

# Generalized regularity spaces. Part II: on the trail of Calderon and Zygmund

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Les journées du GdR Analyse Multifractale

19 Septembre 2018



## Definition

$f \in T_u^p(x_0)$  ( $p \in [1, \infty]$ ,  $u \geq -\frac{d}{p}$ ) if  $f \in L^p(\mathbb{R}^d)$  and there exists a polynomial  $P$  of degree strictly less than  $u$  and  $C > 0$  such that for all  $r > 0$

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An *elliptic partial differentiable equation* at  $x_0 \in \mathbb{R}^d$  of order  $m \in \mathbb{N}$  is a partial differentiable equation of the form

$$\mathcal{E}f = \sum_{|\alpha| \leq m} a_\alpha D^\alpha f = g$$

where for all  $|\alpha| \leq m$ ,  $a_\alpha$  is an  $s \times r$  matrix of functions,  $f$  and  $g$  are vector valued functions with  $f_j \in W_m^p(\mathbb{R}^d)$  for all  $j \in \{1, \dots, r\}$ .

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$$\mu(x_0) := \inf_{|\xi|=1} \det\left[\left(\sum_{|\alpha|=m} a_\alpha^*(x_0) \xi^\alpha\right) \left(\sum_{|\alpha|=m} a_\alpha(x_0) \xi^\alpha\right)\right] > 0$$

is the ellipticity constant of  $\mathcal{E}$  at  $x_0$ .

## Calderon &amp; Zygmund (1961)

Let  $\mathcal{E}f = g$  be an equation with coefficients in  $T_u^\infty(x_0)$ ,  $u > 0$ , which is elliptic at  $x_0$ . Let  $1 < p < \infty$ ,  $u \geq v \geq -\frac{d}{p}$  and  $v$  be non-integral. If  $f \in W_p^m(\mathbb{R}^d)$  and  $g \in T_v^p(x_0)$ , then for all  $j \in \{1, \dots, r\}$  and  $|\alpha| \leq m$ ,

$$\|D^\alpha f_j\|_{T_{v+m-|\alpha|}^q(x_0)} \leq C \left( \sum_{k=1}^s \|g_k\|_{T_v^p(x_0)} + \sum_{k=1}^r \|f_k\|_{m,p} \right)$$

with

- $\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{m-\alpha}{d}$  if  $\frac{1}{p} > \frac{m-\alpha}{d}$ ,
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and  $C$  depends on  $v, p, r, s, \mu(x_0)$  and the least upper bound of the norms in  $T_u^\infty(x_0)$  of the coefficients of  $\mathcal{E}$ .

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A function  $\phi : ]0, +\infty[ \rightarrow ]0, +\infty[$  belongs to the set of *Boyd's function* if  $\phi(1) = 1$ ,  $\phi$  is continuous and for all  $r \in ]0, +\infty[$ ,

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$$\underline{b}(\phi) = \sup_{r \in ]0, 1[} \frac{\log(\bar{\phi}(r))}{\log(r)} = \lim_{r \rightarrow 0} \frac{\log(\bar{\phi}(r))}{\log(r)}$$

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- ⑤  $\bar{b}(\phi) < 0 \Rightarrow \int_1^{+\infty} \frac{\bar{\phi}(x)}{x} dx < \infty$  and  
 $\underline{b}(\phi) > 0 \Rightarrow \int_0^1 \frac{\bar{\phi}(x)}{x} dx < \infty$ ,

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## Proposition

Let  $E$  be a closed subset of  $\mathbb{R}^d$  and  $\phi \in \mathcal{B}$  with  $\underline{b}(\phi) > 0$ , if there exists  $M > 0$  such that  $f \in T_\phi^p(x_0)$  with  $\|f\|_{T_\phi^p(x_0)} \leq M$  for all  $x_0 \in E$

