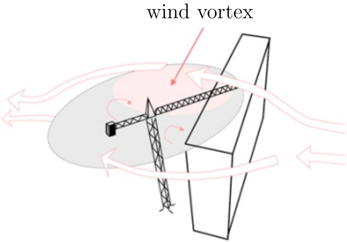
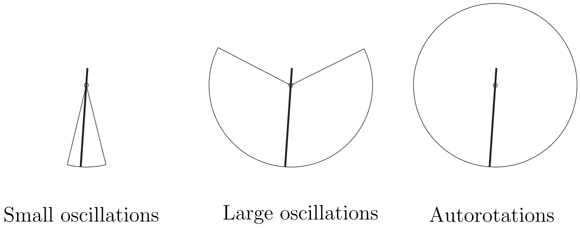
The first-passage time as an analysis tool for the reliability of stochastic oscillators

Vanvinckenroye Hélène
PhD Dissertation Defense

Motivation



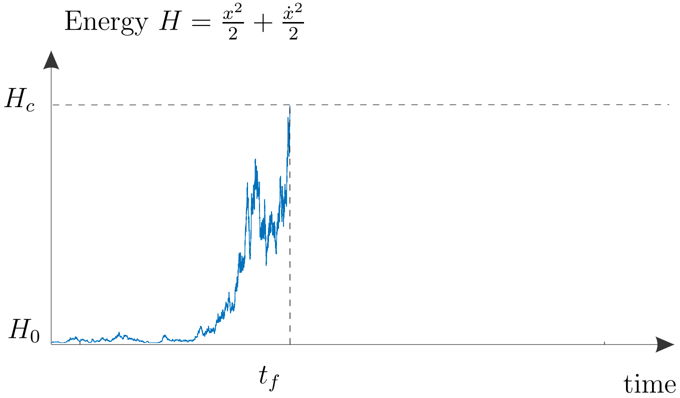
Etudes des effets du vent sur les grues à tour
Voisin, 2003.



Motivation

Movie

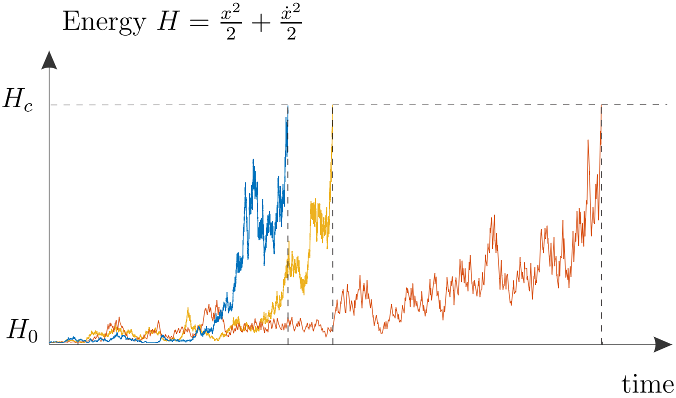
The first-passage time



How much time ?

First – passage time t_f to go from H_0 to H_c

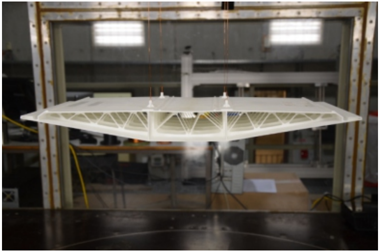
The first-passage time



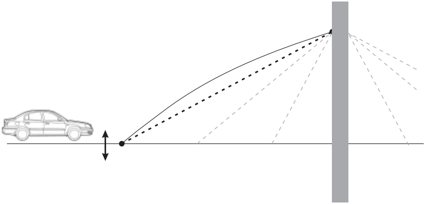
How much time ?

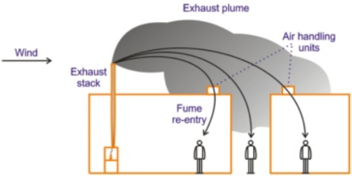
First – passage time t_f to go from H_0 to H_c

The first-passage time



Mitigation of the torsional flutter phenomenon of a bridge deck section during a lifting phase, Andrianne T. and de Ville de Goyet V., 2016.



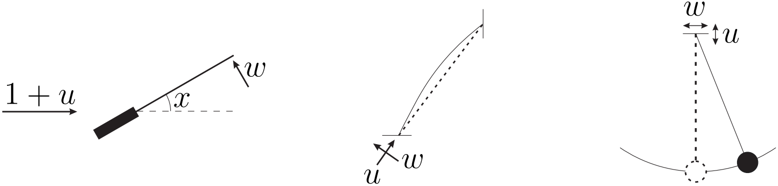


CFD simulation of micro-scale pollutant dispersion in the built environment, Blocken et al. (2013)

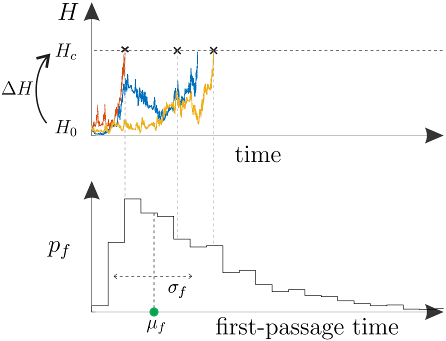
The Mathieu oscillator

$$\ddot{x}(t) + 2\xi\dot{x}(t) + (1 + u(t))\, x(t) = w(t)$$

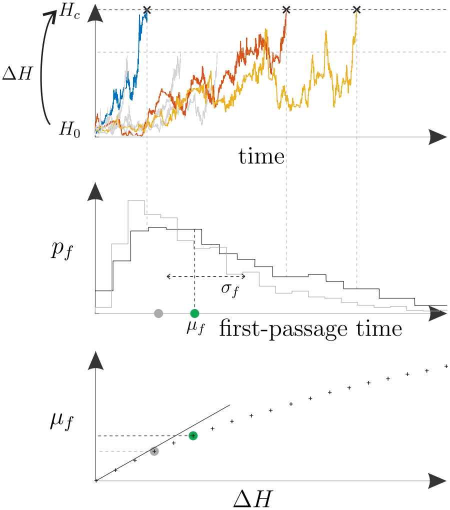
Parametric ($u(t)$) Forced ($w(t)$)



