Generalized Regularity Spaces
Part I: A wavelet-based approach

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GDR Analyse Multifractale

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Road map

- The pointwise Hölder spaces
- Generalized Hölder spaces
- A wavelet characterization of the generalized Hölder spaces
- Irregularity spaces
- A Multifractal formalism for the generalized Besov spaces

Let $x_0 \in \mathbb{R}^d$; a function $f \in L^\infty(\mathbb{R}^d)$ belongs to the Hölder space $\Lambda^\alpha(x_0)$ ($\alpha > 0$) if there exist $C > 0$ and a polynomial P_{x_0} of degree less than α s.t., for j large enough,

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Alternatively, let

$$\Delta_h^1 f(x) = f(x+h) - f(x) \qquad \text{and} \qquad \Delta_h^{n+1} = \Delta_h^1 \Delta_h^n f(x),$$

and $B_h^M(x_0, 2^{-j}) = \{x : [x_0, x_0 + (M+1)h] \subset B(x_0, 2^{-j})\}$.

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We have $f \in \Lambda^\alpha(x_0)$ iff

$$\sup_{|h| \leq 2^{-j}} \|\Delta^{[\alpha]+1} f\|_{L^\infty(B_h^{[\alpha]}(x_0, 2^{-j}))} \leq C 2^{-j\alpha}.$$

The pointwise Hölder spaces allow to define a notion of regularity of f at x_0: the Hölder exponent of f at x_0 is defined as

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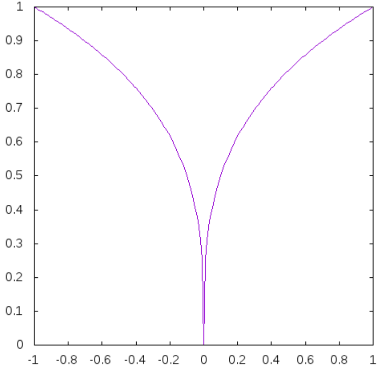
A good idea is to study the Hölder exponent using global quantities; the spectrum of f is defined as

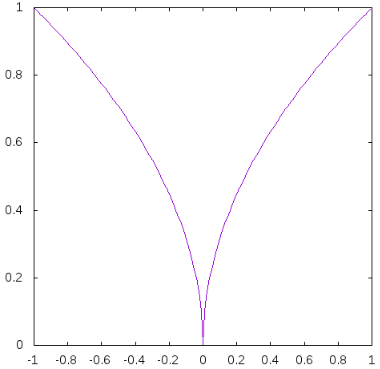
$$D^{(\infty)}(h) = \dim_{\mathcal{H}}\{x : h^{(\infty)}(x) = h\},$$

where $\dim_{\mathcal{H}}$ stands for the Hausdorff dimension.

