On some generalisations of the Fréchet functional equations

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Abstract

In this paper we study the Fréchet functional equation in the n-dimensional Euclidian space as well as in the context of distributions. We also generalise the Cauchy functional equation for distributions to any natural order.

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1 Introduction

An additive function is a function $f : \mathbf{R} \to \mathbf{R}$ satisfying the equation

$$f(x + y) = f(x) + f(y), \tag{1}$$

for any x and any y belonging to the real line. This equation has been studied by several mathematicians during the 17th century and Cauchy provided the first significant result, stating that an additive function that is continuous is necessarily linear [5]; equation (1) is usually referred as the Cauchy functional equation. It can be shown that an additive function that is not linear is necessarily discontinuous everywhere [7] and Hamel provided such a solution using Zorn's lemma to get a basis (of $\mathbf{R}$ as a $\mathbf{Q}$-vector space) which now bears his name [10]. One of the charms of this equation lies in the form of the solutions: they are either extremely regular or "pathological"; see [13] for a deeper study of this problem.

The Cauchy functional equation can be generalised as follows: given a function $f : \mathbf{R} \to \mathbf{R}$, let us define the (forward) difference operator of order one Δ^1 as

$$\Delta^1_h f(x) = f(x + h) - f(x), \tag{2}$$

for any x and any h belonging to $\mathbf{R}$. Equation (1) becomes

$$\Delta^1_y f(x) = f(y). \tag{3}$$

Then, by defining the difference operator of order $m > 1$ by the recurrence formula

$$\Delta^m_h f(x) = \Delta^1_h \Delta^{m-1}_h f(x),$$

the Cauchy functional equation leads to $\Delta_h^2 f(x) = 0$. This equation is not equivalent to (1), since any polynomial of degree at most 1 is a solution. The interesting fact is that the continuous solutions of this last equation are such polynomials. Indeed, Fréchet proved that the continuous solutions of the equation

$$\Delta_h^m f(x) = 0 \tag{4}$$

are the polynomials of degree at most $m-1$ [8]. As for Cauchy, equation (4) is today named after Fréchet. In general, when studying such an equation, one tries to obtain the weakest hypothesis under which the solutions are not "pathological". Most often, the result associated to Fréchet functional equation reads as follows: "the solutions of Fréchet functional equation (4) that are locally integrable are the polynomials of order at most $m-1$", although other conditions exist [17, 18, 15, 3].

A generalisation of (3) is given by the following equation:

$$\Delta_h^m f(x) = m! f(h). \tag{5}$$

Even though less studied, it is well known that the solutions of (5) which are locally integrable are the homogeneous polynomials of degree m [16, 2].

In this paper, we study equations (4) and (5) from various points of view. First we give very weak assumptions under which the solutions of the local Fréchet functional equation are locally polynomials: we consider functions that are bounded almost everywhere which satisfies the equation for almost every h small enough (with respect to the Lebesgue measure). This requirement is somehow similar to the one given in [3], albeit weaker, but we use here very different techniques. Next, we generalise this result by considering the equation on the Euclidian space $\mathbf{R}^n$, where x and h both belong to this space in the definition of the difference operator (2). We also characterise the solutions to the functional equation (5) in the same framework. Next, we consider the Fréchet functional equation in the setting of the distributions. Finally, we also study equation (5) for distributions; more precisely, we generalise the Cauchy functional equation for distributions given in [14] to any order m in the natural setting of the pullbacks.

In the sequel, we will assume that the reader is well versed in the theories of finite differences (see e.g. [4, 12]) and distributions (see e.g. [11, 9]). For example, we will regularly use the formula

$$\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(x + (m-j)h). \tag{6}$$

We set $\mathbf{E} = \mathbf{R}^n$ and $B(x,r)$ will stand for the open ball centered at x with radius $r > 0$. We denote by $\mathcal{D}(X)$ the space of infinitely differentiable functions on X with compact support and by $\mathcal{D}'(X)$ the topological dual of $\mathcal{D}(X)$. The space of locally integrable functions on X will be designated by $L^1_{\mathrm{loc}}(X)$ and D is the differentiation operator, whereas D_j ($j \in \{1, \dots, n\}$) is the partial differentiation operator following the direction of the j-th component e_j of the canonical basis of $\mathbf{E}$.

2 The Fréchet functional equation on a real interval

Let us first sail in *terra cognita* by considering the usual case of the local version of the Fréchet functional equation on the real line.