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$$f_\alpha(x) = \sum_{|\beta| < \underline{b}(\phi) - |\alpha|} \frac{f_{\alpha+\beta}(x_0)}{|\beta|!} (x - x_0)^\alpha + R_\alpha(x, x_0)$$

with

$$|R_\alpha(x, x_0)| \leq C\phi(|x - x_0|)|x - x_0|^{-|\alpha|}.$$

for all  $x, x_0 \in E$ , is checked

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Let  $E \subset \mathbb{R}^d$  be a closed set,  $U = \{x \in \mathbb{R}^d \mid d(x, E) < 1\}$ ,  $n \in \mathbb{N}_0$  and  $\phi \in \mathcal{B}$  with  $n < \underline{b}(\phi)$ . If there exists  $M > 0$  such that  $f \in T_\phi^p(x_0)$  with  $\|f\|_{T_\phi^p(x_0)} \leq M$  for all  $x_0 \in E$ , then there exists  $F \in C^n(U)$  such that  $F = f$  almost everywhere on  $E$ .

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$$|\Delta_h^{m-n} D^\alpha F(x)| \leq C \phi(|h|) |h|^{-n}.$$

$$\Delta_h^1 g(x) = g(x + h) - g(x) \quad \text{and} \quad \Delta_h^{l+1} g(x) = \Delta_h^1 \Delta_h^l g(x)$$

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Inequality  $\phi(rx) \leq \phi(r)\bar{\phi}(x)$  and asymptotic behaviour of  $\phi$  and  $\bar{\phi}$ .

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Proposition (D. Kreit, S. Nicolay - 2012)

Let  $\sigma$  be an admissible sequence and  $n, m$  two natural numbers such that  $n < \underline{s}(\sigma^{-1}) \leq \bar{s}(\sigma^{-1}) < m$ , then

$$\Lambda^\sigma(\mathbb{R}^d) = \{f \in C^n(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) : \\ \sup_{|h| \leq 2^{-j}} \|\Delta_h^{m-n} D^\alpha f\|_{L^\infty} \leq C\sigma_j 2^{jn} \forall j \in \mathbb{N}_0, |\alpha| = n\}$$

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Whitney (1934) : Compatibility condition  $\Rightarrow$  Differentiability

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Calderon-Zygmund (1961) :  $T_u^p \Rightarrow$  Compatibility condition (bis)  $\Rightarrow$  Hölder-type condition

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$\mathcal{E}$  elliptic with constant coefficients of the same order  $m$

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### Theorem

Let  $x_0 \in \mathbb{R}^d$ ,  $p \in ]1, \infty]$ ,  $\phi \in \mathcal{B}$  such that  $\underline{b}(\phi) > -\frac{d}{p}$  and  $s > 0$  such that  $\bar{b}(\phi) + s < 0$  or there exists  $n \in \mathbb{N}_0$  for which  $n < \underline{b}(\phi) + s \leq \bar{b}(\phi) + s < n + 1$ . We have that  $\mathcal{J}^s$  maps continuously  $T_\phi^p(x_0)$  into  $T_{\phi_s}^q(x_0)$  where :

- $\frac{1}{p} \geq \frac{1}{q} \geq \frac{1}{p} - \frac{s}{d}$  if  $p < \frac{d}{s}$ ,
- $p \leq q \leq \infty$  if  $\frac{d}{s} < p \leq \infty$ ,
- $p \leq q < \infty$  if  $\frac{d}{s} = p$ .

$$f \in T_\phi^p(0)$$

$$f \in T_\phi^p(0), f = P + R$$

$$r^{-\frac{d}{p}} \|R\|_{L^p(B(0,r))} \leq |f|_{T_\phi^p(x_0)} \phi(r) \quad \forall r > 0.$$

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$$\int_0^1 \frac{\bar{\phi}(\xi) \xi^{d-u}}{\xi} d\xi \quad \text{and} \quad \int_1^{+\infty} \frac{\bar{\phi}(\xi) \xi^{d-u}}{\xi} d\xi$$

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## Theorem

$$\mathcal{K}f(\cdot) = p.v. \int k(\cdot - y)f(y) dy,$$

where the kernel  $k \in C^{\infty}(\mathbb{R}^d \setminus \{0\})$  is homogeneous of degree  $-d$  and has mean value zero on the sphere  $\Sigma$ .