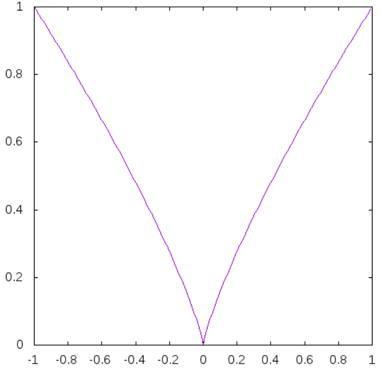
$$|x|^{1/3}$$



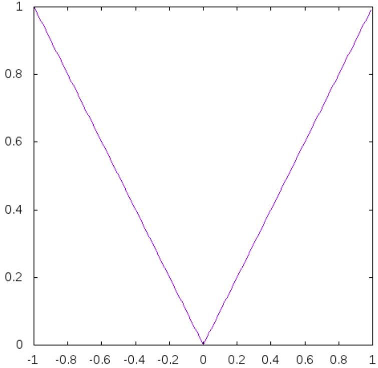
$$\sqrt{|x|}$$



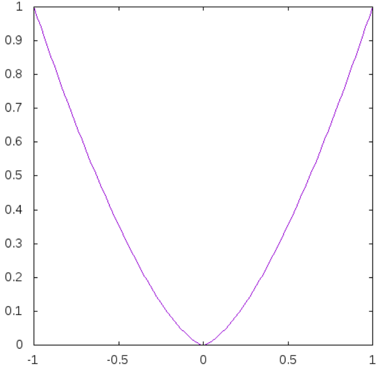
$|x|^{0.8}$



$|x|$



$|x|^{3/2}$



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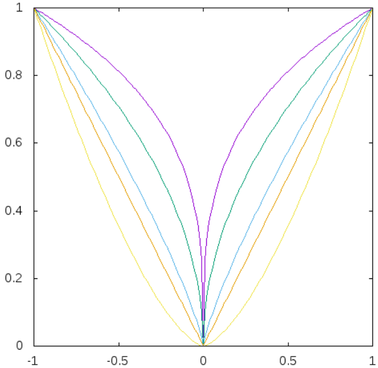

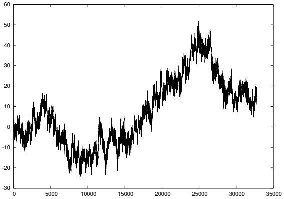

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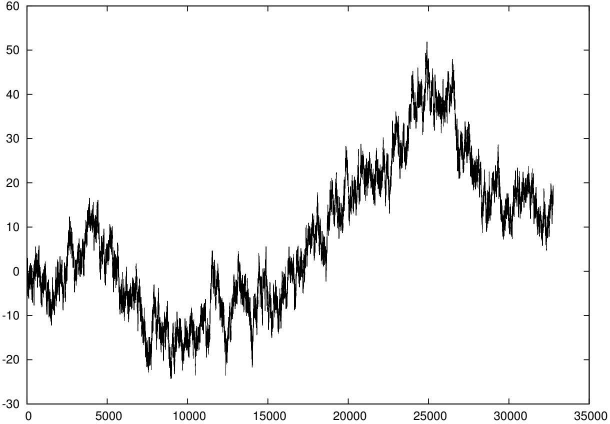

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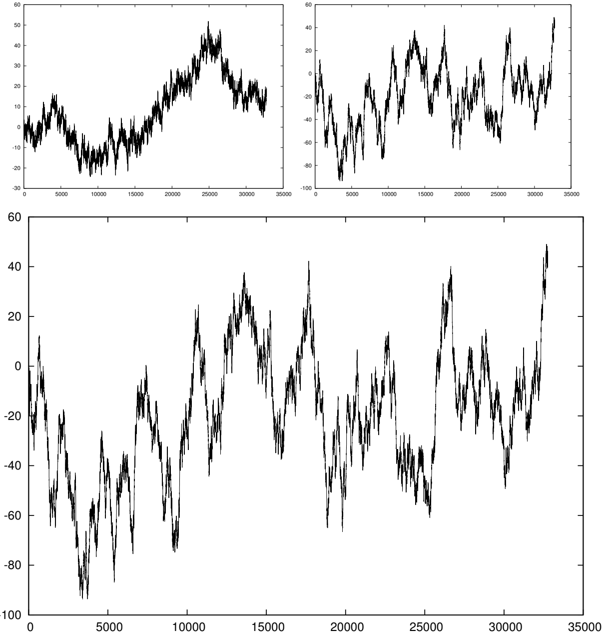
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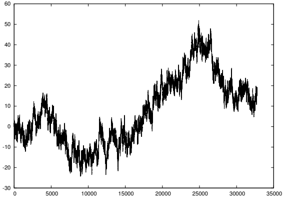
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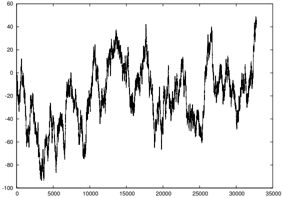
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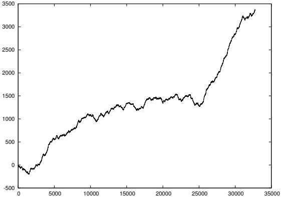
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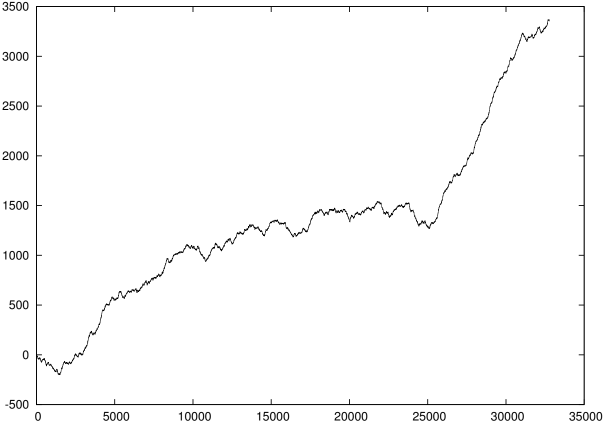
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Cantor's bijection: If $x \in (0,1)$ is an irrational number,
$$f(x) = f([0, x_1, x_2, x_3, x_4, \ldots]) = [0, x_1, x_3, \ldots].$$

