbital wave functions depending on x^2, y^2 , and λ^2 . This is a restricted choice, and in terms of direct channel coordinates, it corresponds to taking $R(x)$ to be a function of $\rho^2 + \rho'^2, \rho \cdot \rho'$, and x^2 . Later in the paper, we put the same restriction on the choice of the orbital wave functions $R(x)$ and $R(y)$.

Including spin and color degrees of freedom and using the notation of Ref. [4], one can define four channel wave functions with total spin $S=0$ as follows:

$$\begin{aligned} \Psi_{PP} &= \frac{N_{PP}}{\sqrt{2}} [R_P(x) |1_{13}1_{24}\rangle |P_{13}P_{24}\rangle \\ &\quad + R_P(y) |1_{14}1_{23}\rangle |P_{14}P_{23}\rangle], \end{aligned} \quad (18)$$

$$\begin{aligned} \Psi_{VV} &= \frac{N_{VV}}{\sqrt{2}} [R_V(x) |1_{13}1_{24}\rangle |V_{13}V_{24}\rangle \\ &\quad + R_V(y) |1_{14}1_{23}\rangle |V_{14}V_{23}\rangle], \end{aligned} \quad (19)$$

$$\begin{aligned} \Psi_{C_0C_0} &= \frac{N_{C_0C_0}}{\sqrt{2}} [R_{C_0}(x) |8_{13}8_{24}\rangle |P_{13}P_{24}\rangle \\ &\quad + R_{C_0}(y) |8_{14}8_{23}\rangle |P_{14}P_{23}\rangle], \end{aligned} \quad (20)$$

$$\begin{aligned} \Psi_{C_1C_1} &= \frac{N_{C_1C_1}}{\sqrt{2}} [R_{C_1}(x) |8_{13}8_{24}\rangle |V_{13}V_{24}\rangle \\ &\quad + R_{C_1}(y) |8_{14}8_{23}\rangle |V_{14}V_{23}\rangle]. \end{aligned} \quad (21)$$

They are symmetric under permutations (12) and (34). This is because the antisymmetry of the total wave function can be established by multiplying each of them with an appropriate flavor function. As we are considering only u and d quarks, the appropriate flavor function would be

$$\frac{1}{2}(ud - du)(\bar{u}\bar{d} - \bar{d}\bar{u}), \quad (22)$$

which has isospin $T=0$.

The notation for the color wave functions $|1_{13}1_{24}\rangle, |1_{14}1_{23}\rangle, |8_{13}8_{24}\rangle, |8_{14}8_{23}\rangle$ and spin wave functions $|P_{13}P_{24}\rangle, |P_{14}P_{23}\rangle, |V_{13}V_{24}\rangle, |V_{14}V_{23}\rangle$ corresponds

to those in Ref. [4], and the definitions are collected in Appendix A. Each of the channel wave functions (18)–(21) has a direct and an exchange part. The wave function Ψ_{PP} corresponds to the pseudoscalar-meson–pseudoscalar-meson (PP) channel, while Ψ_{VV} corresponds to the vector-meson–vector-meson (VV) channel. The third and fourth wave functions represent closed (hidden color) channels. They are formed by coupling two color octet $q\bar{q}$ pairs to a single $2q-2\bar{q}$ state. When the spin of the q and \bar{q} in each pair couples to zero, the channel is denoted by C_0C_0 , and when it couples to one, the notation is C_1C_1 . In this paper we study the influence of the hidden color states.

Using the results from Appendix A, one can see that the wave functions (18)–(21) are not orthogonal to each other. This raises the question as to whether or not they are linearly independent. It can be answered by rewriting the wave functions in terms of the four-dimensional orthogonal color spin basis

$$\begin{aligned}\phi_1 &= |\bar{3}_{12}3_{34}\rangle\chi_+, & \phi_2 &= |\bar{3}_{12}\bar{3}_{34}\rangle\chi_-, \\ \phi_3 &= |6_{12}\bar{6}_{34}\rangle\chi_+, & \phi_4 &= |6_{12}\bar{6}_{34}\rangle\chi_-.\end{aligned}\quad (23)$$

The question is discussed in Appendix B where using (23) we show that the functions (20) and (21) are linearly dependent on (18) and (19) if the orbital wave functions are expressed in terms of a complete set of functions. In that case the hidden color wave functions are redundant. On the other hand, the set (18)–(21) is not normally linearly dependent if the orbital functions are linear combinations of a finite set of functions with a restricted form. In that case the hidden color channels can introduce important new components into the wave function and lower the variational energy (cf. Sec. V).

IV. MATRIX ELEMENTS

This section contains expressions for the overlaps of the wave functions (18)–(21) and for the matrix elements of the kinetic energy and of the confining and spin-spin interaction potentials of the Hamiltonian (1)–(3). The spin and color parts of the wave functions are integrated out, and we give expressions for the matrix elements in terms of integrals containing the orbital parts of (18)–(21). In general, each of the orbital wave functions $R_i(x)$ ($i=P, V, C_0, C_1$) can be expanded in a suitable basis:

$$R_i(x) = \sum_b a_i(b)R(x, b), \quad (24)$$

where b is a set of parameters and $a_i(b)$ are some coefficients to be found variationally. Then it will be enough to express the orbital matrix elements in terms of matrix elements of the basis functions $R(x, b)$.

A. Overlap matrix elements

Because we work in a nonorthogonal basis, we need the overlap matrix elements of the wave functions (18)–(21):

$$B_{ij} = \langle \Psi_i | \Psi_j \rangle, \quad (25)$$

with $i, j = PP, VV, C_0C_0$, or C_1C_1 . The integrals appearing in B_{ij} referring to the orbital wave functions are

$$O^d(b, b') = \langle R(x, b) | R(x, b') \rangle, \quad (26)$$

$$O^e(b, b') = \langle R(x, b) | R(y, b') \rangle, \quad (27)$$

where d and e stand for the direct and exchange integrals. Table I exhibits the matrix elements of B_{ij} as linear combinations of O^d and O^e . The coefficients of the linear combinations come from the integration in the color-spin space. In Appendix C we give the analytic expressions of (26) and (27) for the particular case

$$R(x, b) = e^{-a^2\rho^2} e^{-a^2\rho'^2} e^{-b^2x^2}. \quad (28)$$

This simple choice incorporates a convenient form for the internal wave functions of the $q\bar{q}$ pairs in the asymptotic channels.