The Mathieu oscillator



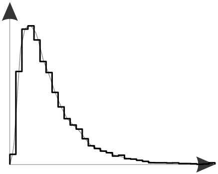
The Mathieu oscillator



The Mathieu oscillator

	Monte Carlo simulations
+	versatile
-	time consuming
	hard to analyze
	needs to be repeated

Analytical approach	Numerical approach
good understanding	versatile
large validity	time efficient
complex	
not always possible	hard to analyze
	needs to be repeated



Governing equations

$$\ddot{x}(t) + 2\xi\dot{x} + (1 + u(t))\ x(t)\ = w(t)$$

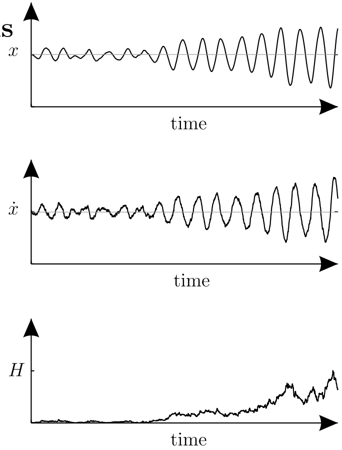
stationary white noise excitations
of intensities $S_u, S_w \ll 1$

slightly damped
$\xi \ll 1$

(very) long transient regime

$\rightarrow$ the energy H evolves slowly

$$H = \frac{\dot{x}^2}{2} + \frac{x^2}{2}$$



Governing equations

$$\ddot{x}(t) + 2\xi\dot{x} + (1 + u(t))\ x(t)$$

Stochastic averaging

$$\dot{H} = m(H) + \sigma(H)\eta(t)$$

white noise of intensity 1

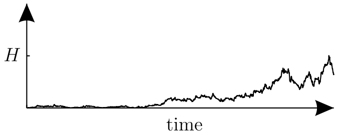
drift $\quad m(H) = \frac{H}{2}S_u + \frac{S_w}{2} - 2\xi H$

and

diffusion $\quad \sigma^2(H) = \frac{H^2}{2}S_u + HS_w$

$\rightarrow$ the energy H evolves slowly

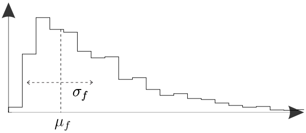
$$H = \frac{\dot{x}^2}{2} + \frac{x^2}{2}$$



Governing equations

Generalized Pontryagin equation

$$m(H_0)\frac{\partial M_k}{\partial H_0} + \frac{1}{2}\sigma^2(H_0)\frac{\partial^2 M_k}{\partial H_0^2} = -M_{k-1} \quad \text{with} \quad M_0 = 1$$

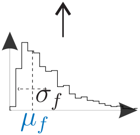


$M_k = \mathcal{E}\{t_f^k\}$
$M_1 = \mu_f$ mean first-passage time
$M_2 = \sigma_f^2 + \mu_f^2$ mean square first-passage time

Average first-passage time of the undamped Mathieu oscillator

$$\ddot{x}(t) + \cancel{2\xi}\dot{x}(t) + (1 + u(t))\; x(t) = w(t)$$

Analytical			Numerical
average FPT	average FPT	**variance of the FPT**	**complete distribution**
undamped	**damped**	undamped	**damped**
linear	linear	linear	**nonlinear**
white noise excitations	white noise excitations	white noise excitations	**evolutionary excitation**



Average first-passage time of the undamped Mathieu oscillator

Pontryagin equation with $k = 1$: $M_1 = \mathcal{E}\left\{t_f\right\} = \mu_f$

$$\left(\frac{H_0}{2}S_u + \frac{S_w}{2}\right)\frac{\partial \mu_f}{\partial H_0} + \left(\frac{H_0^2}{4}S_u + \frac{H_0}{2}S_w\right)\frac{\partial^2 \mu_f}{\partial H_0^2} = -1$$

Asymptotic expansion

$$\mu_f = \frac{4}{S_u}\ln\left(\frac{H_c S_u + 2S_w}{H_0 S_u + 2S_w}\right)$$

$$\Downarrow$$

$$\mu_f \frac{S_u}{4} = \ln\left(1 + \frac{\Delta H^\star}{H_0^\star + 1}\right)$$

with $H_0^\star = \frac{H_0 S_u}{2S_w}$ and $\Delta H^\star = \frac{\Delta H S_u}{2S_w}$.