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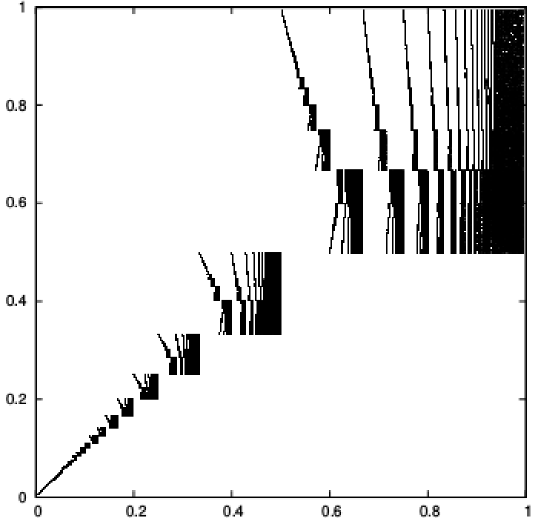
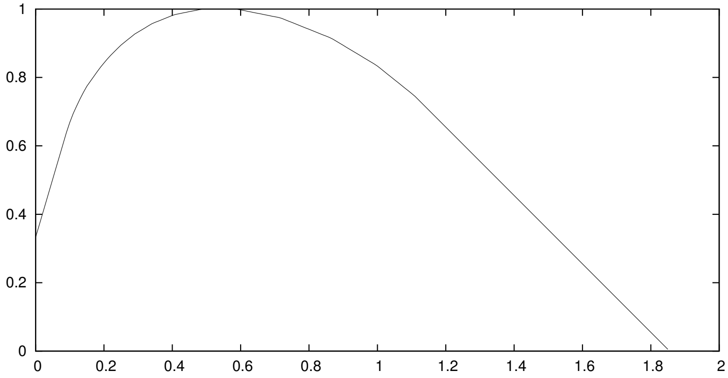


The spectrum $D^{(\infty)}$ associated to Cantor's bijection (numerical computation).



Let

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Theorem

For almost every irrational number $x \in (0,1)$, we have

$$h^{(\infty)}(x) \in \left[\frac{\log K_0}{2\log K_1}, \frac{\log K_1}{2\log K_0}\right].$$

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One sets

$$\underline{s}(\sigma) = \lim_j \frac{\log_2(\inf_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k})}{j}, \qquad \overline{s}(\sigma) = \lim_j \frac{\log_2(\sup_{k\in\mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k})}{j},$$

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so that for any $\epsilon > 0$, there exists $C > 0$ s.t.

$$C^{-1} 2^{j(\underline{s}(\sigma)-\epsilon)} \leq \frac{\sigma_{j+k}}{\sigma_k} \leq C 2^{j(\overline{s}(\sigma)+\epsilon)}.$$

Examples:

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  $\sigma = (2^{-j\alpha}\phi(2^j))_j$ is an admissible sequence, with $\underline{s}(\sigma) = \overline{s}(\sigma) = -\alpha$ (typically, $\phi = |\log|$).