Lemma 1. *If $f : \mathbf{R} \to \mathbf{R}$ is a function that is bounded almost everywhere on (a, b) and which satisfies $\Delta_h^m f = 0$ on (a, b) for almost every h small enough, then f is bounded on (a, b).*

Proof. Remark first that we have, using (6),

$$|f(x)| = |\sum_{j=0}^{m-1}(-1)^j \binom{m}{j} f(x + mh - jh)| \leq 2^m \sup_{y \in \{x+jh : j \in \{1,\dots,m\}\}} |f(y)|,$$

for any $x \in (a, b)$ and say almost every $|h| < \varepsilon$. Let $C > 0$ and $N \subset (a, b)$ be a negligible set such that $|f(x)| \leq C$ for any x belonging to $(a, b) \setminus N$. Given $j \in \{1, \dots, m\}$ and $x \in (a, b)$, let us define

$$A_j = \{h \in (\frac{a-x}{m}, \frac{b-x}{m}) : x + jh \in (a, b) \setminus N\}.$$

For $j \in \{2, \dots, m\}$, we have $A_1 \subset (A_1 \setminus A_j) \cup A_j$, where $A_1 \setminus A_j$ is negligible as a subset of $(N - x)/j$. As a consequence, A_1 is equal to $\cap_{j=1}^m A_j$ almost everywhere. Let us also remark that $A_1 \cap (-\varepsilon, \varepsilon)$ is equal to an interval almost everywhere. Therefore, by choosing h such that $x + jh \in (a, b) \setminus N$ for every $j \in \{1, \dots, m\}$, we get

$$|f(x)| \leq 2^m C,$$

for any $x \in (a, b)$, as expected. □

Theorem 1. *Let $m \in \mathbf{N}$ and $f : \mathbf{R} \to \mathbf{R}$ be a functions that is bounded almost everywhere on a neighborhood of a point $x_0 \in \mathbf{R}$. If equation $\Delta_h^m f = 0$ is satisfied on a neighborhood of x_0 for almost every h sufficiently small, then f can be written as a polynomial of degree at most $m - 1$ on a neighborhood of x_0.*

Proof. It is sufficient to prove the result for $m > 1$. Thanks to Lemma 1, we can suppose that there exists $\eta > 0$ for which f is bounded and satisfies $\Delta_h^m f = 0$ on $(x_0 - \eta, x_0 + \eta)$ for almost every $|h| < \eta$. For the sake of clarity, we will drop the locution "almost every" in the remaining of the proof. Let ε be such that $0 < \varepsilon < \eta/m$ and set $\delta = \eta - m\varepsilon$. We can suppose that we have $|\Delta_h^j f(x)| \leq C$ for any $j \in \{0, \dots, m\}$, any $x \in (x_0 - \delta, x_0 + \delta)$ and $|h| < \varepsilon$. If $x \in (x_0 - \delta, x_0 + \delta)$, let us prove that f is continuous at x. Given $r \in \mathbf{N}$, let h be such that $|rh| < \varepsilon$; Newton's interpolation formula gives

$$f(x + qh) = \sum_{j=0}^{q} \frac{\Delta_h^j f(x)}{j!} (q)_j,$$

for any $q \in \mathbf{N}_0$, where $(q)_j$ denotes the falling factorial,

$$(q)_j = \begin{cases} 1 & \text{if } j = 0 \\ \prod_{k=0}^{j-1}(q-k) & \text{if } j > 0 \end{cases}.$$

We also have, for $q \geq m$,

$$f(x+qrh) = \sum_{j=0}^{q} \frac{\Delta_{rh}^j f(x)}{j!}\,(q)_j,$$

which implies

$$\sum_{j=0}^{qr} \frac{\Delta_h^j f(x)}{j!}\,(qr)_j = \sum_{j=0}^{q} \frac{\Delta_{rh}^j f(x)}{j!}\,(q)_j.$$

If we denote by $s(j,k)$ the Stirling numbers of the first kind, we can rewrite the last equality as follows:

$$\sum_{j=0}^{qr} \frac{\Delta_h^j f(x)}{j!} \sum_{k=0}^{j} s(j,k)(qr)^k = \sum_{j=0}^{q} \frac{\Delta_{rh}^j f(x)}{j!} \sum_{k=0}^{j} s(j,k)q^k.$$