B. Kinetic energy matrix elements

The kinetic energy operators can be written in terms of one of the coordinate systems (10)–(12). We have, for example,

$$T = -\frac{\hbar^2}{2m}(\Delta_\rho + \Delta_{\rho'} + \Delta_x) = -\frac{\hbar^2}{2m}(\Delta_x + \Delta_y + \Delta_\lambda), \quad (29)$$

where the mass m is taken from Eq. (5). The matrix elements $\langle \Psi_i | T | \Psi_j \rangle$ of (29) can be expressed in terms of a direct and an exchange integral

$$K^d(b, b') = \langle R(x, b) | T | R(x, b') \rangle, \quad (30)$$

$$K^e(b, b') = \langle R(x, b) | T | R(x, b') \rangle. \quad (31)$$

TABLE I. Overlap matrix $\langle \psi_i(b) | \psi_j(b') \rangle$ of Eq. (25), where O^d and O^e are given in Eqs. (26) and (27).

	$P'P'$	$V'V'$	$C'_0C'_0$	$C'_1C'_1$
PP	$N_{PP}N_{P'P'}(O^d + \frac{1}{6}O^e)$	$-\frac{\sqrt{3}}{6}N_{PP}N_{V'V'}O^e$	$\frac{\sqrt{2}}{3}N_{PP}N_{C'_0C'_0}O^e$	$-(\frac{2}{3})^{1/2}N_{PP}N_{C'_1C'_1}O^e$
VV		$N_{VV}N_{V'V'}(O^d - \frac{1}{6}O^e)$	$-(\frac{2}{3})^{1/2}N_{VV}N_{C'_0C'_0}O^e$	$-\frac{\sqrt{2}}{3}N_{VV}N_{C'_1C'_1}O^e$
C_0C_0			$N_{C_0C_0}N_{C'_0C'_0}(O^d - \frac{1}{6}O^e)$	$\frac{\sqrt{3}}{6}N_{C_0C_0}N_{C'_1C'_1}O^e$
C_1C_1				$N_{C_1C_1}N_{C'_1C'_1}(O^d + \frac{1}{6}O^e)$

TABLE II. Matrix of the kinetic energy operator (29). K^d and K^e are defined by Eqs. (30) and (31).

	$P'P'$	$V'V'$	$C'_0C'_0$	$C'_1C'_1$
PP	$N_{PP}N_{P'P'}(K^d + \frac{1}{6}K^e)$	$-\frac{\sqrt{3}}{6}N_{PP}N_{V'V'}K^e$	$\frac{\sqrt{2}}{3}N_{PP}N_{C'_0C'_0}K^e$	$-(\frac{2}{3})^{1/2}N_{PP}N_{C'_1C'_1}K^e$
VV		$N_{VV}N_{V'V'}(K^d - \frac{1}{6}K^e)$	$-(\frac{2}{3})^{1/2}N_{VV}N_{C'_0C'_0}K^e$	$-\frac{\sqrt{2}}{3}N_{VV}N_{C'_1C'_1}K^e$
C_0C_0			$N_{C_0C_0}N_{C'_0C'_0}(K^d - \frac{1}{6}K^e)$	$\frac{\sqrt{3}}{6}N_{C_0C_0}N_{C'_1C'_1}K^e$
C_1C_1				$N_{C_1C_1}N_{C'_1C'_1}(K^d + \frac{1}{6}K^e)$

The resulting matrix elements are shown in Table II. They can also be obtained from Table I by making the replacement $O \rightarrow K$ because the integrals in the color space are the same. The particular form which these integrals take for the choice (28) are also given in Appendix C.

C. Confinement potential

Integration in the color-spin space allows the matrix elements of the confining potential (2) to be expressed in terms of the following direct or exchange integrals:

$$V_1(b, b') = \frac{1}{2} \langle R(x, b) | r_{13}^2 + r_{24}^2 | R(x, b') \rangle, \quad (32)$$

$$V_2(b, b') = -\frac{1}{4} \langle R(x, b) | r_{12}^2 + r_{34}^2 - r_{13}^2 - r_{24}^2 - r_{14}^2 - r_{23}^2 | R(y, b') \rangle, \quad (33)$$

$$V_3(b, b') = \frac{1}{2} \langle R(x, b) | 2(r_{12}^2 + r_{34}^2) - r_{13}^2 - r_{24}^2 + 7(r_{14}^2 + r_{23}^2) | R(x, b') \rangle, \quad (34)$$

$$V_4(b, b') = -\frac{1}{9} \langle R(x, b) | 5(r_{12}^2 + r_{34}^2) - \frac{1}{2}(r_{13}^2 + r_{24}^2 + r_{14}^2 + r_{23}^2) | R(y, b') \rangle, \quad (35)$$

$$V_5(b, b') = \frac{1}{9} \langle R(x, b) | r_{12}^2 + r_{34}^2 + \frac{7}{2}(r_{13}^2 + r_{24}^2 + r_{14}^2 + r_{23}^2) | R(y, b') \rangle. \quad (36)$$

In the above expressions, one can use the equalities

$$\langle r_{12}^2 \rangle_{d,e} = \langle r_{34}^2 \rangle_{d,e}, \quad (37)$$

$$\langle r_{13}^2 \rangle_{d,e} = \langle r_{24}^2 \rangle_{d,e}, \quad (38)$$

$$\langle r_{14}^2 \rangle_{d,e} = \langle r_{23}^2 \rangle_{d,e}, \quad (39)$$

where $\langle O \rangle_d = \langle R(x, b) | O | R(x, b') \rangle$ and $\langle O \rangle_e = \langle R(x, b) | O | R(y, b') \rangle$ stand for direct and exchange matrix elements. The relation (37) expresses the fact that the quark pair 12 and the antiquark pair 34 are identical in orbital space. The relations (38) and (39) hold for identical pairs of quarks and imply identical behavior of any

$q\bar{q}$ in the direct or exchange channels.