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$$\|\mathcal{K}f\|_{T_\phi^p(x_0)} \leq C_{\phi,p} M \|f\|_{T_\phi^p(x_0)} \quad (3)$$

where

$$M = \sup_{\substack{|x|=1 \\ 0 \leq |\alpha| \leq \lceil \bar{b}(\phi) \rceil_{\mathbb{N}_0}}} |D^\alpha k(x)|.$$

## Example

$$j \in \{1, \dots, d\}$$

$$\mathcal{R}_j f(x) = p.v. \frac{-i\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}} \int \frac{(x_j - y_j)}{|x - y|^{d+1}} f(y) dy.$$

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$$\Lambda = i \sum_{j=1}^d \mathcal{R}_j D_j,$$

# How to find the regularity of the solution ?

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- $p_3 \in ]1, \infty[$ ,  $\mathcal{H}g \in T_\psi^{p_3}(x_0)$ ,  $\|\mathcal{H}g\|_{T_\psi^{p_3}(x_0)} \leq N\|g\|_{T_\psi^{p_3}(x_0)}$ ,

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- $f = \mathcal{J}^m h \in W_m^{p_2}(\mathbb{R}^d)$  ( $p_2 \in ]1, \infty[$ ) and for  $|\alpha| < m$

$$\begin{aligned} \|D^\alpha f\|_{T_\rho^{p_2}(x_0)} &\leq C_{p_2, \psi} \|\mathcal{J}^{m-|\alpha|} h\|_{T_\psi^{p_2}(x_0)} \\ &\leq C_{p_2, \rho, \psi} \|\mathcal{J}^{m-|\alpha|} h\|_{T_{\rho^{m-|\alpha|}}^{p_2}(x_0)} \\ &\leq C \|h\|_{T_\rho^{p_2}(x_0)} \end{aligned}$$

$$\bar{b}(\psi) < \underline{b}(\rho) + 1, \quad l \in \mathbb{Z} \text{ such that } l < \underline{b}(\rho) \leq \bar{b}(\rho) < l + 1$$

$$h = \mathcal{H}g + \mathcal{H}(\mathcal{E}_{x_0} - \mathcal{E})f + L_1(D)f$$

$$\mathcal{E}_1 = \sum_{|\alpha| < m} a_{\alpha} D^{\alpha} \quad \text{and} \quad \mathcal{E}_2 = \sum_{|\alpha| = m} (a_{\alpha}(x_0) - a_{\alpha}) D^{\alpha}$$

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Let  $x_0 \in \mathbb{R}^d$ ,  $p_1, p_2 \in [1, \infty]$  such that  $0 \leq \frac{1}{p_3} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1$  and  $\phi, \psi \in \mathcal{B}$  such that  $\underline{b}(\phi) > 0$ ,  $\underline{b}(\psi) \geq -\frac{d}{p_2}$ ,  $\phi \prec \psi$  and  $\underline{b}(\psi) \leq 0$  or there exists  $n \in \mathbb{N}_0$  for which  $n < \underline{b}(\psi) \leq \bar{b}(\psi) < n + 1$ . If  $f_1 \in T_\phi^{p_1}(x_0)$ ,  $f_2 \in T_\psi^{p_2}(x_0)$  then  $f_1 f_2 \in T_\psi^{p_3}(x_0)$  with

$$\|f_1 f_2\|_{T_\psi^{p_3}(x_0)} \leq C_{d,p_1,p_2,\phi,\psi} \|f_1\|_{T_\phi^{p_1}(x_0)} \|f_2\|_{T_\psi^{p_2}(x_0)}$$