Average first-passage time of the undamped Mathieu oscillator

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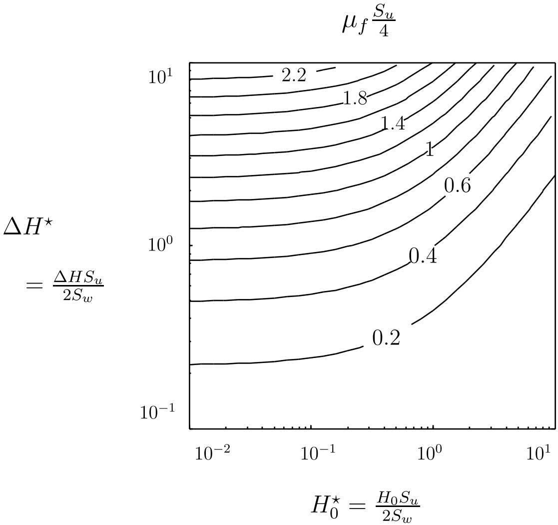
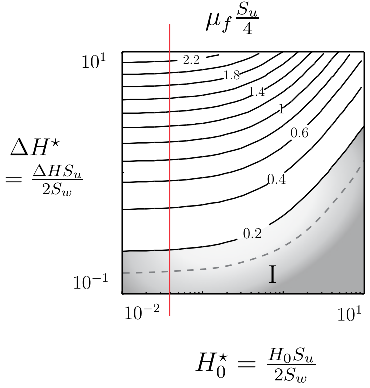
Average first-passage time of the undamped Mathieu oscillator

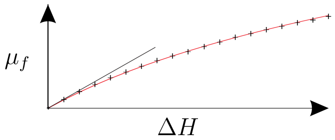


Average first-passage time of the undamped Mathieu oscillator - Regimes

$$\ddot{x}(t) + (1 + u(t))\, x(t) = w(t)$$



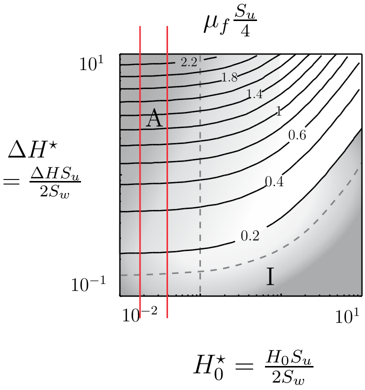
Incubation regime



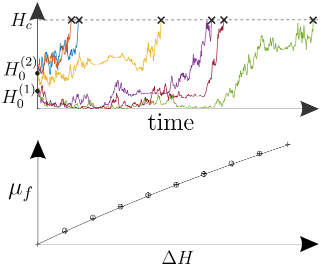
$$\mu_f = \frac{4}{S_u} \frac{\Delta H^\star}{H_0^\star + 1}$$

Average first-passage time of the undamped Mathieu oscillator - Regimes

$$\ddot{x}(t) + (1 + u(t))\, x(t) = w(t)$$



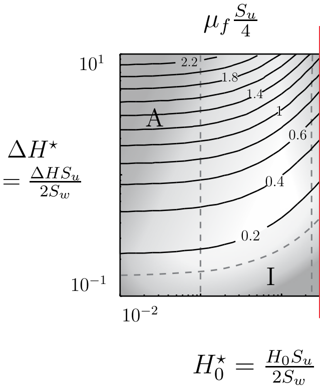
Additive regime

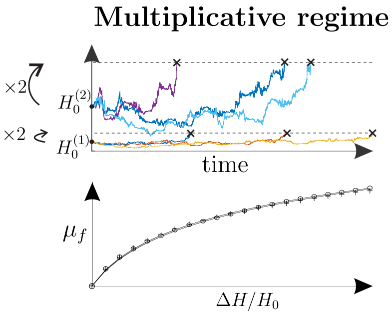


$$\mu_f = \frac{4}{S_u} \ln(1 + \Delta H^\star)$$

Average first-passage time of the undamped Mathieu oscillator - Regimes

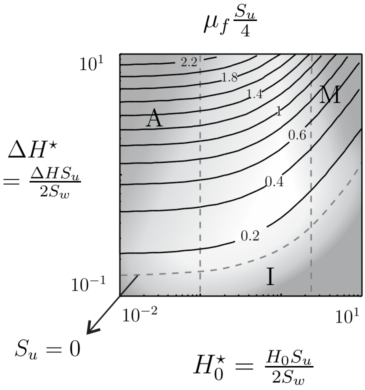
$$\ddot{x}(t) + (1 + u(t))\ x(t) = w(t)$$