This implies

$$\sum_{j=k}^{qr} \frac{\Delta_h^j f(x)}{j!}\,s(j,k) = \sum_{j=k}^{q} \frac{\Delta_{rh}^j f(x)}{j!}\,s(j,k)r^{-k},$$

for $k \in \{0,\ldots,m-1\}$. Now, if $S(j,k)$ denotes the Stirling numbers of the second kind, the equality

$$\sum_{k=l}^{m-1}\sum_{j=k}^{qr} \frac{\Delta_h^j f(x)}{j!}\,s(j,k)S(k,l) = \sum_{k=l}^{m-1}\sum_{j=k}^{q} \frac{\Delta_{rh}^j f(x)}{j!}\,s(j,k)r^{-k}S(k,l),$$

allows to write (since $\sum_{k=l}^{j} s(j,k)S(k,l) = \delta_{jl}$)

$$\frac{\Delta_h^l f(x)}{l!} = \sum_{k=l}^{m-1}\sum_{j=k}^{q} \frac{\Delta_{rh}^j f(x)}{j!}\,s(j,k)r^{-k}S(k,l),$$

for $l \in \{0,\ldots,m-1\}$. Consequently, we have

$$\left|\frac{\Delta_h^l f(x)}{l!}\right| \leq C' \sum_{k=l}^{m-1} r^{-k},$$

for some constant C'. Now, given $0 < |h| < \varepsilon$, let r be the natural number such that $r|h| < \varepsilon \leq (r+1)|h|$. The last inequality gives

$$\left|\frac{\Delta_h^l f(x)}{l!}\right| \leq C' \sum_{k=l}^{m-1} \frac{|h|^k}{(\varepsilon - |h|)^k},$$

for any $l \in \{0, \dots, m-1\}$. In particular, we have $\lim_{h \to 0} \Delta_h^1 f(x) = 0$, so that f is continuous on $(x_0 - \delta, x_0 + \delta)$.

Given $0 < |h| < \delta/m$, there exists a unique polynomial P_h of degree at most $m-1$ such that $P_h(x_0 + jh) = f(x_0 + jh)$, for $j \in \{0, \dots, m\}$. For $r \in \mathbf{N}$, the same arguments as in the beginning of the proof lead to the existence of a polynomial $P_{h/r}$ such that $P_{h/r}(x_0 + jh/r) = f(x_0 + jh/r)$, for $j \in \{0, \dots, mr\}$. Since P_h and $P_{h/r}$ are equal on m different points, they are equal on $\mathbf{R}$. So P_h is equal to f on $\{x_0 + qh : q \in \mathbf{Q},\ 0 \leq q \leq m\}$, which allow to conclude, using the continuity of f. □

Remark 1. If one assumes the measurability of f, the proof of theorem 1 becomes much simpler (one can proceed as in the proof of theorem 2).

3 The Fréchet functional equation on the Euclidian space $\mathbf{R}^n$

The previous result can be generalised to $\mathbf{E}$.

Lemma 2. *The polynomials*

$$P(x) = \sum_{|\alpha|=m} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

are solutions to the equation $\Delta_h^m f(x) = m! f(h)$.

Proof. By linearity, it is sufficient to prove that, given a multi-index $\alpha \in \mathbf{N}^n$ such that $|\alpha| = m$, the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a solution to the considered equation. This claim is easy to check for $m = 1$ and $m = 2$. If the result has been obtained for $m = k$, let us show that it is still valid for $k+1$. Let α be a multi-index such that $|\alpha| = k+1$; we can suppose that $\alpha_n \geq 1$ in order to write

$$\begin{aligned}
&\Delta_h^k(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n - 1} x_n)(x) \\
&= \sum_{j=0}^{k} \binom{k}{j} \Delta_h^j (x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n - 1})(x)\ (\Delta_h^{k-j} x_n)(x + jh) \\
&= \binom{k}{k-1} \Delta_h^{k-1}(x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n - 1})(x) h_n + k! h_1^{\alpha_1} \cdots h_{n-1}^{\alpha_{n-1}} h_n^{\alpha_n - 1}(x_n + k h_n),
\end{aligned}$$

which allows to write

$$\begin{aligned}
\Delta_h^{k+1}(x_1^{\alpha_1} \cdots x_n^{\alpha_n})(x) &= \binom{k}{k-1} \Delta_h^k (x_1^{\alpha_1} \cdots x_n^{\alpha_n - 1})(x) h_n + k! h_1^{\alpha_1} \cdots h_n^{\alpha_n - 1} h_n \\
&= (k+1)! h_1^{\alpha_1} \cdots h_n^{\alpha_n},
\end{aligned}$$

as desired. □

As a consequence, the polynomials of degree at most $m-1$ satisfy the Fréchet functional equation.