$$\|\mathcal{H}\mathcal{E}_1 f\|_{T_\psi^{p_3}(x_0)} \leq C_{p_1,p_2,\phi,\psi} N M \sum_{|\alpha| < m} \|D^\alpha f\|_{T_\psi^{p_2}(x_0)} \leq C \|h\|_{T_\rho^{p_2}(x_0)}$$

$$\mathcal{E}_2 = \sum_{|\alpha|=m} (a_\alpha(x_0) - a_\alpha) D^\alpha,$$

Let  $x_0 \in \mathbb{R}^d$ ,  $p_1, p_2 \in [1, \infty]$  such that  $0 \leq \frac{1}{p_3} := \frac{1}{p_1} + \frac{1}{p_2} \leq 1$  and  $\phi, \rho \in \mathcal{B}$  such that  $-\frac{d}{p_2} \leq \underline{b}(\rho)$ ,  $0 < \underline{b}(\phi)$ . If  $f_1 \in T_\phi^{p_1}(x_0)$  and  $f_2 \in T_\rho^{p_2}(x_0)$  and if  $x_0$  is a Lebesgue-point of  $f_1$ , then for all  $\psi \in \mathcal{B}$  such that  $\underline{b}(\psi) > -\frac{d}{p_2}$ ,  $\phi \prec \psi$  and

- $\bar{b}(\psi) - \underline{b}(\rho) < \underline{b}(\phi)$  if  $\underline{b}(\phi) \leq 1$ ,
- $\bar{b}(\psi) - \underline{b}(\rho) < 1$  if  $\underline{b}(\phi) > 1$  and  $\bar{b}(\psi) < 1$  or there exists  $n \in \mathbb{N}$  for which  $n < \underline{b}(\psi) \leq \bar{b}(\psi) < n + 1$ .

If  $f_2 \in L^{p_3}(\mathbb{R}^d)$ , then  $(f_1 - f_1(x_0))f_2 \in T_\psi^{p_3}(x_0)$  with

$$\|(f_1 - f_1(x_0))f_2\|_{T_\psi^{p_3}(x_0)} \leq C \|f_1\|_{T_\phi^{p_1}(x_0)} (\|f_2\|_{T_\rho^{p_2}(x_0)} + \|f_2\|_{L^{p_3}(\mathbb{R}^d)}).$$

$$f \in W_m^{p_3}(\mathbb{R}^d)$$

$$\begin{aligned} \|\mathcal{H}\mathcal{E}_2 f\|_{T_\psi^{p_3}(x_0)} &\leq C_{p_1, p_2, \phi, \psi} NM \sum_{|\alpha|=m} (\|D^\alpha f\|_{T_\rho^{p_2}(x_0)} + \|D^\alpha f\|_{L^{p_3}(\mathbb{R}^d)}) \\ &\leq C_{p_1, p_2, \phi, \rho, \psi} NM (\|h\|_{T_\rho^{p_2}(x_0)} + \|f\|_{W_m^{p_3}(\mathbb{R}^d)}). \end{aligned}$$

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$$\|L_1(D)f\|_{T_\psi^{p_3}(x_0)} \leq C_{p_2, p_3, \rho, \psi} (\|h\|_{T_\rho^{p_2}(x_0)} + \|f\|_{W_m^{p_3}(\mathbb{R}^d)})$$

$\mathcal{E}f = g$  elliptic at  $x_0$ , order= $m$ ,

- $f \in W_m^p(\mathbb{R}^d)$ ,  $p \in ]1, \infty[$ ,
- $g \in T_\psi^{p_1}(x_0)$ , with  $n < \underline{b}(\psi) + \bar{b}(\psi) < n + 1$  ( $n \in \mathbb{Z}$ ),  
 $-\frac{d}{p} < \underline{b}(\psi)$ ,
- Coefficients of  $\mathcal{E}$  are in  $T_\phi^q(x_0)$ ,  $x_0$  is a Lebesgue point,  
 $0 < \underline{b}(\phi)$ ,  $q \in ]1, \infty]$
- $\frac{1}{p_1} = \frac{1}{p} + \frac{1}{q}$ ,
- $\phi \prec \psi$