$$\mu_f = \frac{4}{S_u} \ln(1 + \frac{\Delta H}{H_0})$$

Average first-passage time of the undamped Mathieu oscillator - limit cases

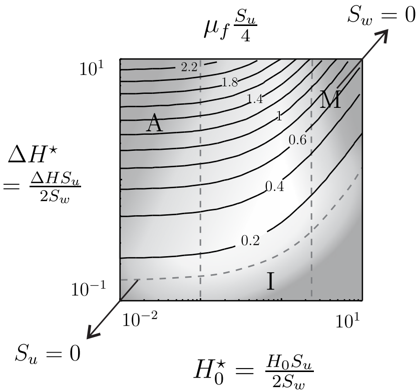


- $S_u = 0$ No parametric

$$\ddot{x}(t) + (1 + \cancel{u(t)})\, x(t) = w(t)$$

$$\mu_f = \frac{4}{S_u} \Delta H^\star = \frac{2}{S_w} \Delta H$$

Average first-passage time of the undamped Mathieu oscillator - limit cases



- $S_u = 0$ No parametric

$$\ddot{x}(t) + (1 + \cancel{u(t)})\, x(t) = w(t)$$

$$\mu_f = \frac{4}{S_u}\Delta H^\star = \frac{2}{S_w}\Delta H$$

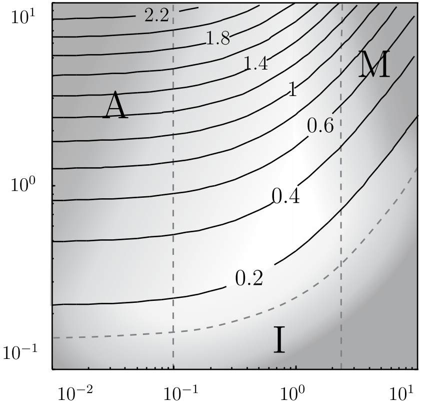
- $S_w = 0$ No forced

$$\ddot{x}(t) + (1 + u(t))\, x(t) = \cancel{w(t)}$$

$$\mu_f = \frac{4}{S_u}\ln\left(1 + \frac{\Delta H}{H_0}\right)$$

Average first-passage time of the undamped Mathieu oscillator

$\mu_f \frac{S_u}{4}$

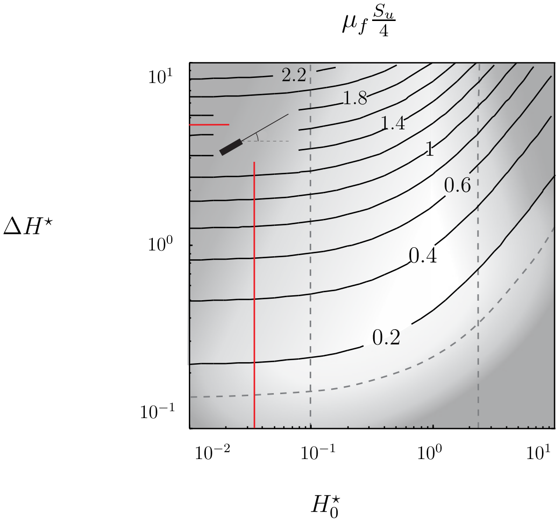


$\Delta H^\star$

$= \frac{\Delta H S_u}{2 S_w}$

$H_0^\star = \frac{H_0 S_u}{2 S_w}$

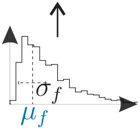
Average first-passage time of the undamped Mathieu oscillator



Average first-passage time of the damped Mathieu oscillator

$$\ddot{x}(t) + 2\xi\,\dot{x}(t) + (1 + u(t))\;x(t) = w(t)$$

Analytical			Numerical
average FPT	average FPT	**variance of the FPT**	**complete distribution**
undamped	**damped**	undamped	**damped**
linear	linear	linear	**nonlinear**
white noise excitations	white noise excitations	white noise excitations	**evolutionary excitation**



Average first-passage time of the damped Mathieu oscillator

Pontryagin equation with $k = 1$: $M_1 = \mathcal{E}\left\{t_f\right\} = \mu_f$

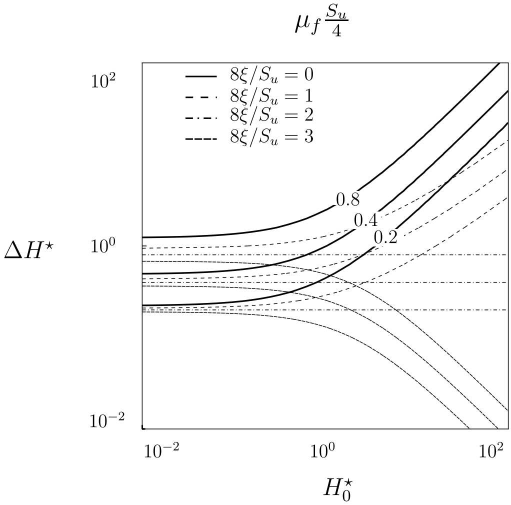
$$\left(\frac{H_0}{2}S_u + \frac{S_w}{2} - 2\xi H\right)\frac{\partial \mu_f}{\partial H_0} + \left(\frac{H_0^2}{4}S_u + \frac{H_0}{2}S_w\right)\frac{\partial^2 \mu_f}{\partial H_0^2} = -1$$

Asymptotic expansion

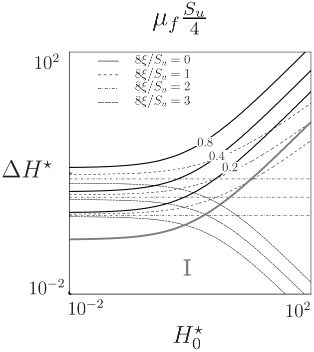
$$\mu_f \frac{S_u}{4} = \mathsf{fct}(H_0^\star, \Delta H^\star, 8\xi/S_u)$$

Analytical expression is established.

Average first-passage time of the damped Mathieu oscillator



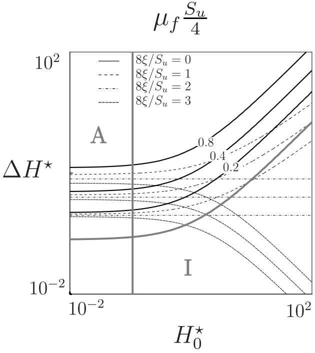
Average first-passage time of the damped Mathieu oscillator - Regimes



Incubation regime

μ_f scales linearly with $\Delta H^\star$ for given $H_0^\star$

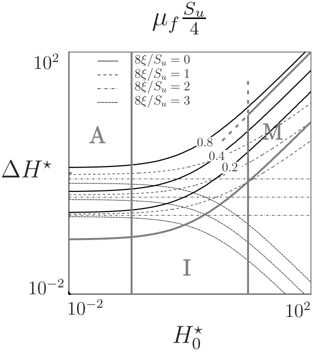
Average first-passage time of the damped Mathieu oscillator - Regimes