The origin of the linear combinations (32)–(36) appearing in the confinement potential matrix elements is explained in Appendix C where the direct and exchange channels are treated in two different coordinate systems. The analytic form of (32)–(36) for the choice (28) can be also found in Appendix C. The matrix elements of the confinement potential expressed as linear combinations of V_1, V_2, \dots, V_5 are shown in Table III. To each matrix element, the constant $\frac{8}{3}e_0$ times the corresponding overlap matrix element should be added to include the contribution of the first term in Eq. (2).

TABLE III. Matrix elements of the confining potential (2). The quantities V_i ($i = 1, 2, \dots, 5$) are defined in Eqs. (32)–(36). The constant $\frac{8}{3}e_0$ times the overlap matrix should be added to this matrix.

	$P'P'$	$V'V'$	$C'_0C'_0$	$C'_1C'_1$
PP	$N_{PP}N_{P'P'} \frac{4k}{9} (3V_1 + V_2)$	$-N_{PP}N_{V'V'} \frac{4\sqrt{3}}{9} kV_2$	$N_{PP}N_{C'_0C'_0} \frac{k\sqrt{2}}{4} V_5$	$-N_{PP}N_{C'_1C'_1} \frac{k\sqrt{6}}{4} V_5$
VV		$N_{VV}N_{V'V'} \frac{4k}{9} (3V_1 - V_2)$	$-N_{VV}N_{C'_0C'_0} \frac{k\sqrt{6}}{4} V_5$	$-N_{VV}N_{C'_1C'_1} \frac{k\sqrt{2}}{4} V_5$
C_0C_0			$N_{C_0C_0}N_{C'_0C'_0} \frac{k}{12} (2V_3 + 3V_4)$	$-N_{C_0C_0}N_{C'_1C'_1} \frac{k\sqrt{3}}{4} V_4$
C_1C_1				$N_{C_1C_1}N_{C'_1C'_1} \frac{k}{12} (2V_3 - 3V_4)$

D. Hyperfine interaction

The hyperfine interaction (3) contains the spin-spin term of the well-known Breit-Fermi interaction. Hadron spectroscopy studies indicate that the tensor term is generally much weaker than the spin-spin term, and for the present purpose it can safely be neglected. The spin-spin interaction we use here has a regularized form factor f_{ij} given by (4) and (5). After integration in the color-spin space, the matrix elements of (3) reduce to linear combinations of integrals containing f_{ij} . These are

$$\begin{aligned}
 I_0 &= \langle R(x,b) | f_{12} | R(x,b') \rangle \\
 &= \langle R(x,b) | f_{34} | R(x,b') \rangle \\
 &= \langle R(x,b) | f_{14} | R(x,b') \rangle \\
 &= \langle R(x,b) | f_{23} | R(x,b') \rangle ,
 \end{aligned} \tag{40}$$

$$\begin{aligned}
 I_1 &= \langle R(x,b) | f_{13} | R(x,b') \rangle \\
 &= \langle R(x,b) | f_{24} | R(x,b') \rangle ,
 \end{aligned} \tag{41}$$

$$\begin{aligned}
 I_2 &= \langle R(x,b) | f_{12} | R(y,b') \rangle \\
 &= \langle R(x,b) | f_{34} | R(y,b') \rangle ,
 \end{aligned} \tag{42}$$

$$\begin{aligned}
 I_3 &= \langle R(x,b) | f_{14} | R(y,b') \rangle \\
 &= \langle R(x,b) | f_{23} | R(y,b') \rangle ,
 \end{aligned} \tag{43}$$

$$\begin{aligned}
 I_4 &= \langle R(x,b) | f_{13} | R(y,b') \rangle \\
 &= \langle R(x,b) | f_{24} | R(y,b') \rangle .
 \end{aligned} \tag{44}$$

The various equalities in (40)–(44) result from the symmetry properties of $R(x,b)$ used in Sec. IV C and also from the particular form (4) of f_{ij} . The analytic expressions of the integrals (40)–(44) with $R(x,b)$ given by (28) are reproduced in Appendix C. The matrix elements of the spin-spin interaction (3) written as linear combinations of $I_0, I_1, I_2, I_3,$ and I_4 are given in Table IV. Each of the matrix elements from this table should be multiplied by the constant

$$C^{SS} = -\frac{8\pi\alpha_s}{3m^2} . \tag{45}$$

The solution of the variational problem related to the Hamiltonian (1) is discussed in the next section.

V. RESULTS

Here we search for a variational solution of the Hamiltonian (1) with wave functions of the type (24). The coefficients are obtained by diagonalizing the Hamiltonian in a chosen basis (Ritz-Rayleigh variational principle). This means that we solve the eigenvalue matrix equation

$$\langle H \rangle C_n = E_n B C_n , \tag{46}$$

where $\langle H \rangle$ is the Hamiltonian matrix, B the overlap matrix as described in the previous section, and C_n a column matrix formed with the coefficients of a variational solution. We need only the lowest solution E_0 from which we subtract $2m_\pi$ where m_π is given by (8).

We are interested in the following problems: (1) the

TABLE IV. Matrix elements of the spin-spin interaction Eq. (3) in terms of the integrals (40)–(44). Each matrix element should be multiplied by the constant C^{SS} from Eq. (45).

	$P'P'$	$V'V'$	$C_0' C_0'$	$C_1' C_1'$
PP	$\frac{N_{PP} N_{P'P'}}{3} (6I_1 + I_2 + I_3 + I_4)$	$N_{PP} N_{V'V'} \frac{\sqrt{3}}{9} (I_2 - I_3 - I_4)$	$N_{PP} N_{C_0' C_0'} \frac{\sqrt{2}}{24} [7(I_3 + I_4) - 2I_2]$	$-N_{PP} N_{C_1' C_1'} \frac{\sqrt{6}}{72} [12I_0 + 2I_2 + 25(I_3 + I_4)]$
VV	$\frac{N_{VV} N_{V'V'}}{9} (-6I_1 + 5I_2 + I_3 + I_4)$	$-N_{VV} N_{C_0' C_0'} \frac{\sqrt{6}}{72} [24I_0 + 2I_2 - 11(I_3 + I_4)]$	$N_{VV} N_{C_1' C_1'} \frac{\sqrt{2}}{72} [-10I_2 + 7(I_3 + I_4)]$	$N_{VV} N_{C_1' C_1'} \frac{\sqrt{2}}{72} [-10I_2 + 7(I_3 + I_4)]$
$C_0 C_0$		$\frac{N_{C_0 C_0} N_{C_0' C_0'}}{24} (-6I_1 + 10I_2 + I_3 + I_4)$		$N_{C_0 C_0} N_{C_1' C_1'} \frac{\sqrt{3}}{72} (-30I_0 + 5I_2 - I_3 - I_4)$
$C_1 C_1$				$\frac{N_{C_1 C_1} N_{C_1' C_1'}}{72} (108I_0 + 6I_1 + 50I_2 + I_3 + I_4)$

importance of the spin-spin interaction, (2) the explicit role of the hidden color states C_0 and C_1 , and (3) the choice of good but simple variational functions Ψ_j .