Given an admissible sequence, let

$$B_h(x_0, 2^{-j}) = \{x : [x, x + ([\bar{s}(\sigma^{-1})] + 1)h] \subset B(x_0, 2^{-j})\}$$

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Let $p \geq 1$, $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and σ be an admissible sequence; $f \in \Lambda^\sigma_p(x_0)$ if there exists $C > 0$ s.t., for j sufficiently large,

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We have $\Lambda^\alpha(x_0) = \Lambda^\sigma_\infty(x_0)$, with $\sigma = (2^{-j\alpha})_j$.

Proposition

Let $p \geq 1$, $f \in L^p_{\text{loc}}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and σ be an admissible sequence s.t. $\underline{s}(\sigma^{-1}) \geq 0$; we have $f \in \Lambda^\sigma_p(x_0)$ iff there exists a sequence of Polynomials $(P_{j,x_0})_j$ of degree less or equal to $[\overline{s}(\sigma^{-1})]$ s.t., for j sufficiently large,

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We recover the usual characterization.

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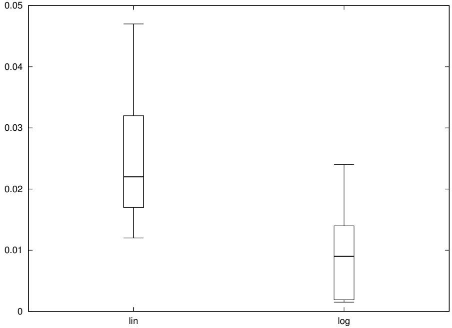
This space give rise to a method for detecting the logarithmic correction existing in stochastic processes, such as the Brownian motion.



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The "same result" holds here.

Under some general assumptions, there exist a real-valued function ϕ and $2^d - 1$ real-valued functions $(\psi^{(i)})_{1\leq i<2d}$ defined on $\mathbb{R}^d$, called wavelets, such that

$$\{\phi(\cdot - k) : k \in \mathbb{Z}^d\} \cup \{\psi^{(i)}(2^j \cdot -k) : 1 \leq i < 2^d,\ k \in \mathbb{Z}^d,\ j \in \mathbb{N}_0\}$$

form an orthogonal basis of $L^2(\mathbb{R}^d)$.

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Any function $f \in L^2(\mathbb{R}^d)$ can be decomposed as

$$f(x) = \sum_{k \in \mathbb{Z}^d} C_k \phi(x - k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_{i=1}^{2^d - 1} c_{j,k}^{(i)} \psi(2^j x - k).$$

We also need a notion of minimal regularity.

Local Besov spaces: $f \in B^s_{p,q}(x_0)$ if there exists $A > 0$ for which there exist $C > 0$ and $\epsilon \in \ell^q$ s.t., for any j,

$$\Big(\sum_{|l-2^j x_0| \leq A 2^j} (2^{(s-\frac{d}{p})j} |c_{j,l}^{(k)}|)^p \Big)^{1/p} \leq C \epsilon_j.$$

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$B^s_{p,q}(x_0)$ is a Xu space.

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A dyadic cube at scale j is a cube of the form

$$\lambda = \frac{k}{2^j} + \frac{i}{2^{j+1}} + [0, \frac{1}{2^{j+1}})^d \qquad (k \in \mathbb{Z}^d, 1 \leq i < 2^d).$$

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We will write c_λ instead of $c_{j,k}^{(i)}$.

Let us set

$$d_{\lambda,p} = \sup_{j' \geq j} \Big(\sum_{\substack{\lambda' \text{ at scale } j' \\ \lambda' \subset \lambda}} (2^{-\frac{d}{p}(j'-j)} |c_{\lambda'}|)^p \Big)^{1/p}.$$

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The p-wavelet leader of x_0 at scale j is defined by

$$d_{j,p}(x_0) = \sup_{\lambda \in 3\lambda_j(x_0)} d_{\lambda,p},$$

where $\lambda_j(x_0)$ is the cube at scale j containing x_0.

