Convergence of Pascal-Like Triangles in Parry–Bertrand Numeration Systems

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Abstract

We pursue the investigation of generalizations of the Pascal triangle based on binomial coefficients of finite words. These coefficients count the number of times a finite word appears as a subsequence of another finite word. The finite words occurring in this paper belong to the language of a Parry numeration system satisfying the Bertrand property, i.e., we can add or remove trailing zeroes to valid representations. It is a folklore fact that the Sierpiński gasket is the limit set, for the Hausdorff distance, of a convergent sequence of normalized compact blocks extracted from the classical Pascal triangle modulo 2. In a similar way, we describe and study the subset of $[0,1]\times[0,1]$ associated with the latter generalization of the Pascal triangle modulo a prime number.

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1 Introduction

Several generalizations and variations of the Pascal triangle exist and lead to interesting combinatorial, geometrical or dynamical properties [1, 2, 7, 8, 9]. This paper is inspired by a series of papers based on generalizations of Pascal triangles to finite words [9, 10, 11, 12].

1.1 Binomial coefficients of words and Pascal-like triangles

In this short subsection, we briefly introduce the concepts we use in this paper. For more definitions, see section 2. A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*. The *binomial coefficient* $\binom{u}{v}$ of two finite words u and v is the number of times v occurs as a subsequence of u (meaning as a “scattered” subword). It is worth noticing that for any finite word u, the binomial coefficient $\binom{u}{\varepsilon}$ is 1 for the only occurrence of ε in u corresponds to the empty sequence.

Let A be a totally ordered alphabet, and let $L \subset A^*$ be an infinite language over A. We order the words of L by increasing genealogical order (i.e., first by length, then by the classical lexicographic ordering for words of the same length using the order on A) and we write $L = \{w_0 < w_1 < w_2 < \cdots\}$. Associated with the language L, we define a Pascal-like triangle $\mathrm{P}_L : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ represented as an infinite table. The entry $\mathrm{P}_L(m,n)$ on the mth row and nth column of P_L is the integer $\binom{w_m}{w_n}$.

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1.2 Previous work

Let b be an integer greater than 1. We let $\mathrm{rep}_b(n)$ denote the (greedy) base-b expansion of $n \in \mathbb{N} \setminus \{0\}$ starting with a non-zero digit. We set $\mathrm{rep}_b(0)$ to be the empty word denoted by ε. We let $L_b = \{1, \ldots, b-1\}\{0, \ldots, b-1\}^* \cup \{\varepsilon\}$ be the set of base-b expansions of the non-negative integers. In [9], we study the particular case of $L = L_2$. The increasing genealogical order thus coincides with the classical order in $\mathbb{N}$. For example, see Table 1 for the first few values[2] of P_2. Clearly, P_b contains several subtables corresponding to

	ε	1	10	11	100	101	110	111
ε	**1**	0	0	0	0	0	0	0
1	**1**	**1**	0	0	0	0	0	0
10	1	1	1	0	0	0	0	0
11	**1**	**2**	0	**1**	0	0	0	0
100	1	1	2	0	1	0	0	0
101	1	2	1	1	0	1	0	0
110	1	2	2	1	0	0	1	0
111	**1**	**3**	0	**3**	0	0	0	**1**

Table 1: The first few values in the generalized Pascal triangle P_2 (A282714).

the usual Pascal triangle. For instance, it contains $(b-1)$ copies of the usual Pascal triangle obtained when only considering words of the form a^m with $a \in \{1, \ldots, b-1\}$ and $m \geq 0$ since $\binom{a^m}{a^n} = \binom{m}{n}$. In Table 1, a copy of the classical Pascal triangle is written in bold.

Considering the intersection of the lattice $\mathbb{N}^2$ with $[0, 2^n] \times [0, 2^n]$, the first 2^n rows and columns $(\binom{i}{j} \bmod 2)_{0 \leq i,j < 2^n}$ of the usual Pascal triangle modulo 2 provide a coloring of this lattice. If we normalize this compact set by a homothety of ratio $1/2^n$, we get a sequence of subsets of $[0,1] \times [0,1]$ which converges, for the Hausdorff distance, to the Sierpiński gasket when n tends to infinity. In the extended context described above, the case when $b = 2$ gives similar results and the limit set, generalizing the Sierpiński gasket, is described using a simple combinatorial property called $(\star)$ [9].

Inspired by [9], we study the sequence $(S_b(n))_{n \geq 0}$ which counts, on each row m of P_b, the number of words of L_b occurring as subwords of the mth word in L_b, i.e., $S_b(m) = \#\{n \in \mathbb{N} \mid \mathrm{P}_b(m, n) > 0\}$. This sequence is shown to be b-regular [10, 12]. We also consider the summatory function $(A_b(n))_{n \geq 0}$ of the sequence $(S_b(n))_{n \geq 0}$ and study its behavior [11, 12].

So far, the setting is the one of integer bases. As a first extension, we handle the case of the Fibonacci numeration system, i.e., with the language $L_F = \{\varepsilon\} \cup 1\{0, 01\}^*$ [10, 11]. It turns out that the sequence $(S_F(n))_{n \geq 0}$ counting the number of words in L_F occurring as subwords of the nth word in L_F has properties similar to those of $(S_b(n))_{n \geq 0}$. Finally, the summatory function $(A_F(n))_{n \geq 0}$ of the sequence $(S_F(n))_{n \geq 0}$ has a behavior similar to the one of $(A_b(n))_{n \geq 0}$.

1.3 Our contribution

The Fibonacci numeration system belongs to an extensively studied family of numeration systems called *Parry–Bertrand numeration systems*, which are based on particular sequences $(U(n))_{n \geq 0}$ (the precise definitions are given later). In this paper, we fill in the gap between integer bases and the Fibonacci numeration systems by extending the results of [9] to every Parry–Bertrand numeration system. First, we generalize the construction of Pascal-like triangles to every Parry–Bertrand numeration system. For a given Parry–Bertrand numeration system based on a particular sequence $(U(n))_{n \geq 0}$, we consider the intersection of the lattice $\mathbb{N}^2$ with $[0, U(n)] \times [0, U(n)]$. Then the first $U(n)$ rows and columns of the corresponding generalized

[2] Some of the objects discussed here are stored in Sloane's *On-Line Encyclopedia of Integer Sequences* [18]. See sequences A007306, A282714, A282715, A282720, A282728, A284441, and A284442.

Pascal triangle modulo 2 provide a coloring of this lattice regarding the parity of the corresponding binomial coefficients. If we normalize this compact set by a homothety of ratio $1/U(n)$, we get a sequence in $[0,1] \times [0,1]$ which converges, for the Hausdorff distance, to a limit set when n tends to infinity. Again, the limit set is described using a simple combinatorial property extending the one from [9].

Compared to the integer bases, new technicalities have to be taken into account to generalize Pascal triangles to a larger class of numeration systems. The numeration systems occurring in this paper essentially have two properties. The first one is that the language of the numeration system comes from a particular automaton. The second one is the Bertrand condition which allows to delete or add ending zeroes to valid representations.

This paper is organized as follows. In Section 2, we collect necessary background. Section 3 is devoted to a special combinatorial property that extends the $(\star)$ condition from [9]. This new condition allows us to define a sequence of compact sets, which is shown to be a Cauchy sequence in Section 4. In Section 5, using the property of the latter sequence, we define a limit set which is the analogue of the Sierpiński gasket in the classical framework. We show that the sequence of subblocks of the generalized Pascal triangle modulo 2 in a Parry–Bertrand numeration converges to this new limit set. As a final remark, we consider the latter sequence of compact sets modulo any prime number.

2 Background and particular framework

We begin this section with well-known definitions from combinatorics on words; see, for instance, [16]. Let A be an alphabet, i.e., a finite set. The elements of A are called *letters*. A finite sequence over A is called a *finite word*. The length of a finite word w, denoted by $|w|$, is the number of letters belonging to w. The only word of length 0 is the empty word ε. The set of finite words over the alphabet A including the empty word (resp., excluding the empty word) is denoted by A^* (resp., A^+). The set of words of length n over A is denoted by A^n. If u and v are two finite words belonging to A^*, the *binomial coefficient* $\binom{u}{v}$ of u and v is the number of occurrences of v as a subsequence of u, meaning as a scattered subword. The sequences over A indexed by $\mathbb{N}$ are the *infinite words* over A. If w is a finite non-empty word over A, we let $w^\omega := www\cdots$ denote the infinite word obtained by concatenating infinitely many copies of w. If $L \subset A^*$ is a set of finite words and $u \in A^*$ is a finite word, we let $u^{-1}.L$ denote the set of words $\{v \in A^* \mid uv \in L\}$. Let A be totally ordered. If $u, v \in A^*$ are two words, we say that *u is less than v in the genealogical order* and we write $u < v$ if either $|u| < |v|$, or if $|u| = |v|$ and there exist words $p, q, r \in A^*$ and letters $a, b \in A$ with $u = paq$, $v = pbr$ and $a < b$. By $u \leq v$, we mean that either $u < v$, or $u = v$.

In the first part of this section, we gather two" results on binomial coefficients of finite words and integers. The proof of the first lemma is easy, but still can be found in [13, Chap. 6].