The importance of the spin-spin interaction is shown in Fig. 2 where the energy $E_0 - 2m_\pi$ is plotted as a function of the parameter b of Eq. (28). The matrix $\langle H \rangle$ is 4×4 ; i.e., it contains PP , VV , C_0C_0 , and C_1C_1 channels. The parameter $1/b$ plays the role of a deformation parameter. At small b the two $q\bar{q}$ pairs are far apart and the lowest eigenstate of the $2q-2\bar{q}$ system approaches $2m_\pi$. At large b the two quark pairs get close together and the interaction becomes repulsive. In fact, at large b all quarks get close together. In the limit of zero separations $r_{ij}=0$, V_{ij}^{conf} reduces to the term $-e_0/4 \sum \hat{\lambda}_i \cdot \hat{\lambda}_j$. This term cancels identically in $E_0 - 2m_\pi$. Hence the repulsion at short separations is entirely due to the kinetic term. In the case where the spin-spin term is suppressed, $E_0 - 2m_\pi$ increases steadily from zero to infinity as a function of b . When the spin-spin is included, the energy surface gets an attractive pocket which shows that the spin-spin interaction is crucial in obtaining a bound $2q-2\bar{q}$ system with the Hamiltonian (1). Such a result is entirely consistent with the findings of Ref. [4].

The minimum in the attractive pocket is at $b = 1.1 \text{ fm}^{-1}$ or $1/b \cong 0.9 \text{ fm}$. This is greater than twice the pseudoscalar $q\bar{q}$ rms radius as given in Eq. (9), suggesting a molecular-type structure for the $2q-2\bar{q}$ system.

With respect to problems (2) and (3), we are close to Kamimura's approach [14] and also to his technique based on the Gaussian basis variational method. That technique proved very successful in treating few-body nuclei [15]. The principle is to use simple orbital wave functions (Gaussians) and to include as many channels as possible. In our case we can take into account four distinct channels PP , VV , C_0C_0 , and C_1C_1 as explained in the previous section.

As we deal with a nonorthogonal basis in the eigenvalue problem (46), it is not convenient to use the expansion coefficients in order to test the hidden color content of

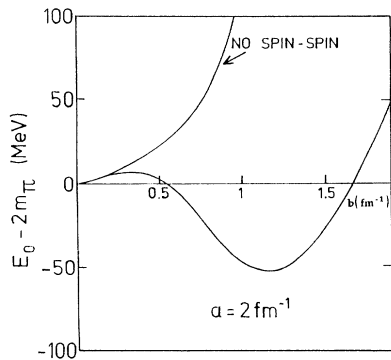


FIG. 2. Interaction energy of a $2q-2\bar{q}$ system calculated with the trial orbital wave function (28) in the four-channel basis (18)–(21). The parameter $a = 2 \text{ fm}^{-1}$ is kept fixed, and the interaction energy is plotted against the relative position parameter b (fm^{-1}). The upper curve is calculated without the hyperfine interaction (3), while the lower curve is with the full Hamiltonian (1).

TABLE V. Minimum in the interaction energy of a $2q-2\bar{q}$ system showing the effect of adding new channels as explained in the text.

Channels	b (fm^{-1})	b' (fm^{-1})	E_{min} (MeV)
PP, VV	1.2		-28.9
PP, VV, C_0C_0, C_1C_1	1.1		-51.9
$PP, VV, C_0C_0, C_1C_1, P'P'$	1.1	0.4	-60.9
$PP, VV, C_0C_0, C_1C_1, P'P', V'V'$	1.1	2.2	-86.2
$PP, VV, P'P', V'V'$	1.1	2.2	-84.6
$PP, VV, C_0C_0, C_1C_1, P'P', V'V', C'_0C'_0$	1.1	2.2	-100
$PP, VV, C_0C_0, C_1C_1, P'P', V'V', C'_0C'_0, C'_1C'_1$	1.1	2.2	-102

the variational function. Instead, we plot in Fig. 3 the energy $E_0 - 2m_\pi$ as a function of b for the case where the hidden color channels have been neglected (PP and VV only) to be compared with the case where they are included. One can see that the hidden color channels are efficient only beyond $b > 0.4 \text{ fm}^{-1}$. Thus they couple strongly to the physical channels only at shorter separation distance $d < 2.5 \text{ fm}$ between $q\bar{q}$ pairs. The minimum energy is lowered from -28.9 to -51.9 MeV at separations of the order of $\sim 0.9 \text{ fm}$. This is consistent with the nature of the hidden color states: They correspond to closed channels. A similar behavior has been encountered in the nucleon-nucleon (NN) problem [16].

The next step is to take a succession of Gaussians with $b_1 \neq b_2 \neq \dots \neq b_n$, which leads to a $4n$ -dimensional basis. The coefficients of the variational solution are obtained by solving (46). Here we consider an eight-dimensional basis at most. Our results are shown in Table V. All calculations are made with $a = 2 \text{ fm}^{-1}$, which minimizes the pseudoscalar meson mass, as explained above Eq. (8).