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 $\phi \prec \psi_j$ ,  $n_j < \underline{b}(\psi_j) \leq \bar{b}(\psi_j) < n_j + 1$ ,  $\psi_0 = r^{-\frac{d}{p}}$ ,  $\psi_k = \psi$ ,

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$$0 < \frac{1}{p_i} := \frac{k_p(\phi, \psi)}{q} + \frac{1}{p} < 1 \quad \text{and} \quad f \in W_m^{p'}(\mathbb{R}^d) \forall p' \in [p_i, p]$$

## Theorem

Let  $p \in ]1, \infty[$ ,  $q \in ]1, \infty]$ ,  $x_0 \in \mathbb{R}^d$  and  $\phi, \psi \in \mathcal{B}$  be such that  $-\frac{d}{p} < \underline{b}(\psi)$ ,  $0 < \underline{b}(\phi)$  and there exists  $n \in \mathbb{Z}$  such that  $n < \underline{b}(\psi) \leq \bar{b}(\psi) < n + 1$ . Let  $\mathcal{E}f = g$  be an elliptic differentiable equation of order  $m$  at  $x_0$  such that the coefficients of  $\mathcal{E}$  are functions in  $T_\phi^q(x_0)$  whose  $x_0$  is a Lebesgue point. Let us suppose that :

- $g \in T_\psi^{p_1}(x_0)$  with  $\frac{1}{p_1} := \frac{1}{p} + \frac{1}{q}$
- $\phi \prec \psi$  and  $\bar{b}(\psi) \leq \underline{b}(\psi)$  or  $\bar{b}(\psi) - \underline{b}(\psi) < \min\{1, \underline{b}(\phi)\}$ ,
- $0 < \frac{1}{p'} := \frac{k_p(\phi, \psi)}{q} + \frac{1}{p} < 1$ ,
- $f \in W_m^{p^*}(\mathbb{R}^d)$  for all  $p^* \in [p', p]$

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## Theorem

There exists  $C_{p',\phi,\psi,m}$  such that for all  $|\alpha| \leq m$ ,  $D^\alpha f \in T_{\psi_{m-|\alpha|}}^{q'}(x_0)$  and

$$\begin{aligned} \|D^\alpha f\|_{T_{\psi_{m-|\alpha|}}^{q'}(x_0)} &\leq C_{p',\phi,\psi} (M(1+MN))^{k_p(\phi,\psi)-1} \|g\|_{T_\psi^q(x_0)} \\ &+ k_p(\phi,\psi) (1+MN)^{k_p(\phi,\psi)} (\|f\|_{W_m^p(\mathbb{R}^d)} + \|f\|_{W_m^{p'}(\mathbb{R}^d)}) \end{aligned}$$

for all  $q' \geq 1$  such that

- $\frac{1}{p'} \geq \frac{1}{q'} \geq \frac{1}{p'} - \frac{m-|\alpha|}{d}$  if  $\frac{1}{p'} > \frac{m-|\alpha|}{d}$ ,
- $p' \leq q' \leq \infty$  if  $\frac{1}{p'} < \frac{m-|\alpha|}{d}$ ,
- $p' \leq q' < \infty$  if  $\frac{1}{p'} = \frac{m-|\alpha|}{d}$ .