Lemma 1. *Let A be a finite alphabet. Let $u, v \in A^*$ and let $a, b \in A$. Then we have*

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

where $\delta_{a,b}$ is equal to 1 if $a = b$, 0 otherwise. If s, t, w are three words over A, then we also have

$$\binom{sw}{t} = \sum_{\substack{u,v \in A^* \\ uv = t}} \binom{s}{u}\binom{w}{v}.$$

Implied by the previous result, the next useful lemma deals with binomial coefficients of words ending with blocks of zeroes.

Lemma 2. *Let A be a finite alphabet containing 0. For all non-empty words $u, v \in A^*$ and all $k \in \mathbb{N}$, we have*

$$\binom{u0^k}{v0^k} = \sum_{j=0}^{k} \binom{k}{j}\binom{u}{v0^j}.$$

In the last part of this section, we introduce the setting of particular numeration systems that are used in this paper: the Parry–Bertrand numeration systems. First of all, we recall several definitions and results about representations of real numbers. To that aim, let $\lceil\cdot\rceil$ denote the *ceiling function* defined by $\lceil x\rceil = \inf\{z \in \mathbb{Z} \mid z \geq x\}$. For more details, see, for instance, [3, Chap. 2], [14, Chap. 7] or [17].

Definition 3. Let $\beta \in \mathbb{R}_{>1}$ and let $A_\beta = \{0, 1, \ldots, \lceil\beta\rceil - 1\}$. Every real number $x \in [0, 1)$ can be written as a series

$$x = \sum_{j=1}^{+\infty} c_j \beta^{-j}$$

where $c_j \in A_\beta$ for all $j \geq 1$. The infinite word $c_1c_2\cdots$ is called a *β-representation* of x. Among all the β-representations of x, we define the *β-expansion* $d_\beta(x)$ of x obtained in a greedy way, i.e., for all $j \geq 1$, we have $c_j\beta^{-j} + c_{j+1}\beta^{-j-1} + \cdots < \beta^{-j+1}$. We also make use of the following convention: if $w = w_n \cdots w_0$ is a finite word (resp., $w = w_1w_2\cdots$ is an infinite word) over A_β, the notation $0.w$ has to be understood as the real number $\sum_{j=0}^{n} w_j\beta^{j-n-1}$ (resp., $\sum_{j=1}^{+\infty} w_j\beta^{-j}$); it actually corresponds to the value of the word w in base β.

In an analogous way, the *β-expansion* $d_\beta(1)$ of 1 is the following infinite word over A_β

$$d_\beta(1) := \begin{cases} (\beta-1)^\omega, & \text{if } \beta \in \mathbb{N}; \\ (\lceil\beta\rceil - 1)d_\beta(1 - (\lceil\beta\rceil - 1)/\beta), & \text{otherwise.} \end{cases}$$

In other words, if β is not an integer, the first digit of the β-expansion of 1 is $\lceil\beta\rceil - 1$ and the other digits are derived from the β-expansion of $1 - (\lceil\beta\rceil - 1)/\beta$.

Let $d_\beta(1) = (t_n)_{n\geq 1}$ be the β-expansion of 1. Observe that $t_1 = \lceil\beta\rceil - 1$. We define the *quasi-greedy β-expansion* $d^*_\beta(1)$ of 1 as follows. If $d_\beta(1) = t_1 \cdots t_m$ is finite, i.e., $t_m \neq 0$ and $t_j = 0$ for all $j > m$, then $d^*_\beta(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$, otherwise $d^*_\beta(1) = d_\beta(1)$.

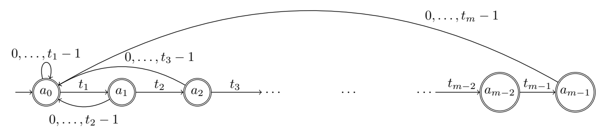
A real number $\beta > 1$ is a *Parry number* if $d_\beta(1)$ is ultimately periodic. Note that if $d_\beta(1)$ is finite, then β is called a *simple Parry number*. In this case, Proposition 5 gives an easy way to decide if an infinite word is the β-expansion of a real number [15]. For more details, see, for instance, [14, Chap. 7]. First, let us recall the definition of a deterministic finite automaton.

Definition 4. A deterministic finite automaton (DFA), over an alphabet A is given by a 5-tuple $\mathcal{A} = (Q, q_0, A, \delta, F)$ where Q is a finite set of states, $q_0 \in Q$ is the initial state, $\delta : Q \times A \mapsto Q$ is the transition function and $F \subset Q$ is the set of final states (graphically represented by two concentric circles). The map δ can be extended to $Q \times A^*$ by setting $\delta(q, \varepsilon) = q$ and $\delta(q, wa) = \delta(\delta(q, w), a)$ for all $q \in Q$, $a \in A$ and $w \in A^*$. We also say that a word w is *accepted* by the automaton if $\delta(q_0, w) \in F$.

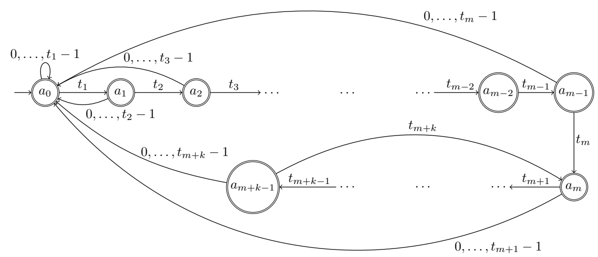
Proposition 5. *Let $\beta \in \mathbb{R}_{>1}$ be a Parry number.*

(a) Suppose that $d_\beta(1) = t_1 \cdots t_m$ is finite, i.e., $t_m \neq 0$ and $t_j = 0$ for all $j > m$. Then an infinite word is the β-expansion of a real number in $[0, 1)$ if and only if it is the label of a path in the automaton $\mathcal{A}_\beta = (\{a_0, \ldots, a_{m-1}\}, a_0, A_\beta, \delta, \{a_0, \ldots, a_{m-1}\})$ depicted in Figure 1a, where the transition function δ is defined as follows: for each $i \in \{1, \ldots, m\}$, $\delta(a_{i-1}, t) = a_0$ for all $t \in \{0, \ldots, t_i - 1\}$; and for every $i < m$, $\delta(a_{i-1}, t_i) = a_i$.

(b) Suppose that $d_\beta(1) = t_1 \cdots t_m(t_{m+1} \cdots t_{m+k})^\omega$ where m, k are taken to be minimal. Then an infinite word is the β-expansion of a real number $[0, 1)$ if and only if it is the label of a path in the automaton $\mathcal{A}_\beta = (\{a_0, \ldots, a_{m+k-1}\}, a_0, A_\beta, \delta, \{a_0, \ldots, a_{m+k-1}\})$ depicted in Figure 1b, where the transition function δ is defined as follows: for each $i \in \{1, \ldots, m+k\}$, $\delta(a_{i-1}, t) = a_0$ for all $t \in \{0, \ldots, t_i - 1\}$; for every $i < m+k$, $\delta(a_{i-1}, t_i) = a_i$, and $\delta(a_{m+k-1}, t_{m+k}) = a_m$.



(a) The case when $d_\beta(1)$ is finite.



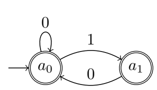
(b) The case when $d_\beta(1)$ is ultimately periodic but not finite.

Figure 1: The automaton $\mathcal{A}_\beta$ in function of the ultimately periodic word $d_\beta(1)$.

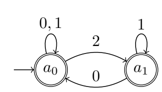
From that result, observe that from any state in the automaton, one can reach the initial state by reading a suitable sequence of zeroes, acting as a reset sequence. Note that if $t_i = 0$, the set $\{0,\ldots,t_i-1\}$ is empty, so several zeroes might actually be required to reach the initial state. Now let us illustrate the previous proposition. For other examples, see, for instance, [5].

Example 6. If $\beta \in \mathbb{R}_{>1}$ is an integer, then $d_\beta(1) = d^*_\beta(1) = (\beta-1)^\omega$. The automaton $\mathcal{A}_\beta$ consists of a single initial and final state a_0 with a loop of labels $0,1,\ldots,\beta-1$.

Consider the golden ratio φ. Since $1 = 1/\varphi + 1/\varphi^2$, we have $d_\varphi(1) = 11$ and $d^*_\varphi(1) = (10)^\omega$. It is thus a Parry number. The automaton $\mathcal{A}_\varphi$ is depicted in Figure 2a. The square φ^2 of the golden ratio is again a Parry number with $d_{\varphi^2}(1) = d^*_{\varphi^2}(1) = 21^\omega$. The automaton $\mathcal{A}_{\varphi^2}$ is depicted in Figure 2b.



(a)



(b)

Figure 2: The automaton $\mathcal{A}_\varphi$ (on the left) and the automaton $\mathcal{A}_{\varphi^2}$ (on the right).

With every Parry number is canonically associated a linear numeration system. Let us recall the definition of such numeration systems.