We can now characterise the solutions to the local Fréchet functional equation for the functions on $\mathbf{E}$ that are bounded almost everywhere on a neighborhood.

Theorem 2. *Let $f : \mathbf{E} \to \mathbf{R}$ be a function that is bounded almost everywhere on a neighborhood of a point $x_0 \in \mathbf{E}$ and such that $\Delta_h^m f = 0$ on a neighborhood of x_0 for almost every h sufficiently small. The function f is equal to a polynomial of degree at most $m-1$ on a neighborhood of x_0.*

Proof. Let us first remark that f is measurable on a neighborhood of x_0. This result from theorem 1: for x chosen in a neighborhood of x_0, the function

$$f_j : t \mapsto f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_n)$$

is continuous on a neighborhood of the j-th component of x_0; this being said, we can conclude using a result from Carathéodory (see e.g. [6, 1]).

Let $\varepsilon > 0$ be such that f is measurable and bounded on $B(x_0, \varepsilon)$ with $\Delta_h^m f(x) = 0$ for every $x \in B(x_0, \varepsilon)$ and almost every $|h| < \varepsilon$. It is well known that there exists a function $\Phi \in \mathcal{D}(\mathbf{E})$ such that $\int \Phi(x)\, dx = 1$ and $f = g * \Phi$ on $B(x_0, \varepsilon/2)$, with

$$g(x) = \begin{cases} f(x) & \text{if } x \in B(x_0, \varepsilon) \\ 0 & \text{otherwise} \end{cases} .$$

As a consequence, f is infinitely continuously differentiable on a neighborhood of x_0 (see Remark 2 for an explicit construction of Φ).

We thus can write

$$D_j^m f(x) = \lim_{h \to 0} \frac{\Delta_{he_j}^m f(x)}{h^k} = 0,$$

for any $x \in B(x_0, \varepsilon/2)$ and $j \in \{1, \ldots, n\}$, which implies that f is a polynomial of degree at most $n(m-1)$ on a neighborhood of x_0.

Let us suppose that f is a polynomial of degree $p \geq m$ on $B(x_0, \varepsilon/2)$ and define the polynomial P such that

$$\begin{aligned} f(x) &= \sum_{|\alpha| \leq p} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{|\alpha| = p} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} + \sum_{|\alpha| < p} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\ &= P(x) + \sum_{|\alpha| < p} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \end{aligned}$$

on $B(x_0, \varepsilon/2)$. We have

$$0 = \Delta_h^p f(x_0) = \Delta_h^p P(x_0) = p! P(h),$$

for h sufficiently small, so that $P = 0$. □

Let us give the classical construction of the function Φ, appearing in the proof of Theorem 2.

Remark 2. Let us give the explicit definition of the function Φ appearing in the proof of Theorem 2. Let r be an odd number multiplied by two strictly greater than m and $\rho \in \mathcal{D}(\mathbf{E})$ be a radial function such that $\operatorname{supp}(\rho) \subset B(0, \varepsilon/2)$, $0 \leq \rho \leq 1$ and $\int \rho(x)\, dx = 1$. We can consider the function $\tilde{\Phi}$ on $\mathbf{E}$ obtained as follows:

$$\tilde{\Phi}(x) = \sum_{j=0}^{r/2-1} (-1)^j \binom{r}{j} \frac{1}{(2j-r)^n} \rho(\frac{x}{2j-r}) \tag{7}$$

and set $C_r = \int \tilde{\Phi}(x)\,dx$ (one can check that $C_r = \binom{r}{r/2}/2$) in order to define the function Φ on $\mathbf{E}$ as $\Phi(x) = \tilde{\Phi}(x)/C_r$. For $x \in B(x_0, \varepsilon/2)$, we have

$$\begin{aligned}
&g * \Phi(x) - f(x) \\
&= \frac{1}{C_r} \sum_{j=0}^{r/2-1} (-1)^j \binom{r}{j} \frac{1}{(2j-r)^n} \int f(x-y)\,\rho(\frac{y}{2j-r})\,dy - f(x) \\
&= \frac{1}{2C_r} \Big(\sum_{\substack{j=0 \\ j \neq r/2}}^{r} (-1)^j \binom{r}{j} \int f(x-(2j-r)y)\,\rho(y)\,dy - 2C_r f(x) \Big) \\
&= \frac{1}{2C_r} \int \Delta_y^r f(x) \rho(y)\,dy = 0,
\end{aligned}$$

as claimed.