In Table V the first row is for a wave function with PP and VV channels only and just one orbital function of type (28), i.e., $R_P(x, b) = R_V(x, b)$. At fixed $a = 2 \text{ fm}^{-1}$, the energy minimizes at $b = 1.2 \text{ fm}^{-1}$. The next row with PP, VV, C_0C_0 , and C_1C_1 channels and

$$R_P(x, b) = R_V(x, b) = R_{C_0}(x, b) = R_{C_1}(x, b) \quad (47)$$

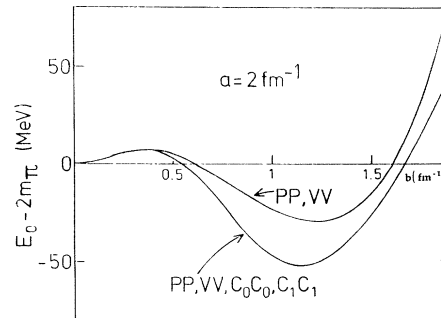


FIG. 3. Interaction energy of a $2q-2\bar{q}$ system as a function of b (fm^{-1}) with the trial orbital wave function (28) in the two-channel basis (18) and (19) and the four-channel basis (18)–(22). As in Fig. 2, we take $a = 2 \text{ fm}^{-1}$.

has a minimum at $b = 1.1 \text{ fm}^{-1}$ and the pocket becomes much more attractive than in the previous case. The other entries in the table show the effect of adding new channels with a different parameter b' . In that case the channels are denoted by P' , V' , C'_0 , and C'_1 . Comparison of the fifth and last rows shows again that the hidden color channels bring a substantial amount of attraction.

The stability of the results at adding a new parameter b' has been tested by varying b at $b' = 2.2 \text{ fm}^{-1}$ fixed. We found that the binding energy is very stable around $b = 1.1 \text{ fm}^{-1}$.

Our basis is different from the one used by Weinstein and Isgur [4] mainly in that the octet (hidden color) channels are included explicitly. As explained in Appendix B, if a complete set of orbital wave functions had been used in Ψ_{PP} and Ψ_{VV} , then the octet channels would have been linearly dependent on the singlet channels and would therefore have been redundant. But with simple orbital wave functions as above the octet states introduce important new components to the wave function and lower the variational energy. This is consistent with Kamimura's findings.

In an eight-dimensional basis (Table V), we get a binding of 102 MeV for the $2q-2\bar{q}$ system as compared to 81 MeV of Ref. [4]. However, a precise comparison is difficult because in the present work the mass of the pseudoscalar $q\bar{q}$ pair is 173 MeV, while in Ref. [4] it is 137 MeV. This could be because Weinstein and Isgur use a more elaborate trial wave function for the $q\bar{q}$ system as compared to the one used here [Eq. (7)]. At this stage comparison with the experiment is meaningless.

VI. CONCLUSIONS

As mentioned in the Introduction, two major questions were at the origin of this work.

One was related to the role played by the spin-spin interaction in obtaining the binding of a $2q-2\bar{q}$ system of equal masses. Based on a simple variational solution of type (28), we obtained a positive energy surface at any separation distance, i.e., no binding (Fig. 2). The addition of a hyperfine interaction led to binding. The (negative) minimum in the energy surface appeared at a separation distance $1/b \sim 0.9 \text{ fm}$, larger than twice the pseudoscalar meson radius, which suggests a molecular-type structure of the bound $2q-2\bar{q}$ system.

The other was related to the role played by the hidden color (octet-octet) states in the variational solution. As demonstrated in Appendix B, these states are not needed if the direct and exchange orbital wave functions are expressed in terms of a complete set. In practice, with a variational solution, this is not the case and one has to make the best choice for the variational wave function. Our proposal of explicitly including the hidden color channels has the same motivation as the variational approach proposed by Kamimura [14] for a few-body system, who showed that the explicit inclusion of all possible channels can be more efficient in obtaining a convergence in the variational energy than by using more elaborate orbital wave functions with fewer channels. Kamimura's technique has been very successful for muonic molecules [14] as well as for ^3H or ^3He . In the particular case [15]

of ^3H and ^3He , this also led to a precise calculation of the wave function over a wide range of separation between the clusters of the relevant physical channels.

ACKNOWLEDGMENTS

We wish to acknowledge helpful discussions with Jack Paton, Klaus Goeke, and Walter Glöckle. This research was sponsored in part by FNRS (Belgium), and the manuscript has been prepared at ECT*, Trento.

APPENDIX A

One can construct a color singlet $2q-2\bar{q}$ system in three ways (see Fig. 1). The corresponding orthonormal bases are

$$|\bar{3}_{12}3_{34}\rangle, |6_{12}\bar{6}_{34}\rangle, \quad (\text{A1})$$

$$|1_{13}1_{24}\rangle, |8_{13}8_{24}\rangle, \quad (\text{A2})$$

$$|1_{14}1_{23}\rangle, |8_{14}8_{23}\rangle. \quad (\text{A3})$$

Here particles 1 and 2 are quarks and 3 and 4 are antiquarks. The $\bar{3}$ and 3 color states are antisymmetric, and the 6 and $\bar{6}$ are symmetric under the transpositions (12) and (32). The bases (A2) and (A3) correspond to the direct and exchange channels. As each of the wave functions (18)–(21) contains one vector of (A2) and another of (A3), it is useful for evaluating matrix elements to express the color states (A2) and (A3) in terms of (A1). The relations are

$$|1_{13}1_{24}\rangle = (\frac{1}{3})^{1/2}|\bar{3}_{12}3_{34}\rangle + (\frac{2}{3})^{1/2}|6_{12}\bar{6}_{34}\rangle, \quad (\text{A4})$$

$$|8_{13}8_{24}\rangle = -(\frac{2}{3})^{1/2}|\bar{3}_{12}3_{34}\rangle + (\frac{1}{3})^{1/2}|6_{12}\bar{6}_{34}\rangle \quad (\text{A5})$$

and

$$|1_{14}1_{23}\rangle = -(\frac{1}{3})^{1/2}|\bar{3}_{12}3_{34}\rangle + (\frac{2}{3})^{1/2}|6_{12}\bar{6}_{34}\rangle, \quad (\text{A6})$$

$$|8_{14}8_{23}\rangle = (\frac{2}{3})^{1/2}|\bar{3}_{12}3_{34}\rangle + (\frac{1}{3})^{1/2}|6_{12}\bar{6}_{34}\rangle. \quad (\text{A7})$$

The situation is analogous in spin space. The three $S=0$ orthonormal basis sets are

$$\chi_+, \chi_-, \quad (\text{A8})$$

$$|P_{13}P_{24}\rangle, |V_{13}V_{24}\rangle, \quad (\text{A9})$$

$$|P_{14}P_{23}\rangle, |V_{14}V_{23}\rangle. \quad (\text{A10})$$

In the notation of Ref. [4], the basis vectors (A8) are defined by

$$\chi_- = |S_{12}S_{34}\rangle \quad \text{and} \quad \chi_+ = |A_{12}A_{34}\rangle.$$