Definition 7. Let $U = (U(n))_{n\geq 0}$ be an increasing sequence of integers such that $U(0) = 1$. We say that U is a *linear numeration system* if U satisfies a linear recurrence relation, i.e., there exist $k \geq 1$ and $a_0,\ldots,a_{k-1} \in \mathbb{Z}$ such that

$$U(n+k) = a_{k-1}\,U(n+k-1) + \cdots + a_0\,U(n) \quad \forall n \geq 0. \tag{1}$$

Let n be a positive integer. By successive Euclidean divisions, there exists $\ell \geq 1$ such that

$$n = \sum_{j=0}^{\ell-1} c_j\, U(j)$$

where the c_j's are non-negative integers and $c_{\ell-1}$ is non-zero. The word $c_{\ell-1}\cdots c_0$ is called the *normal U-representation* of n and is denoted by $\mathrm{rep}_U(n)$. In other words, the word $c_{\ell-1}\cdots c_0$ is the greedy expansion of

n in the considered numeration system. We set $\mathrm{rep}_U(0) := \varepsilon$. Finally, we refer to $L_U := \mathrm{rep}_U(\mathbb{N})$ as the *language of the numeration* and we let A_U denote the minimal alphabet such that $L_U \subset A_U^*$ (since $\sup_{n\geq 0} \frac{U(n+1)}{U(n)}$ is bounded by a constant, such a finite alphabet does exist). If $d_r \cdots d_0$ is a word over an alphabet of digits, then its *U-numerical value* is $\mathrm{val}_U(d_r \cdots d_0) := \sum_{j=0}^{r} d_j\, U(j)$. Observe that, if $\mathrm{val}_U(d_r \cdots d_0) = n$, then the word $d_r \cdots d_0$ is a U-representation of n (but not necessarily its normal U-representation).

Definition 8. Let $\beta \in \mathbb{R}_{>1}$ be a Parry number. We define a particular linear numeration system $U_\beta := (U_\beta(n))_{n\geq 0}$ associated with β as follows. If $d_\beta(1) = t_1 \cdots t_m$ is finite ($t_m \neq 0$), then we set $U_\beta(0) := 1$, $U_\beta(i) := t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1$ for all $i \in \{1, \ldots, m-1\}$ and, for all $n \geq m$,

$$U_\beta(n) := t_1 U_\beta(n-1) + \cdots + t_m U_\beta(n-m).$$

If $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$ (m, k are minimal), then we set $U_\beta(0) := 1$, $U_\beta(i) := t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1$ for all $i \in \{1, \ldots, m+k-1\}$ and, for all $n \geq m+k$,

$$\begin{aligned} U_\beta(n) := {} & t_1 U_\beta(n-1) + \cdots + t_{m+k} U_\beta(n-m-k) + U_\beta(n-k) \\ & - t_1 U_\beta(n-k-1) - \cdots - t_m U_\beta(n-m-k). \end{aligned}$$

The linear numeration system U_β from Definition 8 has an interesting property: it is a Bertrand numeration system.

Definition 9. A linear numeration system $U = (U(n))_{n\geq 0}$ is a *Bertrand numeration system* if, for all $w \in A_U^+$, $w \in L_U \Leftrightarrow w0 \in L_U$.

Bertrand proved that the linear numeration system U_β associated with the Parry number β from Definition 8 is the unique linear numeration system associated with β that is also a Bertrand numeration system [4]. In that case [4], any word w in the set $0^* L_{U_\beta}$ of all normal U_β-representations with leading zeroes is the label of a path in the automaton $\mathcal{A}_\beta$ from Proposition 5.

Finally, every Parry number is a Perron number [14, Chap. 7]. A real number $\beta > 1$ is a *Perron number* if it is an algebraic integer whose conjugates have modulus less than β. Numeration systems based on Perron numbers are defined as follows and have the property (2), which is often used in this paper.

Definition 10. Let $U = (U(n))_{n\geq 0}$ be a linear numeration system. Consider the characteristic polynomial of the recurrence (1) given by $P(X) = X^k - a_{k-1}X^{k-1} - \cdots - a_1 X - a_0$. If P is the minimal polynomial of a Perron number $\beta \in \mathbb{R}_{>1}$, we say that U is a *Perron numeration system*. In this case, the polynomial P can be factored as $P(X) = (X-\beta)(X-\alpha_2)\cdots(X-\alpha_k)$, where the complex numbers $\alpha_2, \ldots, \alpha_k$ are the conjugates of β, and, for all $j > 1$, we have $|\alpha_j| < \beta$. Using a well-known fact regarding recurrence relations, we have $U(n) = c_1\beta^n + c_2\alpha_2^n + \cdots + c_k\alpha_k^n$ for all $n \geq 0$ where $c_1, \ldots, c_k$ are complex numbers depending on the initial values of U. Since $|\alpha_j| < \beta$ for all $j > 1$, we have

$$\lim_{n\to+\infty} \frac{U(n)}{\beta^n} = c_1. \tag{2}$$

Remark 11. Note that if two Perron numeration systems are associated with the same Perron number, then these two systems only differ by the choice of the initial values $U(0), \ldots, U(k-1)$. The choice of those initial values is of great importance. See, for instance, Example 12.

Example 12. The usual integer base system is a special case of a Perron–Bertrand numeration system.

The golden ratio φ is a Perron number whose minimal polynomial is $P(X) = X^2 - X - 1$. A Perron–Bertrand numeration system associated with φ is the Fibonacci numeration system based on the Fibonacci numbers $(F(n))_{n\geq 0}$ defined by $F(0) = 1$, $F(1) = 2$ and $F(n+2) = F(n+1) + F(n)$. If we change the initial conditions and set $F'(0) = 1$, $F'(1) = 3$ and $F'(n+2) = F'(n+1) + F'(n)$, we again get a Perron numeration associated with φ which is not a Bertrand numeration system. Indeed, 2 is a greedy representation, but not 20 because $\mathrm{rep}_{F'}(\mathrm{val}_{F'}(20)) = 102$.

The particular setting of this paper is the following one: we let $\beta \in \mathbb{R}_{>1}$ be a Parry number and we constantly use the special Parry–Bertrand numeration U_β from Definition 8. From Definition 3 and Definition 7, the alphabet A_{U_β} is the set $\{0, 1, \ldots, \lceil\beta\rceil - 1\}$ and the language of the system of numeration U_β is $L_{U_\beta} \subset A^*_{U_\beta}$ (which is defined using the automaton $\mathcal{A}_\beta$ from Proposition 5).

3 The ($\star$) condition

We let $w_n = \mathrm{rep}_{U_\beta}(n)$ denote the nth word of the language L_{U_β} in the genealogical order. The generalized Pascal triangle $\mathrm{P}_{U_\beta} : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (i, j) \mapsto \binom{w_i}{w_j}$ is represented as an infinite table[3] whose entry on the ith row and the jth column is the binomial coefficient $\binom{w_i}{w_j}$. For instance, when $\beta = \varphi$, the first few values in the generalized Pascal triangle P_{U_φ} are given in Table 2 below. Considering the intersection of the lattice $\mathbb{N}^2$ with $[0, U_\beta(n)] \times [0, U_\beta(n)]$, the first $U_\beta(n)$ rows and columns $(\binom{w_i}{w_j} \bmod 2)_{0\le i,j<U_\beta(n)}$ of the generalized Pascal triangle P_{U_β} modulo 2 provide a coloring of this lattice, leading to a sequence of compact subsets of $\mathbb{R}^2$. If we normalize these sets respectively by a homothety of ratio $1/U_\beta(n)$, we define a sequence $(\mathcal{U}^\beta_n)_{n\ge0}$ of subsets of $[0, 1] \times [0, 1]$.

Definition 13. Let $Q := [0, 1] \times [0, 1]$. Consider the sequence $(\mathcal{U}^\beta_n)_{n\ge0}$ of sets in $[0, 1] \times [0, 1]$ defined for all $n \ge 0$ by

$$\mathcal{U}^\beta_n := \frac{1}{U_\beta(n)} \bigcup \left\{ (\mathrm{val}_{U_\beta}(v), \mathrm{val}_{U_\beta}(u)) + Q \mid u, v \in L_{U_\beta}, \binom{u}{v} \equiv 1 \bmod 2 \right\} \subset [0, 1] \times [0, 1].$$

Each $\mathcal{U}^\beta_n$ is a finite union of squares of size $1/U_\beta(n)$ and is thus compact.

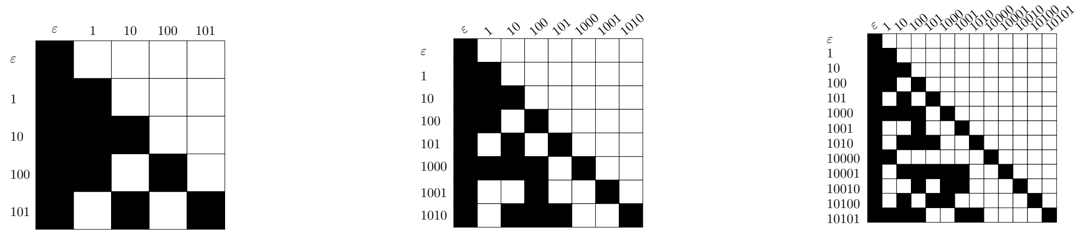
Example 14. When $\beta = \varphi$ is the golden ratio, the first values in the generalized Pascal triangle P_{U_φ} are given in Table 2. The sets $\mathcal{U}^\varphi_3$, $\mathcal{U}^\varphi_4$ and $\mathcal{U}^\varphi_5$ are depicted in Figure 3. The set $\mathcal{U}^\varphi_9$ is depicted in Figure 14 given in the appendix.