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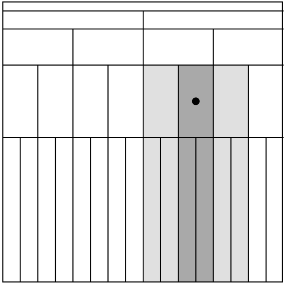

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The converse result cannot be true: in the usual case, a logarithmic correction appears!

Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ and σ be an admissible sequence s.t. $2^{-j\frac{d}{p}}\sigma_j \to 0$; $f \in \Lambda^{\sigma}_{p,\log}(x_0)$ if there exists $C > 0$ s.t., for j sufficiently large,

$$2^{j\frac{d}{p}} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{[\bar{s}(\sigma^{-1})]+1} f\|_{L^p(B_h(x_0, 2^{-j}))} \leq C\sigma_j |\log_2(2^{-j\frac{d}{p}}\sigma_j)|.$$

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Theorem

Let $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ and σ be an admissible sequence s.t. $2^{-j\frac{d}{p}}\sigma_j \to 0$; if there exists $\epsilon > 0$ s.t. $f \in B^{\epsilon}_{p,\infty}(x_0)$ and if there exists $C > 0$ s.t., for j sufficiently large,

$$d_{j,p}(x_0) \leq C\sigma_j,$$

then $f \in \Lambda^{\sigma}_{p,\log}(x_0)$.

If one replaces $f \in B^{\epsilon}_{p,\infty}(x_0)$ with $f \in B^{\epsilon}_{p,\infty}(\mathbb{R}^d)$, these results hold with other wavelets.

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To formalize it, we need the notion of prevalence.

In infinite dimensional Banach spaces, there is no σ-finite translation invariant measure.

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Let E be a complete metric vector space; a Borel set $B \subset E$ is Haar-null if there exists a Borel probability measure μ, strictly positive on some compact set $K \subset E$ such that $\mu(B + x) = 0$ for any $x \in E$.

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A subset of E is Haar-null if it is included in a Haar-null Borel subset of E; the complement of a Haar-null set is called a prevalent set.

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A subset of E is Haar-null if it is included in a Haar-null Borel subset of E; the complement of a Haar-null set is called a prevalent set.

In this setting, a properties that holds on a prevalent set is satisfied almost everywhere.

Theorem

From the prevalence point of view, a.e. function satisfying the previous hyptohesis with $p = \infty$, i.e.

$$d_{j,\infty}(x_0) \leq C\sigma_j,$$

does not belong to $\Lambda^{\sigma}_{\infty}(x_0)$.

Road map

- The pointwise Hölder spaces
- Generalized Hölder spaces
- A wavelet characterization of the generalized Hölder spaces
- Irregularity spaces
- A Multifractal formalism for the generalized Besov spaces

The irregularity spaces allow to define the notion of strongly mono-Hölder function.

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They allow a better understanding of the Hölder spaces.

We say that $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $I^\sigma_p(x_0)$ if there exist $C > 0$ s.t., for j sufficiently large,

$$2^{j\frac{d}{p}} \sup_{|h| \leq 2^{-j}} \|\Delta_h^{[\overline{s}(\sigma^{-1})]+1} f\|_{L^p(B_h(x_0, 2^{-j}))} \geq C\sigma_j.$$

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Proposition

Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and σ be an admissible sequence; if there exists $C > 0$ s.t., for j sufficiently large,

$$d_{j,p}(x_0) \geq C\sigma_j,$$

then $f \in I^\sigma_p(x_0)$.

The function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ belongs to $\Upsilon^\sigma_p(x_0)$ if it both belongs to $\Lambda^\sigma_p(x_0)$ and $I^\sigma_p(x_0)$.