First assume  $\underline{b}(\psi) = \bar{b}(\psi)$ ,

$$k_p(\phi, \psi) := \min\{k \in \mathbb{N}_0 \mid \frac{1}{k}(\underline{b}(\psi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\}\},$$

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For all  $j \in \{0, \dots, k\}$ ,  $\psi_j$  defined by

- $\psi_0 : r \mapsto r^{-\frac{d}{p}}$ ,
- $\psi_j : r \mapsto r^{-\frac{d}{p}}(\psi(r)r^{\frac{d}{p}})^{\frac{j+\varepsilon}{k}}$  for all  $1 \leq j < k$ ,
- $\psi_k = \psi$ .

$0 \leq \varepsilon < 1$  such that

- $0 < \frac{(1-\varepsilon)}{k}(\underline{b}(\psi) + \frac{d}{p}) \leq \frac{(1+\varepsilon)}{k}(\underline{b}(\psi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\}$ ,
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$$\bar{b}(\psi_{j+1}) - \underline{b}(\psi_j) < \min\{1, \underline{b}(\phi)\}$$

For all  $j \in \{0, \dots, k\}$

$$\frac{1}{p_j} := \frac{j}{q} + \frac{1}{p}.$$

if  $f \in W_m^{p_j}(\mathbb{R}^d)$

$$\|h\|_{T_{\psi_j}^{p_j}(x_0)} \leq \|\mathcal{H}g\|_{T_{\psi_j}^{p_j}(x_0)} + C_j(1 + MN)(\|h\|_{T_{\psi_{j-1}}^{p_{j-1}}(x_0)} + \|f\|_{W_m^{p_j}(\mathbb{R}^d)}).$$

If  $f \in W_m^{p'}(\mathbb{R}^d)$  for all  $p' \in [p_i, p_s]$  with

$$\frac{1}{p_1} = \frac{1}{p_s} + \frac{1}{q} \quad \text{and} \quad 0 < \frac{1}{p_i} = \frac{k_p(\phi, \psi)}{q} + \frac{1}{p_s} < 1$$

$$\begin{aligned} \|h\|_{T_{\psi}^{p_i}(x_0)} &\leq C_{p,p',\phi,\psi} (M(1 + MN)^{k-1} \|g\|_{T_{\psi}^{p_1}(x_0)} \\ &\quad + k(1 + MN)^k (\|f\|_{W_m^{p_i}(\mathbb{R}^d)} + \|f\|_{W_m^{p_s}})). \end{aligned}$$

If  $n < \underline{b}(\psi) < \bar{b}(\psi) < n + 1$ , two answers :

- If  $\bar{b}(\psi) \leq \underline{b}(\psi)$

$$k_1 = \min\left\{k \in \mathbb{N}_0 \mid \frac{1}{k}(\underline{b}(\psi) + \frac{d}{p}) < \min\{1, \underline{b}(\phi)\}\right\}$$

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and  $\psi_{k_2} = \psi$ ,  $\bar{b}(\psi_{j+1}) - \underline{b}(\psi_j) < \min\{1, \underline{b}(\phi)\}$ ,  $\bar{b}(\psi_j) < \underline{b}(\phi)$   
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$$\begin{aligned} \|h\|_{T_{\psi_0}^{p_0}(x_0)} &\leq C_{p, p_0, \phi, \psi} (M(1 + MN)^{k_1 - 1} \|g\|_{T_{\psi}^q(x_0)} \\ &\quad + (k_1)(1 + MN)^{k_1} (\|f\|_{W_m^{p'}(\mathbb{R}^d)} + \|f\|_{W_m^p})). \end{aligned}$$

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with  $\frac{1}{p_0} = \frac{k_1}{q} + \frac{1}{p}$ . We conclude by induction for  
 $0 < \frac{1}{p_i} = \frac{k_1 + k_2}{q} + \frac{1}{p} < 1$  and  $\psi$ .

- If  $\bar{b}(\psi) > \underline{b}(\phi)$  but  $\bar{b}(\psi) - \underline{b}(\psi) < \min\{1, \underline{b}(\psi)\}$

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$$\psi : r \mapsto r^\alpha$$

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