	$\binom{w_i}{w_j}$	ε	1	10	100	j 101	1000	1001	1010
	ε	1	0	0	0	0	0	0	0
	1	1	1	0	0	0	0	0	0
	10	1	1	1	0	0	0	0	0
i	100	1	1	2	1	0	0	0	0
	101	1	2	1	0	1	0	0	0
	1000	1	1	3	3	0	1	0	0
	1001	1	2	2	1	2	0	1	0
	1010	1	2	3	1	1	0	0	1

Table 2: The first few values in the generalized Pascal triangle P_{U_φ}.

Remark 15. Each pair (u, v) of words of length at most n with an odd binomial coefficient gives rise to a square region in $\mathcal{U}^\beta_n$. More precisely, we have the following situation. Let $n \ge 0$ and $u, v \in L_{U_\beta}$ such that $0 \le |v| \le |u| \le n$ and $\binom{u}{v} \equiv 1 \bmod 2$. We have $((\mathrm{val}_{U_\beta}(v), \mathrm{val}_{U_\beta}(u)) + Q)/U_\beta(n) \subset \mathcal{U}^\beta_n$ as depicted in Figure 4.

[3]Using the notation $\binom{u}{v}$, the rows (resp., columns) of P_{U_β} are indexed by the words u (resp., v).



(a) The set $\mathcal{U}_3^{\varphi}$. (b) The set $\mathcal{U}_4^{\varphi}$. (c) The set $\mathcal{U}_5^{\varphi}$.

Figure 3: The sets $\mathcal{U}_3^{\varphi}$, $\mathcal{U}_4^{\varphi}$ and $\mathcal{U}_5^{\varphi}$ when $\beta = \varphi$ is the golden ratio.

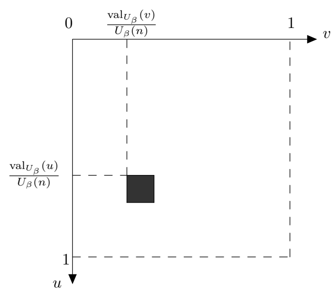


Figure 4: Visualization of a square region in $\mathcal{U}_n^{\beta}$.

Let us now introduce some notations about the Hausdorff metric. We assume that d is the Euclidean distance on $\mathbb{R}^2$. We let $B(x, \epsilon)$ denote the open ball of radius $\epsilon \geq 0$ centered at $x \in \mathbb{R}^2$ and, if $S \subset \mathbb{R}^2$, we let $[S]_\epsilon := \cup_{x \in S} B(x, \epsilon)$ denote the ϵ-*fattening* of S. We consider the space $(\mathcal{H}(\mathbb{R}^2), d_h)$ of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff metric d_h induced by d. It is well known that $(\mathcal{H}(\mathbb{R}^2), d_h)$ is complete [6].

Our aim is to show that the sequence $(\mathcal{U}_n^{\beta})_{n \geq 0}$ of compact subsets of $[0, 1] \times [0, 1]$ is converging and to provide an elementary description of its limit set. The idea is the following one. Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ be a pair of words having an odd binomial coefficient. On the one hand, some of those pairs are such that $\binom{ua}{va} \equiv 0 \bmod 2$ for all letters a such that $ua, va \in L_{U_\beta}$. In other words, those pairs of words create a black square region in $\mathcal{U}_{|u|}^{\beta}$ while the corresponding square region in $\mathcal{U}_{|u|+1}^{\beta}$ is white. As an example, take $\beta = \varphi$, $u = 1010$ and $v = 101$. We have $\binom{u0}{v0} = 2$ (see Figure 3). On the other hand, some of those pairs create a more stable pattern, i.e., $\binom{uw}{vw} \equiv 1 \bmod 2$ for all words w such that $uw, vw \in L_{U_\beta}$. Roughly, those pairs create a diagonal of square regions in $(\mathcal{U}_n^{\beta})_{n \geq 0}$. For instance, take $\beta = \varphi$, $u = 101$ and $v = 10$. In this case, $\binom{uw}{vw} \equiv 1 \bmod 2$ for all admissible words w. In particular, the pairs of words (u, v), $(u0, v0)$ and $(u00, v00)$, $(u01, v01)$ have odd binomial coefficients (see Figure 3) and create a diagonal of square regions. With the second type of pairs of words, we define a new sequence of compact subsets $(\mathcal{A}_n^{\beta})_{n \geq 0}$ of $[0, 1] \times [0, 1]$ which converges to some well-defined limit set $\mathcal{L}^{\beta}$. Then, we show that the first sequence of compact sets $(\mathcal{U}_n^{\beta})_{n \geq 0}$ also converges to this limit set. The remaining of this paper is dedicated to formalize and prove those statements.

To reach that goal, for all non-empty words $u, v \in L_{U_\beta}$, we first define the least integer p such that $u0^p w, v0^p w$ belong to L_{U_β} for all words $w \in 0^* L_{U_\beta}$. In other terms, any word w can be read after $u0^p$ and $v0^p$ in the automaton $\mathcal{A}_\beta$. Then, some pairs of words $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ have the property that not only $\binom{u}{v} \equiv 1 \bmod 2$ but also $\binom{u0^p w}{v0^p w} \equiv 1 \bmod 2$ for all words $w \in 0^* L_{U_\beta}$; see Corollary 23. Such a property creates a particular pattern occurring in $\mathcal{U}_n^{\beta}$ for all sufficiently large n, as shown in Remark 25.

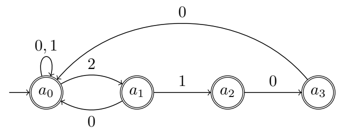


Figure 5: The automaton $\mathcal{A}_\beta$ for the dominant root β of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$.

Proposition 16. *For all non-empty words $u, v \in L_{U_\beta}$, there exists a smallest nonnegative integer $p(u, v)$ such that*

$$(u0^{p(u,v)})^{-1}.L_{U_\beta} = (v0^{p(u,v)})^{-1}.L_{U_\beta} = 0^* L_{U_\beta}.$$

Proof. Using Proposition 5, take $p(u, v)$ to be the least nonnegative integer p such that $\delta(a_0, u0^p) = a_0 = \delta(a_0, v0^p)$. Then, for any word $w \in 0^* L_{U_\beta}$, the words $u0^{p(u,v)}w, v0^{p(u,v)}w$ are labels of paths in $\mathcal{A}_\beta$. Consequently, they are words in L_{U_β}. Conversely, if the words $u0^{p(u,v)}w, v0^{p(u,v)}w$ are labels of paths in $\mathcal{A}_\beta$, then $w \in 0^* L_{U_\beta}$. $\square$

In the following, we will be using $p(\varepsilon, \varepsilon)$. Observe that, using Proposition 5, $\delta(a_0, \varepsilon) = a_0$. We naturally set $p(\varepsilon, \varepsilon) := 0$ and we thus have $(\varepsilon 0^{p(\varepsilon,\varepsilon)})^{-1}.L_{U_\beta} = L_{U_\beta}$.

Example 17. If $\beta > 1$ is an integer, then $p(u, v) = 0$ for all $u, v \in L_{U_\beta}$. See Example 6

If $\beta = \varphi$ is the golden ratio, then $p(u, v) = 0$ if and only if u and v end with 0 or $u = v = \varepsilon$, otherwise $p(u, v) = 1$.

The integer of Proposition 16 can be greater than 1 as illustrated in the following example.

Example 18. Let β be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$. Then $\beta \approx 2.47098$ is a Parry number with $d_\beta(1) = 2101$ and $d^*_\beta(1) = (2100)^\omega$. The automaton $\mathcal{A}_\beta$ is depicted in Figure 5. For instance, $p(101, 21) = 2$.

Definition 19. Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$. We say that (u, v) satisfies the $(\star)$ condition if either $u = v = \varepsilon$, or $|u| \geq |v| > 0$ and

$$\binom{u0^{p(u,v)}}{v0^{p(u,v)}} \equiv 1 \bmod 2 \quad \text{and} \quad \binom{u0^{p(u,v)}}{v0^{p(u,v)}a} = 0 \quad \forall\, a \in A_{U_\beta}$$

where $p(u, v)$ is defined by Proposition 16. Observe that, if $(u, v) \neq (\varepsilon, \varepsilon)$, then $v0^{p(u,v)}a \in L_{U_\beta}$ for all $a \in A_{U_\beta}$. It is worth noticing that if only one of the two words u or v is empty, then the pair (u, v) never satisfies $(\star)$.

The next easy lemma shows that all diagonal elements of $\mathcal{U}_n^\beta$ satisfy $(\star)$.