Additive regime

μ_f is a function of $\Delta H^\star$ only

Average first-passage time of the damped Mathieu oscillator - Regimes

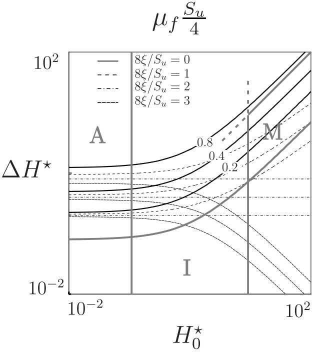


Multiplicative regime

Parallel straight lines

Slope decreases with $\frac{8\xi}{S_u}$

Average first-passage time of the damped Mathieu oscillator



Effect of damping

μ_f increases

Slope changes in M

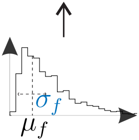
Little effect elsewhere

Different toplogy

Variance of the first-passage time of the undamped Mathieu oscillator

$$\ddot{x}(t) + \cancel{2\xi}\dot{x}(t) + (1 + u(t))\; x(t) = w(t)$$

Analytical			Numerical
average FPT	average FPT	**variance of the FPT**	**complete distribution**
undamped	**damped**	undamped	**damped**
linear	linear	linear	**nonlinear**
white noise excitations	white noise excitations	white noise excitations	**evolutionary excitation**



Variance of the first-passage time of the undamped Mathieu oscillator

Pontryagin equation with $k = 2$: $M_2 = \mathcal{E}\left\{t_f^2\right\}$

$$\left(\frac{H_0}{2}S_u + \frac{S_w}{2}\right)\frac{\partial M_2}{\partial H_0} + \left(\frac{H_0^2}{4}S_u + \frac{H_0}{2}S_w\right)\frac{\partial^2 M_2}{\partial H_0^2} = -M_1 = -\mu_f$$

Asymptotic expansion

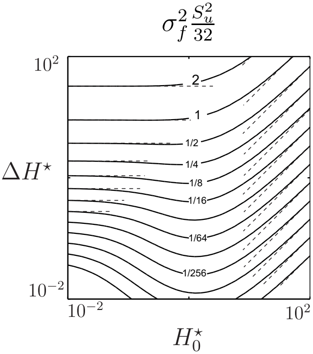
$$M_2\frac{S_u^2}{32} = \mathsf{fct}(H_0^\star, \Delta H^\star)$$

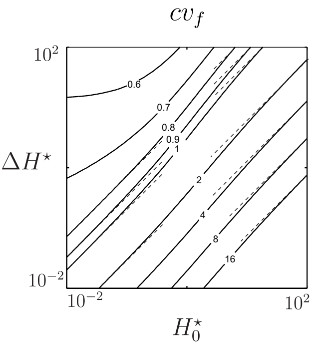
Analytical expression is established.

Variance of the first-passage time of the undamped Mathieu oscillator

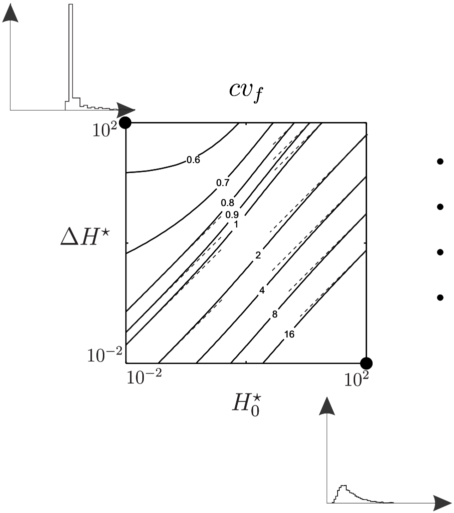
$$\sigma_f^2 = \mathcal{E}\left\{t_f^2\right\} - \mu_f^2$$

$$cv_f = \sigma_f/\mu_f$$



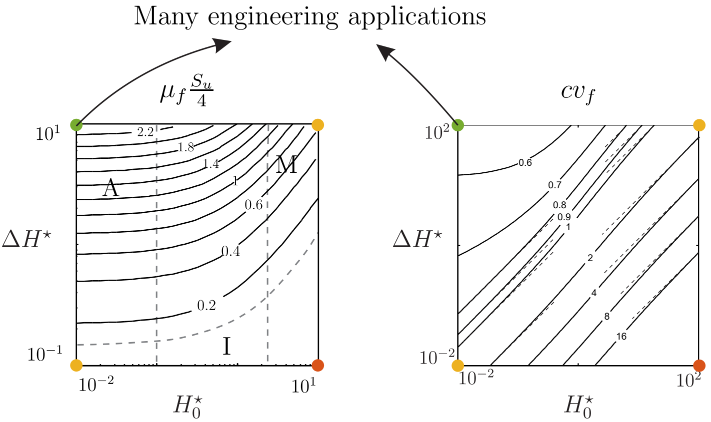


Variance of the first-passage time of the undamped Mathieu oscillator



- Quasi straight lines
- Additive regime
- Large cv_f = spread pdf
- Small cv_f = sharp pdf

Variance of the first-passage time of the undamped Mathieu oscillator

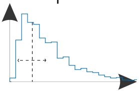


A numerical approach for the distribution of the first-passage time of more complex systems

$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) + \varepsilon z(x, \dot{x}) = w(t)$$