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Theorem

Let $f \in L^p_{\text{loc}}(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and σ be an admissible sequence s.t. $-d/p < \underline{s}(\sigma^{-1})$; let us suppose that there exists $\epsilon_0 > 0$ s.t. $f \in B^{\epsilon_0}_{p,\infty}(x_0)$. If $f \in \Upsilon^\sigma_p(x_0)$, then there exist $C, C' > 0$ s.t., for j sufficiently large

$$C \frac{\sigma_j}{|\log_2(2^{-j\frac{d}{p}}\sigma_j)|} \leq d_{j,p}(x_0) \leq C'\sigma_j.$$

Let $\alpha > 0$,
a function f belongs to $\Lambda^\alpha(\mathbb{R}^d)$ if $f \in \Lambda^\alpha(x)$ for every x with a uniform constant,
a function f belongs to $I^\alpha_\infty(\mathbb{R}^d)$ if $f \in I^\alpha_\infty(x)$ for every x with a uniform constant.

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A function f belongs to $\Upsilon^\sigma_\infty(\mathbb{R}^d)$ if $f \in \Lambda^\alpha(\mathbb{R}^d) \cap I^\sigma_\infty(\mathbb{R}^d)$.

Theorem

For any $\alpha > 0$, $\Upsilon^\alpha_\infty(\mathbb{R}^d)$ is a prevalent subset of $\Lambda^\alpha(\mathbb{R}^d)$.

Road map

- The pointwise Hölder spaces
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Let $p \geq 1$; a function $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ belongs to $T^p_\alpha(x_0)$ ($\alpha \geq -d/p$) if there exist $C > 0$ and a polynomial P_{x_0} of degree less than α s.t., for h sufficiently small,

$$\frac{1}{h^d}\Big(\int_{B(x_0,h)} |f - P_{x_0}|\, dx\Big)^{1/p} \leq C h^\alpha.$$

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The p-exponent of f at x_0 is defined as

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The p-exponent of f at x_0 is defined as

$$h^{(p)}(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}.$$

The multifractal of f p-spectrum is defined as

$$D^{(p)}(h) = \dim_{\mathcal{H}}\{x : h^{(p)}(x) = h\}.$$

If $p = \infty$ (with the usual modifications in the definition), we recover the "classical notion" of regularity based on the Hölder spaces :

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We aim at finding conditions (functional spaces) under which

$$D^{(p)}(h) \approx d + q(h - \underline{s}(\sigma^{-1})).$$

We can generalize the oscillation spaces:

Let $p \geq 1$, $q > 0$ and σ be an admissible sequence; f belongs to $\mathcal{O}^{\sigma^{-1}}_{p,q}(\mathbb{R}^d)$ if $(C_j) \in \ell^q(\mathbb{Z}^d)$ and there exists $C > 0$ s.t., for any j,

$$\Big(\sum_{\lambda \text{ at scale } j} (\sigma_j 2^{-\frac{d}{p}j} d^q_{\lambda,p} \Big)^{1/q} \leq C.$$

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$$\Big(\sum_{\lambda \text{ at scale } j} (\sigma_j 2^{-\frac{d}{p}j} d^q_{\lambda,p} \Big)^{1/q} \leq C.$$

Proposition

Let $p \geq 1$, $q > 0$ and σ be an admissible sequence and $f \in B^{\epsilon}_{p,q}(x_0)$ for some $\epsilon > 0$; if $f \in \mathcal{O}^{\sigma^{-1}}_{p,q}(\mathbb{R}^d)$ then

$$\dim_{\mathcal{H}}\{x : h^{(p)}(x) \leq h\} \leq d + q(h - \underline{s}(\sigma^{-1})).$$

Indeed, it won't be necessary...

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Proposition

Let $p \geq 1$, $q > 0$ and σ be an admissible sequence s.t. $\underline{s}(\sigma^{-1}) > 0$ and $\underline{s}(\sigma^{-1}) > d(\frac{1}{q} - \frac{1}{p})$; we have $\mathcal{O}^{\sigma^{-1}}_{p,q}(\mathbb{R}^d) = B^{\sigma^{-1}}_{q,\infty}(\mathbb{R}^d)$.