Lemma 20. *For any word $u \in L_{U_\beta}$, the pair (u, u) satisfies $(\star)$.*

Proof. If $u = \varepsilon$, the result is clear using Definition 19. Suppose u is non-empty and let $p := p(u, u)$ denote the integer from Proposition 16. Then, for all $a \in A_{U_\beta}$, we get $\binom{u0^p}{u0^p} = 1 \equiv 1 \bmod 2$ and $\binom{u0^p}{u0^p a} = 0$ for we have $|u0^p a| > |u0^p|$. $\square$

If a pair of words satisfies $(\star)$, it has the following two properties. First, its binomial coefficient is odd, as stated in the following proposition. Secondly, it creates a special pattern in $\mathcal{U}_n^\beta$ for all large enough n; see Proposition 22, Corollary 23 and Remark 25.

Proposition 21. *Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$. Then $\binom{u}{v} \equiv 1 \bmod 2$.*

Proof. If $u = v = \varepsilon$, the result is clear by definition. Suppose that u and v are non-empty. Let us proceed by contradiction and suppose that $\binom{u}{v}$ is even. For the sake of clarity, let us set $p := p(u,v)$ from Proposition 16. On the one hand, by Definition 19, we know that $\binom{u0^p}{v0^p} \equiv 1 \bmod 2$ and, on the other hand, Lemma 2 states that

$$\binom{u0^p}{v0^p} = \sum_{j=1}^{p} \binom{p}{j}\binom{u}{v0^j} + \binom{u}{v}.$$

Consequently, we have

$$\sum_{j=1}^{p} \binom{p}{j}\binom{u}{v0^j} \equiv 1 \bmod 2 > 0$$

and there must exist $i \in \{1,\ldots,p\}$ such that $\binom{u}{v0^i} > 0$. Using again Lemma 2, we also have

$$\binom{u0^p}{v0^p0} = \sum_{j=1}^{p+1} \binom{p}{j-1}\binom{u}{v0^j} \geq \binom{p}{i-1}\binom{u}{v0^i} > 0,$$

which contradicts Definition 19. □

Proposition 22. *Let $u, v \in L_{U_\beta}$ be two non-empty words such that (u,v) satisfies $(\star)$. For any letter $a \in A_{U_\beta}$, the pair of words $(u0^{p(u,v)}a, v0^{p(u,v)}a) \in L_{U_\beta} \times L_{U_\beta}$ satisfies $(\star)$.*

Proof. Set $p := p(u,v)$. Let a be a letter in A_{U_β} and also set $p' := p(u0^pa, v0^pa)$. By Lemmas 1 and 2,

$$\binom{u0^pa0^{p'}}{v0^pa0^{p'}} = \sum_{j=1}^{p'} \binom{p'}{j}\binom{u0^pa}{v0^pa0^j} + \binom{u0^p}{v0^pa} + \binom{u0^p}{v0^p}.$$

Since (u,v) satisfies $(\star)$, $\binom{u0^p}{v0^pa} = 0$. We now show that all the coefficients $\binom{u0^pa}{v0^pa0^j}$, for $j = 1,\ldots,p'$, are also 0. Let $1 \leq j \leq p'$. From Lemma 1, we know that

$$\binom{u0^pa}{v0^pa0^j} = \binom{u0^p}{v0^pa0^j} + \delta_{a,0}\binom{u0^p}{v0^pa0^{j-1}}.$$

Clearly, the first term $\binom{u0^p}{v0^pa0^j}$ must be 0. Indeed, otherwise it means that the word $v0^pa$ appears as a subword of the word $u0^p$, which contradicts $(\star)$. The second term $\binom{u0^p}{v0^pa0^{j-1}}$ only appears if $a = 0$. In that case, this term becomes $\binom{u0^p}{v0^p0^j} = 0$, for otherwise there is an occurrence of the word $v0^p0$ in $u0^p$, contradicting $(\star)$. Consequently, using Definition 19, we get

$$\binom{u0^pa0^{p'}}{v0^pa0^{p'}} = \binom{u0^p}{v0^p} \equiv 1 \bmod 2.$$

Using the same argument, for any letter $b \in A_{U_\beta}$, we also have

$$\binom{u0^pa0^{p'}}{v0^pa0^{p'}b} = 0.$$

□

Corollary 23. *Let $u, v \in L_{U_\beta}$ be two non-empty words such that (u,v) satisfies $(\star)$. Then*

$$\binom{u0^{p(u,v)}w}{v0^{p(u,v)}w} \equiv 1 \bmod 2 \quad \forall\, w \in 0^*L_{U_\beta}.$$

Proof. Set $p := p(u,v)$. From Proposition 16, $u0^pw, v0^pw$ belong to L_{U_β} for any word $w \in 0^*L_{U_\beta}$. Now proceed by induction on the length of $w \in 0^*L_{U_\beta}$. If $|w| = 0$, then $w = \varepsilon$ is the empty word and the statement is true using Definition 19. If $|w| = 1$, then $w = a$ is a letter belonging to A_{U_β}. Then, by Proposition 22, we know that $(u0^pa, v0^pa)$ satisfies $(\star)$. Using Proposition 21, we have $\binom{u0^pa}{v0^pa} \equiv 1 \bmod 2$. Now suppose that $|w| \geq 2$ and write $w = aw'b \in 0^*L_{U_\beta}$ where a, b are letters. From Lemma 1, we deduce that

$$\binom{u0^pw}{v0^pw} = \binom{u0^paw'}{v0^paw'b} + \binom{u0^paw'}{v0^paw'}.$$

By induction hypothesis, $\binom{u0^paw'}{v0^paw'} \equiv 1 \bmod 2$ since $aw' \in 0^*L_{U_\beta}$ and $|aw'| < |w|$. Furthermore, $\binom{u0^paw'}{v0^paw'b}$ must be 0, otherwise it means that the word $v0^pa$ occurs as a subword of the word $u0^p$, which contradicts the fact that (u,v) satisfies $(\star)$. This ends the proof. $\square$

The next lemma is useful to characterize the pattern created in $\mathcal{U}_n^\beta$, for all sufficiently large n, by pairs of words satisfying $(\star)$, see Remark 25 below. In this result, we make use of the convention given in Definition 3.

Lemma 24. *Let $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$.*

(a) The sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$.

(b) For all $n \geq 0$, let $w = d_n$ denote the prefix of length n of $d_\beta^*(1)$. Then the sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p(u,v)}d_n)}{U_\beta(|u|+p(u,v)+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p(u,v)}d_n)}{U_\beta(|u|+p(u,v)+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v0^{p(u,v)}d_\beta^*(1), 0.u0^{p(u,v)}d_\beta^*(1))$.

Proof. Let $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and set $p := p(u,v)$. We prove the first item as the proof of the second one is similar. The result is trivial if $u = v = \varepsilon$. Suppose that u and v are non-empty words. Let us write $u = u_{|u|-1}u_{|u|-2}\cdots u_0$ where $u_i \in A_{U_\beta}$ for all i. By definition, we have

$$\frac{\mathrm{val}_{U_\beta}(u0^{p+n})}{U_\beta(|u|+p+n)} = \sum_{i=0}^{|u|-1} u_i \frac{U_\beta(i+p+n)}{U_\beta(|u|+p+n)}.$$

Using (2), $U_\beta(i+p+n)/U_\beta(|u|+p+n)$ tends to $\beta^i/\beta^{|u|}$ when n tend to infinity. Consequently,

$$\lim_{n\to+\infty} \frac{\mathrm{val}_{U_\beta}(u0^{p+n})}{U_\beta(|u|+p+n)} = \sum_{i=0}^{|u|-1} u_i\beta^{i-|u|} = 0.u.$$

Using the same reasoning on the word v, we conclude that the sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p(u,v)+n})}{U_\beta(|u|+p(u,v)+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$. $\square$

Remark 25. Let $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and set $p := p(u,v)$. Suppose that u and v are non-empty (the case when $u = v = \varepsilon$ is similar: in the following, replace $0^*L_{U_\beta}$ by L_{U_β} where needed). Using Corollary 23, the pair of words $(u0^pw, v0^pw)$ has an odd binomial coefficient for any word $w \in 0^*L_{U_\beta}$. In particular, the pair of words $(u0^pw, v0^pw)$ corresponds to a square region in $\mathcal{U}^\beta_{|u|+p+n}$ for all $w \in 0^*L_{U_\beta}$ such that $|w| = n \geq 0$. Using Remark 15, this region is

$$\left(\frac{\mathrm{val}_{U_\beta}(v0^pw)}{U_\beta(|u|+p+n)}, \frac{\mathrm{val}_{U_\beta}(u0^pw)}{U_\beta(|u|+p+n)}\right) + \frac{Q}{U_\beta(|u|+p+n)} \subset \mathcal{U}^\beta_{|u|+p+n}.$$

Using Lemma 24, when $w = 0^n$ (the smallest word of length n in $0^*L_{U_\beta}$), the sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^{p+n})}{U_\beta(|u|+p+n)}, \frac{\mathrm{val}_{U_\beta}(u0^{p+n})}{U_\beta(|u|+p+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v, 0.u)$. This point will be the first endpoint of a segment associated with u and v. See Definition 27. Analogously, using Lemma 24, when $w = d_n$ is the prefix of length n of $d^*_\beta(1)$ (the greatest word of length n in $0^*L_{U_\beta}$), then the sequence

$$\left(\left(\frac{\mathrm{val}_{U_\beta}(v0^pd_n)}{U_\beta(|u|+p+n)}, \frac{\mathrm{val}_{U_\beta}(u0^pd_n)}{U_\beta(|u|+p+n)}\right)\right)_{n\geq 0}$$

converges to the pair of real numbers $(0.0^{|u|-|v|}v0^pd^*_\beta(1), 0.u0^pd^*_\beta(1))$. This point will be the second endpoint of the same segment associated with u and v. See again Definition 27. As a consequence, the sequence of sets whose nth term is defined by

$$\bigcup_{\substack{|w|=n \\ w\in 0^*L_{U_\beta}}} \left(\left(\frac{\mathrm{val}_{U_\beta}(v0^pw)}{U_\beta(|u|+p+n)}, \frac{\mathrm{val}_{U_\beta}(u0^pw)}{U_\beta(|u|+p+n)}\right) + \frac{Q}{U_\beta(|u|+p+n)}\right) \tag{3}$$

converges, for the Hausdorff distance, to the diagonal of the square $(0.0^{|u|-|v|}v, 0.u) + Q/\beta^{|u|+p}$.