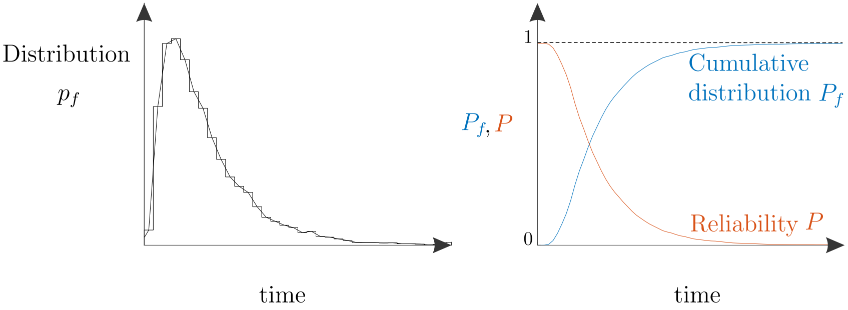
$$S(w; t)$$

Analytical			Numerical
average FPT	average FPT	**variance of the FPT**	**complete distribution**
undamped	**damped**	undamped	**damped**
linear	linear	linear	**nonlinear**
white noise excitations	white noise excitations	white noise excitations	**evolutionary excitation**



Governing equations

$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) + \varepsilon z(x,\dot{x}) = w(t)$$



Governing equations

$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) + \varepsilon z(x, \dot{x}) = w(t)$$

Equivalent linearization

$$\ddot{x}(t) + \beta_e(H)\dot{x}(t) + \omega_e^2(H)x(t) = w(t)$$

with $H = \frac{x^2}{2} + \frac{\dot{x}^2}{2\omega_e^2(H)}$

Governing equations

Stochastic averaging

$$\dot{H} = m(H, t) + \sigma(H, t)\eta(t)$$

Backward-Kolmogorov equation

$$\frac{\partial P}{\partial t} = m(H_0, t)\frac{\partial P}{\partial H_0} + \frac{1}{2}\sigma^2(H_0, t)\frac{\partial^2 P}{\partial H_0^2}$$

Galerkin scheme

$$P(t; H_0) = P_{lin}(t; H_0) + P_{nlin}(t; H_0)$$

Projection of the linear solution in the eigen basis of the confluent hypergeometric functions $\mathcal{M}(-\lambda_i, 1, H)$

$$P_{lin}(t; H_0) = \sum_{i=1}^{\infty} T_i(t)\Phi_i(H_0)$$

Time coefficients $T_i(t)$ given by a set of differential equations

In practice limited to a finite number of terms N

Governing equations

By extension

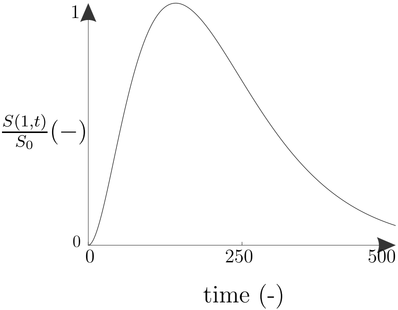
$$P_{nlin}(t; H_0) = \sum_{i=1}^{\infty} c_i(t)\Phi_i(H_0)$$

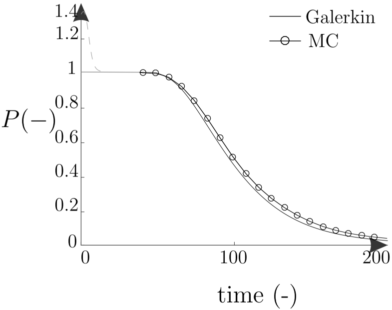
Time coefficients $c_i(t)$ given by a set of N coupled differential equations.

$$\dot{\mathbf{c}}(t) = \mathbf{D}(t)\,\mathbf{c}(t) + \mathbf{e}(t)$$

Example of the Duffing oscillator under seismic excitation

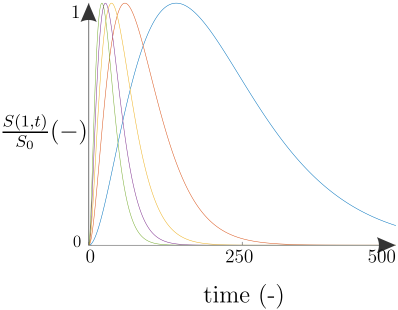
$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) + \varepsilon x^3(t) = w(t)$$

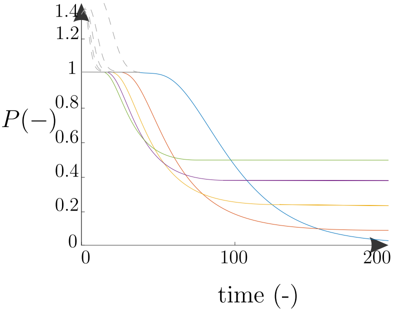




Example of the Duffing oscillator under seismic excitation

$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) + \varepsilon x^3(t) = w(t)$$





Particular case of the undamped oscillator

New basis of eigenfunctions

$$\text{BesselJ}(0, \sqrt{4\lambda_i H}) = \lim_{\xi \to 0} \mathcal{M}(-\lambda_i, 1, H)$$

- Computationally more simple
- Implemented in standard softwares
- Hypergeometric basis is anyway an approximation in the nonlinear case