We directly get the following result.

Corollary 1. *Let $m \in \mathbf{N}$ and $f : \mathbf{E} \to \mathbf{R}$ be a function which is bounded almost everywhere; if f satisfies $\Delta_h^m f = m! f(h)$ on $\mathbf{E}$ for any h, then it can be written as*

$$f(x) = \sum_{|\alpha|=m} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

on $\mathbf{E}$.

4 The Fréchet functional equation for distributions

Given $T \in \mathcal{D}'(\mathbf{E})$, the (forward) difference Δ^m of order $m \in \mathbf{N}$ of T is naturally defined as follows:

$$\Delta_h^m T(\varphi) = T(\Delta_{-h}^m \varphi),$$

for $h \in \mathbf{E}$ and $\varphi \in \mathcal{D}(\mathbf{E})$. If $T \in \mathcal{D}'(\mathbf{R})$, we naturally have

$$\lim_{h \to 0} \frac{\Delta_h^m T}{h^m} = D^m T,$$

so that if T satisfies $\Delta_h^m T = 0$ for any h sufficiently small, the m-th derivative of T is equal to zero, which implies that T is a distribution associated to a polynomial of order at most $m-1$. We aim now at showing that this result still holds if T belongs to $\mathcal{D}'(\mathbf{E})$.

Theorem 3. *Given $m \in \mathbf{N}$, the solutions to the equation $\Delta_h^m T = 0$ for almost every $h \in \mathbf{E}$ with $T \in \mathcal{D}'(\mathbf{E})$ are the distributions associated to a polynomial of degree at most $m-1$.*

Proof. Let $T \in \mathcal{D}'(\mathbf{E})$ be a solution to the Fréchet functional equation. Again, let $r \in \mathbf{N}$ be a natural number greater than m that is equal to an odd number multiplied by two and define Φ as in the proof of theorem 2, using the constant

C_r and the function $\tilde{\Phi}$ satisfying (7). With the same arguments, one can assert the existence of a constant C such that

$$\varphi * \Phi(x) - \varphi(x) = C \int \Delta_y^r \varphi(x)\, \rho(y)\, dy,$$

for any $\varphi \in \mathcal{D}(\mathbf{E})$ and any $x \in \mathbf{E}$. We get

$$\begin{aligned} T(\varphi * \Phi - \varphi) &= C T_x\Big(\int \Delta_y^r \varphi(x)\, \rho(y)\, dy\Big) = C \int T\big(\Delta_y^r \varphi\big)\, \rho(y)\, dy \\ &= C \int \Delta_y^r T\big(\varphi\big)\, \rho(y)\, dy = 0, \end{aligned}$$

so that T is associated to a infinitely continuously differentiable function f. Now, since the distribution associated to $\Delta_h^m f$ is vanishing, we have obtained that $\Delta_h^m f(x) = 0$ for any $x \in \mathbf{E}$ and any $h \in \mathbf{E}$, which is sufficient to conclude. □

As a consequence, we recover the usual solution to the Fréchet functional equation.

Corollary 2. *Given $m \in \mathbf{N}$, if $f \in L^1_{\text{loc}}(\mathbf{E})$ satisfies $\Delta_h^m f = 0$ for almost every h, then f is a polynomial of degree at most $m - 1$.*

5 A generalisation of the Cauchy functional equation for distributions

In this section we generalise ideas from [14] and show that a natural formulation of the results can be achieved through pullbacks. Let us recall this notion. Given two open sets $U \subset \mathbf{R}^m$, $V \subset \mathbf{R}^n$ and a C^∞ map $f : U \to V$ whose differential g is such that $g(x)$ is surjective for every $x \in U$, there exists a unique continuous linear map $f^* : \mathcal{D}'(V) \to \mathcal{D}'(U)$ such that $f^*T = T \circ f$ if $T \in C^0(V)$; f^*T is called the pullback of T by f (for more details, see e.g. [11]).