Example 26. As a first example, when $\beta = 2$, we find back the construction in [9]. As a second example, let us take $\beta = \varphi$ to be the golden ratio. Let $u = 101$ and $v = 10$ (resp., $u' = 100 = v'$). Then $p(u,v) = 1$ (resp., $p(u',v') = 0$); see Example 17. Those pairs of words satisfy $(\star)$. The first few terms of the sequence of sets (3) are respectively depicted in Figure 6 and Figure 7. Observe that when n tends to infinity, the union of black squares in $\mathcal{U}^\varphi_{n+4}$ (resp., $\mathcal{U}^\varphi_{n+3}$) converges to the diagonal of $(0.0v, 0.u) + Q/\varphi^4$ (resp., $(0.v', 0.u') + Q/\varphi^3$).

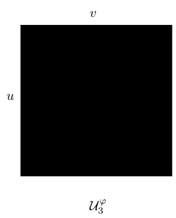
4 The sequence of compact sets $(\mathcal{A}^\beta_n)_{n\geq 0}$

The observation made in Remark 25 leads to the definition of an initial set $\mathcal{A}^\beta_0$. The same technique is applied in [9]. At first, let us define a segment associated with a pair of words.

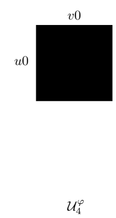
Definition 27. Let (u,v) in $L_{U_\beta} \times L_{U_\beta}$ such that $|u| \geq |v| \geq 0$. We define a closed segment $S_{u,v}$ of slope 1 and of length $\sqrt{2} \cdot \beta^{-|u|-p(u,v)}$ in $[0,1] \times [0,1]$. The endpoints of $S_{u,v}$ are given by $A_{u,v} := (0.0^{|u|-|v|}v, 0.u)$ and

$$B_{u,v} := A_{u,v} + (\beta^{-|u|-p(u,v)}, \beta^{-|u|-p(u,v)}) = (0.0^{|u|-|v|}v0^{p(u,v)}d^*_\beta(1), 0.u0^{p(u,v)}d^*_\beta(1)).$$

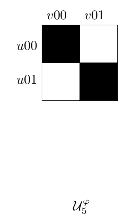
Observe that, if $u = v = \varepsilon$, the associated segment of slope 1 has endpoints $(0,0)$ and $(1,1)$. Otherwise, the segment $S_{u,v}$ lies in $[0,1] \times [1/\beta, 1]$.



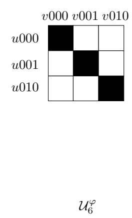
(a) A subset of $\mathcal{U}_3^\varphi$.



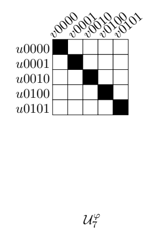
(b) The element $n = 0$ of (3).



(c) The element $n = 1$ of (3).

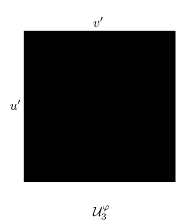


(d) The element $n = 2$ of (3).

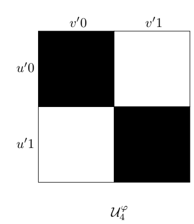


(e) The element $n = 3$ of (3).

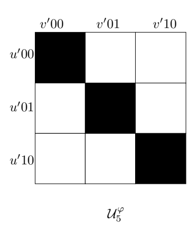
Figure 6: The first few terms of sequence of sets (3) converging to the diagonal of the square $(0.0v, 0.u)+Q/\varphi^4$ for $u = 101$ and $v = 10$.



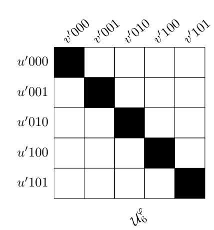
(a) The element $n = 0$ of (3).



(b) The element $n = 1$ of (3).



(c) The element $n = 2$ of (3).



(d) The element $n = 3$ of (3).

Figure 7: The first few terms of sequence of sets (3) converging to the diagonal of the square $(0.v', 0.u')+Q/\varphi^3$ for $u' = 100$ and $v' = 100$.

Definition 28. Let us define the following compact set which is the closure of a countable union of segments

$$\mathcal{A}_0^\beta := \overline{\bigcup_{\substack{(u,v) \\ \text{satisfying}(\star)}} S_{u,v}}.$$

Notice that Definition 27 implies that $\mathcal{A}_0^\beta \subset [0,1] \times [0,1]$. More precisely, $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon} \subset [0,1] \times [1/\beta, 1]$. Furthermore, observe that we take the closure of a union to ensure the compactness of the set.

Example 29. Let $\beta = \varphi$ be the golden ratio. In Figure 8, the segment $S_{u,v}$ is represented for all (u,v) satisfying $(\star)$ and such that $0 \le |v| \le |u| \le 10$.

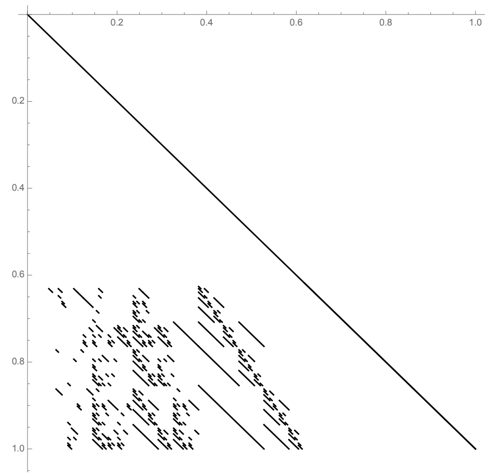


Figure 8: An approximation of $\mathcal{A}_0^\varphi$ computed with words of length ≤ 10.

In the following definition, we introduce another sequence of compact sets obtained by transforming the initial set $\mathcal{A}_0^\beta$ under iterations of two maps. This new sequence, which is shown to be a Cauchy sequence in Proposition 31, allows us to define properly the limit set $\mathcal{L}^\beta$.

Definition 30. We let c denote the homothety of center $(0,0)$ and ratio $1/\beta$ and we consider the map $h : (x,y) \mapsto (x, \beta y)$. We define a sequence of compact sets by setting, for all $n \ge 0$,

$$\mathcal{A}_n^\beta := \bigcup_{\substack{0 \le i \le n \\ 0 \le j \le i}} h^j(c^i(\mathcal{A}_0^\beta)).$$

In Figure 9, we apply c and h at most twice from $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon}$. Let m, n with $m \le n$. Using Figure 9, observe that

$$\mathcal{A}_m^\beta \cap ([1/\beta^{m+1}, 1] \times [0,1]) = \mathcal{A}_n^\beta \cap ([1/\beta^{m+1}, 1] \times [0,1]). \tag{4}$$

The proof of the following proposition is slightly different from the proof of [9, Lemma 25].