Alternative formulation of the energy

$$\ddot{x}(t) + 2\xi\dot{x}(t) + x(t) + \varepsilon z(x, \dot{x}) = w(t)$$

$$\ddot{x}(t) + \beta_e(H)\dot{x}(t) + \omega_e^2(H)x(t) = w(t)$$

$$H = \frac{x^2}{2} + \frac{\dot{x}^2}{2\omega_e^2(H)}$$

amplitude-based formulation

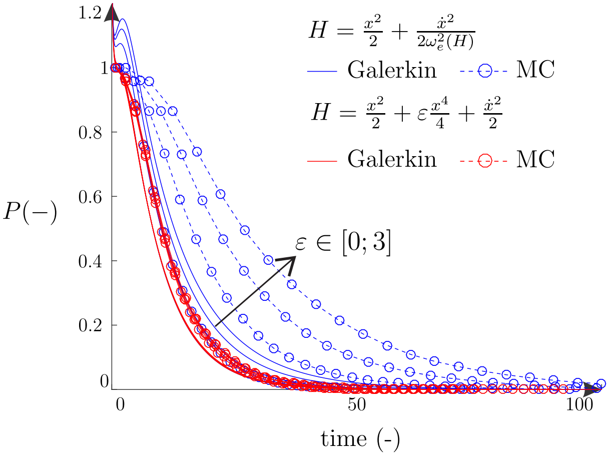
$$u(x) = \int_0^x (y + \varepsilon z(y))dy$$

$$H = u(x) + \frac{\dot{x}^2}{2}$$

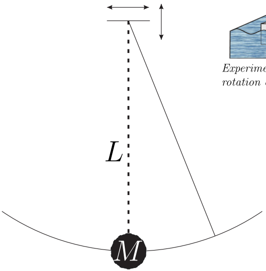
Potential energy envelope formulation

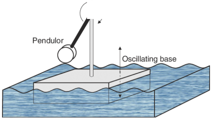
- No statistical linearization
- Restricted to time modulated excitations and nonlinearities in term of stiffness (Duffing)

Alternative formulation of the energy

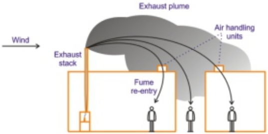


Some applications

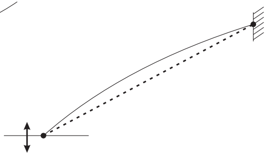


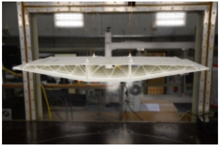


Experimental control for initiating and maintaining rotation of parametric pendulum, Vaziri et al., 2014.



CFD simulation of micro-scale pollutant dispersion in the built environment, Blocken et al.(2013)





Mitigation of the torsional flutter phenomenon of bridge deck section during a lifting phase, Andrianne T. and de Ville de Goyet V., 2016.

The tower crane problem

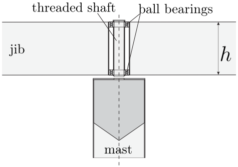
$$I\ddot{\theta} + C\dot{\theta} = M_w = \frac{1}{2}\rho_{air} C_M h B^2 ||\boldsymbol{v}_{rel}||^2$$

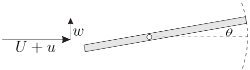
$$\Downarrow$$

$$I\ddot{\theta} + \left(C + M^\star \frac{r}{U}(1 + 2\frac{u}{U})\right)\dot{\theta} + M^\star(1 + 2\frac{u}{U})\theta = M^\star \frac{w}{U}$$

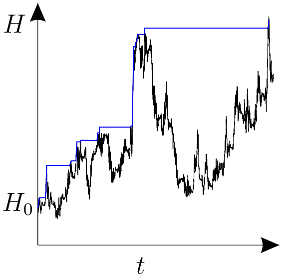
$$\Downarrow$$

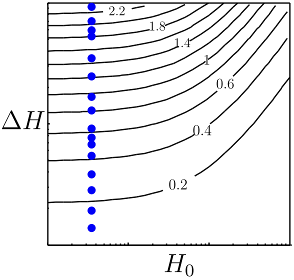
$$\theta'' + 2\xi_s\theta' + (1 + \tilde{u})\theta = -\tilde{w}$$



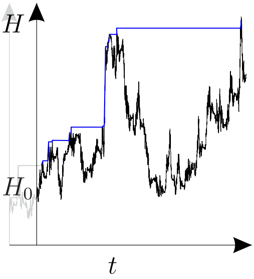


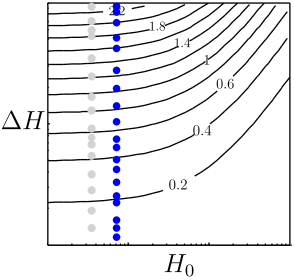
Algorithmic establishment of the first-passage map from experimental data



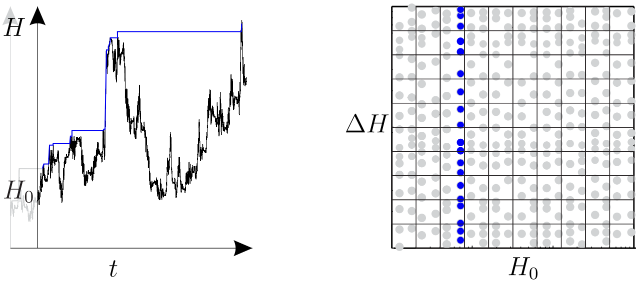


Algorithmic establishment of the first-passage map from experimental data

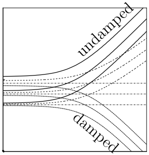


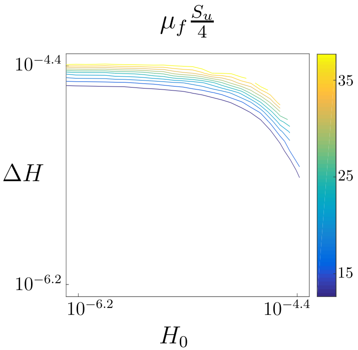


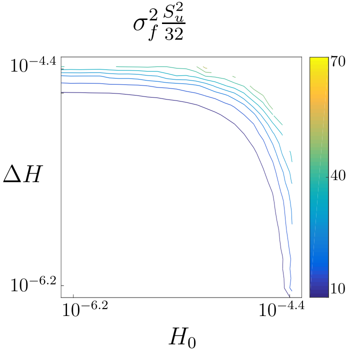
Algorithmic establishment of the first-passage map from experimental data