It is useful to write them explicitly as

$$\chi_+ = \frac{1}{\sqrt{12}}(2\uparrow\uparrow\downarrow\downarrow + 2\downarrow\downarrow\uparrow\uparrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow), \quad (\text{A11})$$

$$\chi_- = \frac{1}{2}(\uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\downarrow\uparrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow), \quad (\text{A12})$$

where \uparrow and \downarrow are the spin states with $s = \frac{1}{2}$ and $s_z = \frac{1}{2}$, $s_z = -\frac{1}{2}$, respectively. The particle order 1,2,3,4 is under-

stood everywhere. One can check the normalization and orthogonality, and see that the symmetry properties with respect to the transpositions (12) and (32) are

$$(12)\chi_+ = (34)\chi_+ = \chi_+, \quad (\text{A13})$$

$$(12)\chi_- = (34)\chi_- = -\chi_-. \quad (\text{A14})$$

In fact, χ_+ and χ_- form the Young-Yamanuchi basis of the two-dimensional irreducible representation [2 2] of S_4 . The corresponding Young tableaux are

$$\chi_+ \rightarrow \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \quad \chi_- \rightarrow \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}. \quad (\text{A15})$$

The asymptotic channel spin functions (A9) or (A10) are obtained by first introducing the two-body zero-spin state

$$\chi_{00} = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

and the one-spin states

$$\chi_{11} = \uparrow\uparrow,$$

$$\chi_{10} = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow),$$

$$\chi_{1,-1} = \downarrow\downarrow,$$

from which one can build the four-particle $S=0$ states

$$P_{13}P_{24} = \chi_{00}(1,3)\chi_{00}(2,4) \\ = \frac{1}{2}(\uparrow\uparrow\downarrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow + \downarrow\downarrow\uparrow\uparrow), \quad (\text{A16})$$

$$P_{14}P_{23} = \chi_{00}(1,4)\chi_{00}(2,3) \\ = \frac{1}{2}(\uparrow\uparrow\downarrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow + \downarrow\downarrow\uparrow\uparrow), \quad (\text{A17})$$

$$V_{13}V_{24} = \sum_m \langle 1m \ 1-m | 00 \rangle \chi_{1m}(1,3)\chi_{1-m}(2,4) \\ = \frac{1}{\sqrt{12}}(2\uparrow\downarrow\uparrow\downarrow + 2\downarrow\uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow\uparrow \\ - \uparrow\downarrow\downarrow\uparrow - \downarrow\downarrow\uparrow\uparrow), \quad (\text{A18})$$

$$V_{14}V_{23} = \sum_m \langle 1m \ 1-m | 00 \rangle \chi_{1m}(1,4)\chi_{1-m}(2,3) \\ = \frac{1}{\sqrt{12}}(2\uparrow\downarrow\uparrow\downarrow + 2\downarrow\uparrow\downarrow\uparrow - \uparrow\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow\uparrow \\ - \uparrow\downarrow\downarrow\uparrow - \downarrow\downarrow\uparrow\uparrow). \quad (\text{A19})$$

One can notice that $V_{13}V_{24}$ and $P_{13}P_{24}$ result from the action of the transposition (14) or (23) on χ_+ and χ_- , respectively,

$$V_{13}V_{24} = (23)\chi_+ = (14)\chi_+ = -\frac{1}{2}\chi_+ + \frac{\sqrt{3}}{2}\chi_-, \quad (\text{A20})$$

$$P_{13}P_{24} = (23)\chi_- = (14)\chi_- = \frac{\sqrt{3}}{2}\chi_+ + \frac{1}{2}\chi_-, \quad (\text{A21})$$

and that

$$V_{14}V_{23} = (13)\chi_+ = (24)\chi_+ = -\frac{1}{2}\chi_+ - \frac{\sqrt{3}}{2}\chi_-, \quad (\text{A22})$$

$$P_{14}P_{23} = -(13)\chi_- = -(24)\chi_- = +\frac{\sqrt{3}}{2}\chi_+ - \frac{1}{2}\chi_-. \quad (\text{A23})$$

APPENDIX B

The purpose of this appendix is to discuss the possible linear dependence of the functions (18)–(21). To do this we introduce a short notation for the color singlet states,

$$\eta_1 = |1_{13}1_{24}\rangle |P_{13}P_{24}\rangle, \quad \eta_2 = |1_{14}1_{23}\rangle |P_{14}P_{23}\rangle, \quad (\text{B1})$$

$$\eta_3 = |1_{13}1_{24}\rangle |V_{13}V_{24}\rangle, \quad \eta_4 = |1_{14}1_{23}\rangle |V_{14}V_{23}\rangle,$$

and a corresponding notation for the color octet states:

$$\zeta_1 = |8_{13}8_{24}\rangle |P_{13}P_{24}\rangle, \quad \zeta_2 = |8_{14}8_{23}\rangle |P_{14}P_{23}\rangle, \quad (\text{B2})$$

$$\zeta_3 = |8_{13}8_{24}\rangle |V_{13}V_{24}\rangle, \quad \zeta_4 = |8_{14}8_{23}\rangle |V_{14}V_{23}\rangle.$$

Then we know from the results of Appendix A that

$$\eta_i = \sum_j A_{ij}\phi_j, \quad \zeta_i = \sum_j B_{ij}\phi_j, \quad (\text{B3})$$

where the ϕ_i are defined in Eq. (25) and A and B are 4×4 matrices. Thus in matrix form one has $\zeta = BA^{-1}\eta$ provided the inverse of A exists (and it does). This means, for example, that

$$|8_{13}8_{24}\rangle |P_{13}P_{24}\rangle \\ = a|1_{13}1_{24}\rangle |P_{13}P_{24}\rangle + b|1_{14}1_{23}\rangle |P_{14}P_{23}\rangle \\ + c|1_{13}1_{24}\rangle |V_{13}V_{24}\rangle + d|1_{14}1_{23}\rangle |V_{14}V_{23}\rangle \quad (\text{B4})$$

and

$$|8_{14}8_{23}\rangle |P_{14}P_{23}\rangle \\ = b|1_{13}1_{24}\rangle |P_{13}P_{24}\rangle + a|1_{14}1_{23}\rangle |P_{14}P_{23}\rangle \\ + d|1_{13}1_{24}\rangle |V_{13}V_{24}\rangle + c|1_{14}1_{23}\rangle |V_{14}V_{23}\rangle, \quad (\text{B5})$$

where a, b, c, d are numbers which can be calculated. One obtains $a = -1/2\sqrt{2}$, $b = 3/4\sqrt{2}$, $c = 0$, and $d = -3\sqrt{3}/4\sqrt{2}$.