Proposition 31. *The sequence $(\mathcal{A}_n^\beta)_{n \ge 0}$ is a Cauchy sequence.*

Proof. Let $\epsilon > 0$ and take $n > m$. We must show that $\mathcal{A}_m^\beta \subset [\mathcal{A}_n^\beta]_\epsilon$ and $\mathcal{A}_n^\beta \subset [\mathcal{A}_m^\beta]_\epsilon$. The first inclusion is easy. Indeed, since $\mathcal{A}_m^\beta \subset \mathcal{A}_n^\beta$, we directly have that $[\mathcal{A}_n^\beta]_\epsilon$ contains $\mathcal{A}_m^\beta$. Let us show the second inclusion. From (4), $\mathcal{A}_m^\beta$ and consequently $[\mathcal{A}_m^\beta]_\epsilon$ both contain $\mathcal{A}_n^\beta \cap ([1/\beta^{m+1}, 1] \times [0,1])$. Now we show that $[\mathcal{A}_m^\beta]_\epsilon$

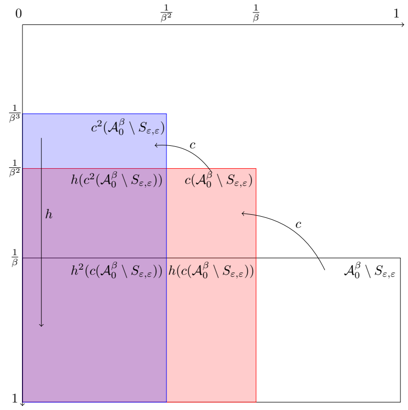


Figure 9: Two applications of c and h from $\mathcal{A}_0^\beta \setminus S_{\varepsilon,\varepsilon}$.

contains $[0, 1/\beta^{m+1}) \times [0, 1]$ if m is sufficiently large, which ends the proof. By Definition 28, $\mathcal{A}_0^\beta$ contains the segment $S_{\varepsilon,\varepsilon}$ of slope 1 with endpoints $(0, 0)$ and $(1, 1)$. Thus, by Definition 30, $\mathcal{A}_m^\beta$ contains the segment $h^m(c^m(S_{\varepsilon,\varepsilon}))$ of slope β^m with endpoints $(0, 0)$ and $(1/\beta^m, 1)$. Let $(x, y) \in [0, 1/\beta^{m+1}) \times [0, 1]$. Then $(y/\beta^m, y)$ belongs to $h^m(c^m(S_{\varepsilon,\varepsilon})) \subset \mathcal{A}_m^\beta$. Consequently, if m is sufficiently large, then

$$d((x, y), \mathcal{A}_m^\beta) \leq d((x, y), (y/\beta^m, y)) \leq x + y/\beta^m < \epsilon. \qquad \square$$

Definition 32. Since the sequence $(\mathcal{A}_n^\beta)_{n\geq 0}$ is a Cauchy sequence in the complete metric space $(\mathcal{H}(\mathbb{R}^2), d_h)$, its limit is a well-defined compact set denoted by $\mathcal{L}^\beta$.

Example 33. Let φ be the golden ratio. We have represented in Figure 10 all the segments of $\mathcal{A}_0^\varphi$ for words of length at most 10 and we have applied the maps $h^j(c^i(\cdot))$ to this set of segments for $0 \leq j \leq i \leq 4$. Thus we have an approximation of $\mathcal{A}_4^\varphi$.

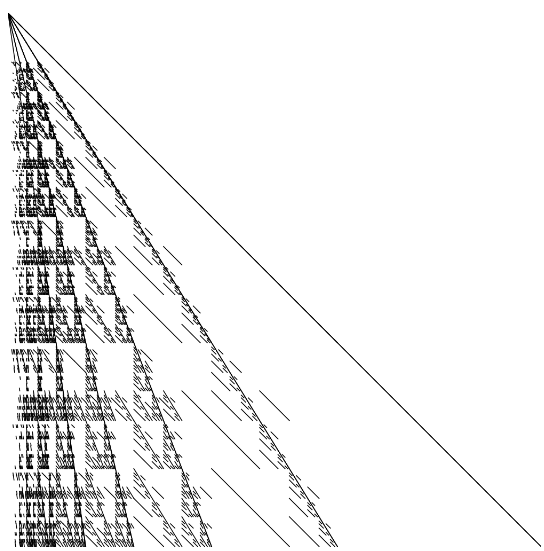


Figure 10: An approximation of the limit set $\mathcal{L}^\varphi$.

5 The limit of the sequence of compact sets $(\mathcal{U}_n^\beta)_{n\geq 0}$

In this section, we show that the sequence $(\mathcal{U}_n^\beta)_{n\geq 0}$ of compact subsets of $[0,1]\times[0,1]$ also converges to $\mathcal{L}^\beta$. The proofs of Lemmas 34 and 39 are essentially the same as the ones from [9] ([9, Lemma 27, Lemma 28]), so we highlight the main differences. The first part is to show that, when ϵ is a positive real number, then $\mathcal{U}_n^\beta \subset [\mathcal{L}^\beta]_\epsilon$ for all sufficiently large n.

Lemma 34. *Let $\epsilon > 0$. For all sufficiently large $n \in \mathbb{N}$, we have $\mathcal{U}_n^\beta \subset [\mathcal{L}^\beta]_\epsilon$.*

Proof. Let $\epsilon > 0$. Take $n \in \mathbb{N}$ and let $(x,y) \in \mathcal{U}_n^\beta$. From Remark 15, there exists $(u,v) \in L_{U_\beta} \times L_{U_\beta}$ such that $\binom{u}{v} \equiv 1 \bmod 2$, $0 \leq |v| \leq |u| \leq n$ and the point (x,y) belongs to the square region

$$((\mathrm{val}_{U_\beta}(v), \mathrm{val}_{U_\beta}(u)) + Q)/U_\beta(n) \subset \mathcal{U}_n^\beta. \tag{5}$$

Let us set $A := (\mathrm{val}_{U_\beta}(v)/U_\beta(n), \mathrm{val}_{U_\beta}(u)/U_\beta(n))$ to be the upper-left corner of the square region (5) in $\mathcal{U}_n^\beta$.

Assume first that (u,v) satisfies $(\star)$. The segment $S_{u,v}$ of length $\sqrt{2}\cdot\beta^{-|u|-p(u,v)}$ having $A_{u,v} = (0.0^{|u|-|v|}v, 0.u)$ as endpoint belongs to $\mathcal{A}_0^\beta$. Now apply $n-|u|$ times the homothety c to this segment. So the segment $c^{n-|u|}(S_{u,v})$ of length $\sqrt{2}\cdot\beta^{-n-p(u,v)}$ of endpoint $B_1 := (0.0^{n-|v|}v, 0.0^{n-|u|}u)$ belongs to $\mathcal{A}_{n-|u|}^\beta$ and thus to $\mathcal{L}^\beta$. Using (2) (the reasoning is similar to the one developed in the proof of Lemma 24), there exists $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, $d(A, B_1) < \epsilon/2$. Hence, for all $n \geq N_1$ such that $\sqrt{2}/U_\beta(n) < \epsilon/2$, we have

$$d((x,y),\mathcal{L}^\beta) \leq d((x,y),B_1) \leq d((x,y),A) + d(A,B_1) \leq \sqrt{2}/U_\beta(n) + d(A,B_1) < \epsilon.$$

Now assume that (u,v) does not satisfy $(\star)$. Since $\binom{u}{v} \equiv 1 \bmod 2$, then either u and v are non-empty words, or u is non-empty and $v = \varepsilon$. First, suppose that u and v are non-empty. Let k be a non-negative integer such that $k > \lceil \log_2 |u| \rceil$ and $2^k > p(u,v)$. By definition of $p(u,v)$, the words $u0^{2^k}1$ and $v0^{2^k}1$ belong to L_{U_β}. As in the proof of [9, Lemma 27], we have

$$\binom{u0^{2^k}1}{v0^{2^k}1} \equiv 1 \bmod 2.$$

Using this result, it is then easy to check that the pair of words $(u0^{2^k}1, v0^{2^k}1)$ satisfies $(\star)$. Finally proceed as in the first part of the proof (namely replace u by $u0^{2^k}1$ and v by $v0^{2^k}1$) to get $d((x,y),\mathcal{L}^\beta) < \epsilon$, as expected.

Assume now that u is non-empty and $v = \varepsilon$. In this case, the point A is on the vertical line of equation $x = 0$. By Definition 28, $\mathcal{A}_0^\beta$ contains the segment $S_{\varepsilon,\varepsilon}$ of slope 1 with endpoints $(0,0)$ and $(1,1)$. Thus, by Definition 30, $\mathcal{A}_n^\beta$ contains the segment $h^n(c^n(S_{\varepsilon,\varepsilon}))$ of slope β^n with endpoints $(0,0)$ and $(1/\beta^n, 1)$. This segment also lies in $\mathcal{L}^\beta$. There exists $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2$, $d(A, h^n(c^n(S_{\varepsilon,\varepsilon}))) \leq 1/\beta^n < \epsilon/2$. Consequently, for all $n \geq N_2$ such that $\sqrt{2}/U_\beta(n) < \epsilon/2$, we have

$$\begin{aligned} d((x,y),\mathcal{L}^\beta) &\leq d((x,y), h^n(c^n(S_{\varepsilon,\varepsilon}))) \leq d((x,y),A) + d(A, h^n(c^n(S_{\varepsilon,\varepsilon}))) \\ &\leq \sqrt{2}/U_\beta(n) + d(A, h^n(c^n(S_{\varepsilon,\varepsilon}))) < \epsilon. \end{aligned}$$

In each of the three cases, we conclude that $(x,y) \in [\mathcal{L}^\beta]_\epsilon$, which proves that $\mathcal{U}_n^\beta \subset [\mathcal{L}^\beta]_\epsilon$ for all sufficiently large n. ☐

If $\epsilon > 0$, it remains to show that $\mathcal{L}^\beta \subset [U_n]_\epsilon$ for all sufficiently large $n \in \mathbb{N}$. To that aim, we need to bound the number of consecutive words, in the genealogical order, that end with 0 in L_{U_β}.

Definition 35. We let $C_\beta := \max\{n \in \mathbb{N} \mid 0^n \text{ is a factor of } d_\beta^*(1)\}$ denote the maximal number of consecutive zeroes in $d_\beta^*(1)$.