Given $j \in \mathbf{N}_0$, let us define

$$p_j : \mathbf{E}^2 \to \mathbf{E} \quad (x, y) \mapsto x + jy \qquad \text{and} \qquad q_j : \mathbf{E}^2 \to \mathbf{E} \quad (x, y) \mapsto jx + y.$$

Obviously, p_0 and q_0 are orthogonal projections and

$$\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} p_j^* f(x, h),$$

for any $f \in C^0(\mathbf{E})$. One can thus define the unique continuous linear map

$$\Delta^m : \mathcal{D}'(\mathbf{E}) \to \mathcal{D}'(\mathbf{E}^2) \quad T \mapsto \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} p_j^* T.$$

In [14], the Cauchy functional equation $\Delta^1 T = q_0^* T$ (with $T \in \mathcal{D}'(\mathbf{E})$) is studied, although this equation not considered as an equation on $\mathbf{E}^2$. Here, we will consider the functional equation

$$\Delta^m T = m! q_0^* T. \tag{8}$$

This equation generalises the previous one in the same way that equation (5) generalises the Cauchy functional equation.

Likewise, we could have defined the operator

$$\Lambda^m : \mathcal{D}'(\mathbf{E}) \to \mathcal{D}'(\mathbf{E}^2) \quad T \mapsto \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} q_j^* T,$$

in order to consider the equation $\Lambda^m T = m! p_0^* T$. A simple calculation show that this equation is equivalent to (8).

Lemma 3. *A distribution $T \in \mathcal{D}'(\mathbf{E})$ satisfies $\Delta^m T = m! q_0^* T$ if and only if it also satisfies $\Lambda^m T = m! p_0^* T$.*

Proof. Let $T \in \mathcal{D}'(\mathbf{E})$ be a solution to (8) and consider the diffeomorphism

$$\pi : \mathbf{E}^2 \to \mathbf{E}^2 \quad (x, y) \mapsto (y, x).$$

One easily checks that the following relation holds:

$$q_j^* f(x, y) = (p_j \circ \pi)^* f(x, y),$$

for any $f \in \mathcal{D}(\mathbf{E})$, which implies $q_j^* = \pi^* p_j^*$ on $\mathcal{D}'(\mathbf{E})$.

Therefore, we get

$$\begin{aligned} \Lambda^m T &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} q_j^* T = \pi^* \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} p_j^* T \\ &= \pi^* \Delta^m T = m! \pi^* q_0^* T = m! p_0^* T \end{aligned}$$

and a symmetric argument allows to conclude. □

Let us now show that the solutions to equation (8) are the distributions associated to the homogeneous polynomials of degree m.

Lemma 4. *A distribution $T \in \mathcal{D}'(\mathbf{E})$ associated to a polynomial of type*

$$P(x) = \sum_{|\alpha| = m} c_\alpha x^\alpha \tag{9}$$

is a solution to the equation (8).

Proof. Let $T \in \mathcal{D}'(\mathbf{E})$ be the distribution associated to the polynomial P of type (9). Given $\varphi \in \mathcal{D}(\mathbf{E}^2)$, we have

$$\begin{aligned} \Delta^m T(\varphi) &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} p_j^* T(\varphi) \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \iint P\big(p_j(x, y)\big) \varphi(x, y)\, dxdy \\ &= \iint \Delta_y^m P(x)\, \varphi(x, y)\, dxdy \\ &= m! \iint q_0^* P(x, y)\, \varphi(x, y)\, dxdy = m! q_0^* T(\varphi), \end{aligned}$$

as expected. □

We have the converse result.

Theorem 4. *Given $m \in \mathbf{N}$, the solutions to equation (8) are the distributions associated to the polynomials P which satisfy equality (9).*

Proof. Let us consider a solution $T \in \mathcal{D}'(\mathbf{E})$ to equation (8). For any j belonging to $\{1, \ldots, n\}$, we have, using the chain rule,

$$D_j p_k^* T = \sum_{l=0}^{n} (D_j [p_k]_l)(p_k^* D_l T) = p_k^* D_j T,$$

as well as $D_j q_0^* T = 0$. This implies $\Delta^m D_j T = 0$.

Now, the unicity of p_j^* implies that, for any $T \in \mathcal{D}'(\mathbf{E})$, we have

$$\Delta^m T(\varphi) = \int \Delta_y^m T_x\big(\varphi(x, y)\big)\, dy,$$

for any $\varphi \in \mathcal{D}(\mathbf{E}^2)$. These relations imply that a solution T necessarily satisfies $\Delta_y^m D_j T = 0$ for almost every y. Hence, $D_j T$ is a distribution associated to a polynomial of degree at most $m - 1$, which is sufficient to conclude. □

As a consequence, we recover the usual solution to equation (5).

Corollary 3. *Given $m \in \mathbf{N}$, if $f \in L^1_{\mathrm{loc}}(\mathbf{E})$ satisfies (8), then f is a polynomial of the form (9).*

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