Now we discuss an octet wave function $\Psi_{C_0C_0}$ as in Eq. (20) of Sec. III. Using (B4) and (B5), we get

$$\Psi_{C_0C_0} = \Psi_{PP} + \Psi_{VV}, \quad (\text{B6})$$

where

$$N_{PP}R_P(x) = N_{C_0C_0}[aR_{C_0}(x) + bR_{C_0}(y)] \quad (\text{B7})$$

and

$$N_{VV}R_V(x) = N_{C_0C_0}[cR_{C_0}(x) + dR_{C_0}(y)]. \quad (\text{B8})$$

For interpreting (B6)–(B8) consider wave functions of the form (18) and (19) and express $R_P(x)$ and $R_V(x)$ in terms of a basis set of $2n$ orbital wave functions $R(x, b)$ as in Eq. (24). Suppose the basis set has the property that the exchange orbital wave function $R(y, b)$ is a member of the set for every $R(x, b)$. Then the octet wave func-

tions are not needed. Any octet wave function with orbital parts given in the same basis can be written as a linear combination of color singlet parts by using Eqs. (B6)–(B8).

An alternative would be to take a basis set $R(x, b)$ of n orbital functions where the exchange functions $R(y, b)$ are independent of the direct wave functions and keep both the color singlet and color octet states. As in the first case, the total basis set has dimension $4n$, the variational space is the same, and the results of a variational calculation will be identical.

APPENDIX C

The integration in the color space can be performed in one of the three orthonormal bases (A1)–(A3) introduced in Appendix A depending on the choice of the internal coordinates. For the direct matrix elements, it is appropriate to use the basis $|1_{13}1_{24}\rangle, |8_{13}8_{24}\rangle$ together with the coordinates ρ, ρ' , and \mathbf{x} given by (10). For the exchange matrix elements, we chose to use the basis $|\bar{3}_{12}3_{34}\rangle, |6_{12}\bar{6}_{34}\rangle$ together with the relative coordinates \mathbf{x}, \mathbf{y} , and λ appearing in (10), (11), and (12). In the spin space, there are also three orthonormal bases (A8)–(A10) available as discussed in Appendix A. The choice of basis is entirely analogous to that of the color basis.

Here we give the analytic form of various matrix elements appearing in the overlap and the Hamiltonian matrix for

$$R(x, b) = e^{-a^2(\rho^2 + \rho'^2) - b^2 x^2} = e^{-a^2(\lambda^2 + y^2) - b^2 x^2} \quad (\text{C1})$$

and

$$R(y, b) = e^{-a^2(a^2 + \alpha^2) - b^2 y^2} = e^{-a^2(\lambda^2 + x^2) - b^2 y^2}. \quad (\text{C2})$$

1. Overlap matrix

The matrix elements of Table I contain O^d and O^e . For (C1) and (C2) these become

$$\begin{aligned} O^d(b, b') &= \langle R(x, b) | R(x, b') \rangle \\ &= \frac{\pi^4 \sqrt{\pi}}{2^3 a^6 (b^2 + b'^2)^{3/2}}, \end{aligned} \quad (\text{C3})$$

$$\begin{aligned} O^e(b, b') &= \langle R(x, b) | R(y, b') \rangle \\ &= \frac{\pi^4 \sqrt{\pi}}{a^3 [2(a^2 + b^2)(b^2 + b'^2)]^{3/2}}. \end{aligned} \quad (\text{C4})$$

2. Kinetic energy matrix

The matrix elements of Table II take the particular form

$$\begin{aligned} K^d(b, b') &= \langle R(x, b) | T | R(x, b') \rangle \\ &= \frac{3\hbar^2}{2m} \frac{\pi^{9/2}}{4} \frac{a^2(b^2 + b'^2) + b^2 b'^2}{a^6 (b^2 + b'^2)^{5/2}}, \end{aligned} \quad (\text{C5})$$

$$\begin{aligned} K^e(b, b') &= \langle R(x, b) | T | R(y, b') \rangle \\ &= \frac{3\hbar^2}{2m} \frac{\pi^{9/2}}{2^{3/2}} \frac{a^4 + 3a^2(b^2 + b'^2) + 5b^2 b'^2}{a[(a^2 + b^2)(a^2 + b'^2)]^{5/2}}. \end{aligned} \quad (\text{C6})$$

3. Confinement potential

The direct matrix elements V_1 and V_3 are best evaluated in the coordinate system $(\rho, \rho', \mathbf{x})$ and the exchange matrix elements V_2, V_4 , and V_5 in the mixed coordinate system $(\mathbf{x}, \mathbf{y}, \lambda)$. The corresponding operators and the analytical expressions for the wave functions (C1) and (C2) are

$$\begin{aligned} V_1 &= \langle R(x, b) | \rho^2 + \rho'^2 | R(x, b') \rangle \\ &= \frac{3\pi^{9/2}}{2^4} \frac{1}{a^8 (b^2 + b'^2)^{3/2}}, \end{aligned} \quad (\text{C7})$$

$$\begin{aligned} V_2 &= \langle R(x, b) | \lambda^2 | R(y, b') \rangle \\ &= \frac{3\pi^{9/2}}{2^{7/2}} \frac{1}{a^5 [(a^2 + b^2)(a^2 + b'^2)]^{3/2}}, \end{aligned} \quad (\text{C8})$$

$$\begin{aligned} V_3 &= \langle R(x, b) | 9x^2 + \frac{7}{2}(\rho^2 + \rho'^2) + \frac{5}{3}\rho \cdot \rho' | R(x, b') \rangle \\ &= \frac{3\pi^{9/2}}{2^4} \frac{1}{a^6 (b^2 + b'^2)^{3/2}} \left[\frac{9}{b^2 + b'^2} + \frac{7}{2a^2} \right], \end{aligned} \quad (\text{C9})$$