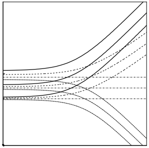
The tower crane problem

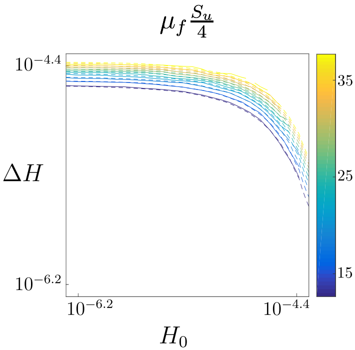


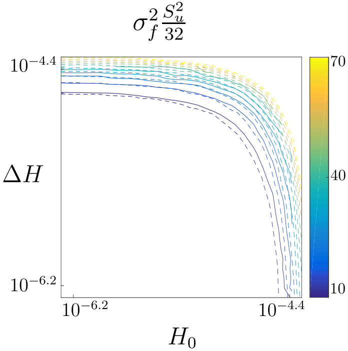




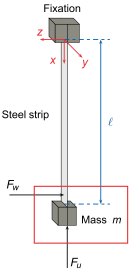
The tower crane problem

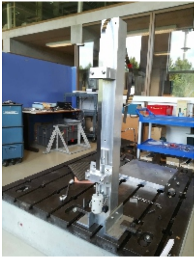


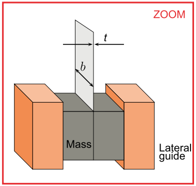


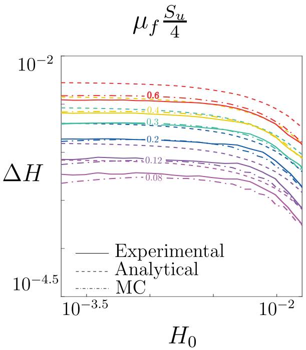


The pre-stressed steel strip



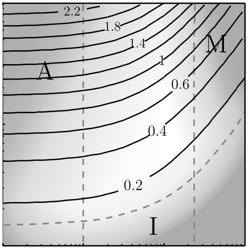


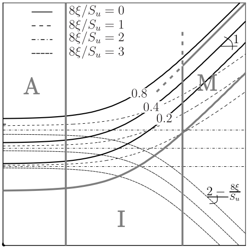


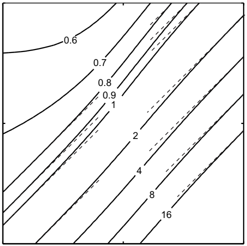


Conclusion

How much time ?





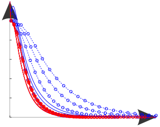


Conclusion

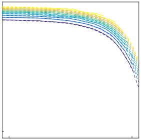
Theoretical frame

analytical solution
maps, reduced energy
regimes

Numerical approach for damped, nonlinear systems under evolutionary excitation



New basis for the undamped case
Two different energy definitions



Experimental investigation of the tower crane

Algorithmic establishment of the first-passage map
Equivalent Mathieu oscillator

Engineering point of view on a mathematical problem

Limitation and perspectives

Theoretical developments limited by increasing complexity
More complex problems can be

reduced to equivalent simple systems
analyzed within developed frame

Perspectives: prediction of equivalent Mathieu oscillator

MDOF systems, colored excitations

analytical expressions for P

Future applications

Monitoring of structures, identification of structural properties, bridge flutter,...

Thank you!

Backup slide

$$\dot{H} = m(H) + \sigma(H)\zeta(t)$$

Generalized Pontryagin equation

$$m(H_0)\frac{\partial M_k}{\partial H_0} + \frac{1}{2}\sigma^2(H_0)\frac{\partial^2 M_k}{\partial H_0^2} = -M_{k-1} \quad \text{with} \quad M_0 = 1$$

Boundary conditions

$$M_k(H_0) = 0, \text{ if } H_0 = H_c \qquad \text{and} \qquad |M_k(H_0 = 0)| < \infty$$

Second condition is qualitative, can be transformed into quantitative condition through

$$\begin{cases} \sigma^2(H) & \to \mathcal{O}(|H - H_l|^{\alpha_l}),\ \alpha_l \geq 0, \qquad H \to H_l \\ m(H) & \to \mathcal{O}(|H - H_l|^{\beta_l}),\ \beta_l \geq 0, \qquad H \to H_l \\ \frac{2m(H)(H-H_l)^{\alpha_l - \beta_l}}{\sigma^2(H)} & \to c_l, \qquad H \to 0 \end{cases}$$

For entrance and repulsively natural boundary classes, the second condition can be replaced by the quantitative condition

$$\mathcal{O}(|m(H_0)M_k^{'}(H_0)|) \sim \mathcal{O}(|M_{k-1}^{'}(H_0)|), \qquad H_0 \to H_l.$$