Let us define these generalized Besov spaces!

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We have

$$B^s_{p,q} = [H^t_p, H^u_p]_{\alpha,q},$$

with $s = (1-\alpha)t + \alpha u$, where H^t_p and H^u_p are Sobolev spaces.

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One can define generalized interpolation spaces by replacing the dyadic sequence with an admissible sequence.

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One can define generalized interpolation spaces by replacing the dyadic sequence with an admissible sequence.

Theorem

Let $r, s \in \mathbb{R}$ and σ be a generalized sequence $\inf_k \frac{\sigma_{1+k}}{\sigma_k} > 1$ and

$$r < \underline{s}(\sigma) < \overline{s}(\sigma) < s;$$

we have

$$B^\sigma_{p,q} = [H^r_p, H^s_p]_{\sigma,q}.$$

It also works with the "classical" Besov spaces W_p^t.

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Theorem

Let $p, q \in [1, \infty]$, σ be an admissible sequence such that $\inf_k \frac{\sigma_{1+k}}{\sigma_k} > 1$ and $k, n \in \mathbb{N}_0$ be such that

$$k < \underline{s}(\sigma) \leq \overline{s}(\sigma) < n;$$

We have

$$B^{\sigma}_{p,q} = \{f \in W^k_p : (2^{-j|\alpha|}\sigma_j \sup_{|h|\leq 2^{-j}} \|\Delta_h^{n-|\alpha|} D^{\alpha} f\|_{L^p})_j \in \ell^q \, \forall |\alpha| = k\}.$$

It also works with the "classical" Besov spaces W_p^t.

Theorem

Let $p, q \in [1, \infty]$, σ be an admissible sequence such that $\inf_k \frac{\sigma_{1+k}}{\sigma_k} > 1$ and $k, n \in \mathbb{N}_0$ be such that

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We have

$$B^{\sigma}_{p,q} = \{f \in W^k_p : (2^{-j|\alpha|}\sigma_j \sup_{|h|\leq 2^{-j}} \|\Delta_h^{n-|\alpha|} D^\alpha f\|_{L^p})_j \in \ell^q \, \forall |\alpha| = k\}.$$

These spaces are still considered nowadays in connection with embeddings, limiting embedding, entropy numbers, probability theory and theory of stochastic processes.

Proposition

Let $p \geq 1$, $q, r > 0$ and σ be an admissible sequence s.t. $\underline{s}(\sigma^{-1}) > 0$ and $\underline{s}(\sigma^{-1}) > d(\frac{1}{q} - \frac{1}{p})$; if $f \in B^{\sigma^{-1}}_{q,r}(\mathbb{R}^d)$ then

- for any $h < \underline{s}(\sigma^{-1}) - d/p$,
  $\{x : h^{(p)}(x) = h\} = \emptyset$,
- for any $h \geq \underline{s}(\sigma^{-1}) - d/p$,
  $\{x : h^{(p)}(x) \leq h\} \leq d + q(h - \underline{s}(\sigma^{-1}))$.

Theorem

Let $p \geq 1$, $q, r > 0$ and σ be an admissible sequence s.t. $\overline{s}(\sigma^{-1}) = \underline{s}(\sigma^{-1}) > 0$ and $\underline{s}(\sigma^{-1}) > d(\frac{1}{q} - \frac{1}{p})$; from the prevalence point of view, for a.e. function $f \in B^{\sigma^{-1}}_{q,r}$, the p-spectrum is defined on $[\underline{s}(\sigma^{-1}) - d/p, \underline{s}(\sigma^{-1})]$ and for any h belonging to this interval, we have

$$\dim_{\mathcal{H}}\{x : h^{(p)}(x) = h\} = d + q(h - \underline{s}(\sigma^{-1})).$$

Moreover, for a.e. x, $h^{(p)}(x) = \underline{s}(\sigma^{-1})$.

For Further Reading

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