$$\begin{aligned} V_4 &= \langle R(x, b) | -x^2 - y^2 + \frac{2}{9}\lambda^2 | R(y, b') \rangle \\ &= -\frac{3\pi^{9/2}}{2^{5/2}} \frac{1}{a^3 [(a^2 + b^2)(a^2 + b'^2)]^{3/2}} \\ &\quad \times \left[\frac{1}{a^2 + b^2} + \frac{1}{a^2 + b'^2} - \frac{1}{9a^2} \right], \end{aligned} \quad (\text{C10})$$

$$\begin{aligned} V_5 &= \langle R(x, b) | x^2 + y^2 + \frac{14}{9}\lambda^2 | R(y, b') \rangle \\ &= \frac{3\pi^{9/2}}{2^{5/2}} \frac{1}{a^3 [(a^2 + b^2)(a^2 + b'^2)]^{3/2}} \\ &\quad \times \left[\frac{1}{a^2 + b^2} + \frac{1}{a^2 + b'^2} + \frac{7}{9a^2} \right]. \end{aligned} \quad (\text{C11})$$

4. Hyperfine interaction

The matrix elements of Table IV are linear combinations of five distinct integrals I_0, I_1, I_2, I_3 , and I_4 containing the radial part (4) of the spin-spin interaction. These are

$$\begin{aligned}
I_0 &= \langle R(x, b) | f_{12} | R(x, b') \rangle \\
&= \langle R(x, b) | f_{34} | R(x, b') \rangle \\
&= \langle R(x, b) | f_{14} | R(x, b') \rangle = \langle R(x, b) | f_{23} | R(x, b') \rangle \\
&= \frac{\pi^3}{2^{3/2}} \frac{\sigma^3}{a^3 [2a^2(b^2 + b'^2) + (2a^2 + b^2 + b'^2)\sigma^2]^{3/2}}, \tag{C12}
\end{aligned}$$

$$I_1 = \langle R(x, b) | f_{13} | R(x, b') \rangle = \langle R(x, b) | f_{24} | R(x, b') \rangle = \left[\frac{\pi}{2} \right]^3 \frac{\sigma^3}{a^3 [(a^2 + \sigma^2)(b^2 + b'^2)]^{3/2}}, \tag{C13}$$

$$\begin{aligned}
I_2 &= \langle R(x, b) | f_{12} | R(y, b') \rangle = \langle R(x, b) | f_{34} | R(y, b') \rangle \\
&= \frac{\pi^3}{2^{3/2}} \frac{\sigma^3}{a^3 [(a^2 + b^2)(a^2 + b'^2) + (2a^2 + b^2 + b'^2)\sigma^2]^{3/2}}, \tag{C14}
\end{aligned}$$

$$\begin{aligned}
I_3 &= \langle R(x, b) | f_{14} | R(y, b') \rangle = \langle R(x, b) | f_{23} | R(y, b') \rangle \\
&= \pi^3 \frac{\sigma^3}{[2a^2(a^2 + b^2)(a^2 + b'^2) + (a^2 + b^2)(3a^2 + b^2)\sigma^2]^{3/2}}, \tag{C15}
\end{aligned}$$

$$\begin{aligned}
I_4 &= \langle R(x, b) | f_{13} | R(y, b') \rangle = \langle R(x, b) | f_{24} | R(y, b') \rangle \\
&= \pi^3 \frac{\sigma^3}{[2a^2(a^2 + b^2)(a^2 + b'^2) + (a^2 + b^2)(3a^2 + b'^2)\sigma^2]^{3/2}}. \tag{C16}
\end{aligned}$$

Note that for $b = b'$ one has

$$I_3 = I_4 \tag{C17}$$

and for $a = b = b'$ one has

$$I_0 = I_1 = I_2 = I_3 = I_4. \tag{C18}$$

One should comment that at $b = a$ or $b' = a$ the matrix resulting from the variational principle is singular and such points should be avoided.

-
- [1] R. L. Jaffe, Phys. Rev. D **15**, 267 (1977); **15**, 281 (1977); R. L. Jaffe and K. Johnson, Phys. Lett. **60B**, 201 (1976); R. L. Jaffe, Phys. Rev. D **17**, 1444 (1978).
- [2] A. T. Aerts, P. J. Mulders, and J. J. de Swart, Phys. Rev. D **21**, 1370 (1980).
- [3] P. Hasenfratz and J. Kuti, Phys. Rep. **C40**, 75 (1978).
- [4] J. Weinstein and N. Isgur, Phys. Rev. D **27**, 588 (1983).
- [5] J. Weinstein and N. Isgur, Phys. Rev. D **41**, 2236 (1990).
- [6] S. Zouzou, B. Silvestre-Brac, C. Gignoux, and J.-M. Richard, Z. Phys. C **30**, 457 (1986), and references therein.
- [7] See, e.g., J. L. Rosner, Phys. Rev. Lett. **21**, 950 (1968); Phys. Rep. **11C**, 89 (1974); M. Fukugita and K. Igi, *ibid.* **31C**, 237 (1977).
- [8] Chan Hong Mo and H. Høgaasen, Phys. Lett. **72B**, 121 (1977); W. W. Buck, C. B. Dover, and J.-M. Richard, Ann. Phys. (N.Y.) **121**, 47 (1979), and references therein.
- [9] M. R. Pennington, in *Proceedings of the Dalitz Conference 1990*, edited by I. J. R. Aitchison, C. W. Llewellyn Smith, and J. E. Paton (World Scientific, Singapore, 1991), p. 66.
- [10] N. A. Tornqvist, Phys. Rev. Lett. **67**, 556 (1991).
- [11] K. Dooley, E. S. Swanson, and T. Barnes, Phys. Lett. B **275**, 478 (1992).
- [12] G. Karl, Int. J. Mod. Phys. E **1**, 491 (1992).
- [13] B. Silvestre-Brac and C. Semay, Z. Phys. C **57**, 273 (1993).
- [14] M. Kamimura, Phys. Rev. A **38**, 621 (1988).
- [15] H. Kameyama, M. Kamimura, and Y. Fukushima, Phys. Rev. C **40**, 974 (1989).
- [16] M. Harvey, Nucl. Phys. **A352**, 326 (1981).