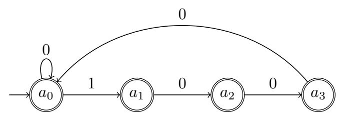


Figure 11: The automaton $\mathcal{A}_\beta$ for the dominant root β of the polynomial $P(X) = X^4 - X^3 - 1$.

In the next proposition, we show that the maximal number of consecutive words ending with 0 in L_{U_β} is $C_\beta + 1$.

Proposition 36. *If we order the words in L_{U_β} by the genealogical order, the maximal number of consecutive words ending with* 0 *in L_{U_β}, i.e., the maximal number of consecutive normal U_β-representations ending with* 0*, is $C_\beta + 1$.*

Proof. Let $n \in \mathbb{N}$ be such that $\operatorname{rep}_{U_\beta}(n)$ ends with 0. We can suppose that $\operatorname{rep}_{U_\beta}(n-1)$ does not end with 0, otherwise we translate n. If $|\operatorname{rep}_{U_\beta}(n+1)| = |\operatorname{rep}_{U_\beta}(n)|$, then $\operatorname{rep}_{U_\beta}(n+1)$ does not end with 0 because $U_\beta(m) \geq 2$ for all $m \geq 1$. Indeed, if a single digit (not the least significant one) is changed, then the value is increased by at least 2. Let $C \geq 1$ be such that, for all $k \in \{0, \ldots, C\}$, $|\operatorname{rep}_{U_\beta}(n+k)| = |\operatorname{rep}_{U_\beta}(n)| + k$ and $|\operatorname{rep}_{U_\beta}(n+C+1)| = |\operatorname{rep}_{U_\beta}(n+C)|$. The normal-$U$ representation preserves the order, i.e., for all integers m_1 and m_2, $m_1 \leq m_2$ if and only if $\operatorname{rep}_{U_\beta}(m_1) \leq \operatorname{rep}_{U_\beta}(m_2)$ (see, for instance, [3]). Thus, the words $\operatorname{rep}_{U_\beta}(n), \ldots, \operatorname{rep}_{U_\beta}(n+C-1)$ are prefixes of $d_\beta^*(1)$, respectively of length $|\operatorname{rep}_{U_\beta}(n)|, |\operatorname{rep}_{U_\beta}(n)| + 1, \ldots, |\operatorname{rep}_{U_\beta}(n)| + C - 1$ (the prefixes of $d_\beta^*(1)$ are the maximal words of different length in L_{U_β}). By Definition 35, we deduce that $C \leq C_\beta$. Consequently, there are at most $C_\beta + 1$ consecutive words ending with 0 in L_{U_β}. $\square$

Let us illustrate the previous proposition.

Example 37. Let φ be the golden ratio. Then $C_\varphi = 1$ since $d_\varphi^*(1) = (10)^\omega$. The first few words of L_{U_φ} are $\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, 10000, 10001, \ldots$. The maximal number of consecutive words ending with 0 in L_{U_φ} is $2 = C_\varphi + 1$.

Example 38. Let β be the dominant root of the polynomial $P(X) = X^4 - X^3 - 1$. Then $\beta \approx 1.38028$ is a Parry number with $d_\beta(1) = 1001$ and $d_\beta^*(1) = (1000)^\omega$. The automaton $\mathcal{A}_\beta$ is depicted in Figure 11. In this example, $C_\beta = 3$. The first few words of L_{U_β} are $\varepsilon, 1, 10, 100, 1000, 10000, 10001, \ldots$. The maximal number of consecutive words ending with 0 in L_{U_β} is $4 = C_\beta + 1$.

Lemma 39. *Let $\epsilon > 0$. For all $(x, y) \in \mathcal{L}^\beta$, $d((x, y), \mathcal{U}_n^\beta) < \epsilon$ for all sufficiently large n.*

Proof. Let $\epsilon > 0$ and let $(x, y) \in \mathcal{L}^\beta$. As in the proof of [9, Lemma 28], there exist nonnegative integers N_1, i, j with $0 \leq j \leq i \leq N_1$, a pair of words $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$, and $(x_0, y_0) \in S_{u,v}$ such that $d((x, y), h^j(c^i((x_0, y_0)))) < \epsilon/2$. Now we will show that $d(h^j(c^i((x_0, y_0))), \mathcal{U}_n^\beta) < \epsilon/2$ for all sufficiently large n, which completes the proof. We will make use of the constants i, j, the words u, v given above and the integer $p := p(u, v)$. Set

$$L_{u,v} := \begin{cases} L_{U_\beta}, & \text{if } u = v = \varepsilon; \\ 0^* L_{U_\beta}, & \text{otherwise.} \end{cases}$$

Since $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfies $(\star)$, the pair of words $(u0^p w, v0^p w)$ has an odd binomial coefficient, for all words $w \in L_{u,v}$, using Lemma 20 and Corollary 23. In particular, this is the case when $w \in L_{u,v}$ is of length n. We can choose n sufficiently large such that $U_\beta(n) \geq C_\beta + 3$ using Proposition 36. In this case, there exist at least two words $w \in L_{u,v}$ with $|w| = n$ and not ending with 0. Furthermore, as soon as w does not end with 0, Lemma 1 shows that

$$\binom{u0^p w0^k}{v0^p w} = \binom{u0^p w}{v0^p w} \equiv 1 \bmod 2 \quad \forall\, k \geq 0.$$

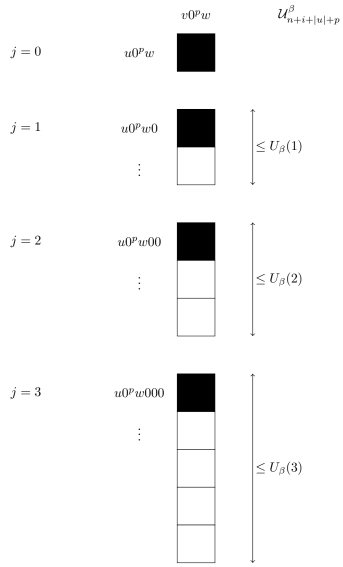


Figure 12: If w does not end with 0 and is such that $|w| = n$, then $\binom{u0^pw0^j}{v0^pw}$ being odd creates a square region in $\mathcal{U}^{\beta}_{n+i+|u|+p}$.

By definition of the sequence U_β, we also have

$$\#\{z \in 0^*L_{U_\beta} \mid u0^pwz \in L_{U_\beta} \text{ and } |z| = k\} \leq U_\beta(k).$$

Thus, for all $j \leq i$, we conclude that at least one of the $U_\beta(j)$ binomial coefficients of the form $\binom{u0^pwz}{v0^pw}$ with w not ending with 0 and $|z| = j$ is odd (indeed, choose $z = 0^j$ for instance). In other terms, at least one of the square regions

$$\left(\frac{\mathrm{val}_{U_\beta}(v0^pw)}{U_\beta(n+i+|u|+p)}, \frac{\mathrm{val}_{U_\beta}(u0^pwz)}{U_\beta(n+i+|u|+p)}\right) + \frac{Q}{U_\beta(n+i+|u|+p)}, \text{ with } |z| = j, \tag{6}$$

is a subset of $\mathcal{U}^{\beta}_{n+i+|u|+p}$, since $|v0^pw|, |u0^pwz| \leq n+i+|u|+p$. This can be visualized in Figure 12.

Now observe that, for any word $w \in L_{u,v}$, each square region of the form (6) is intersected by $h^j(c^i(S_{u,v}))$. Indeed, the latter segment has $A := (0.0^{i+|u|-|v|}v, 0.0^{i-j}u)$ and $B := (0.0^{i+|u|-|v|}v0^pd^*_\beta(1), 0.0^{i-j}u0^pd^*_\beta(1))$ as endpoints and slope β^j. Using (2), if n is sufficiently large, the points

$$\left(\frac{\mathrm{val}_{U_\beta}(v0^p0^n)}{U_\beta(n+i+|u|+p)}, \frac{\mathrm{val}_{U_\beta}(u0^p0^{n+j})}{U_\beta(n+i+|u|+p)}\right) \left(\text{resp., } \left(\frac{\mathrm{val}_{U_\beta}(v0^pd_n)}{U_\beta(n+i+|u|+p)}, \frac{\mathrm{val}_{U_\beta}(u0^pd_{n+j})}{U_\beta(n+i+|u|+p)}\right)\right)$$

and A (resp., B) are close for all $j \leq i$, where d_n denotes the prefix of length n of $d^*_\beta(1)$ for all $n \geq 0$. When u and v are non-empty, this can be seen in Figure 13 where each rectangular gray region contains at least one square region from $\mathcal{U}^{\beta}_{n+i+|u|+p}$ (to draw this picture, we take the particular case of the golden ratio φ and $i = 2$). When $u = v = \varepsilon$, Figure 13 is modified in the following way: simply replace each word of the forms $u0^\ell$, $v0^\ell$ by ε.

Consequently, every point of $h^j(c^i(S_{u,v}))$ is at distance at most

$$\frac{2 \cdot (C_\beta + 2) \cdot U_\beta(j)}{U_\beta(n+i+|u|+p)}$$

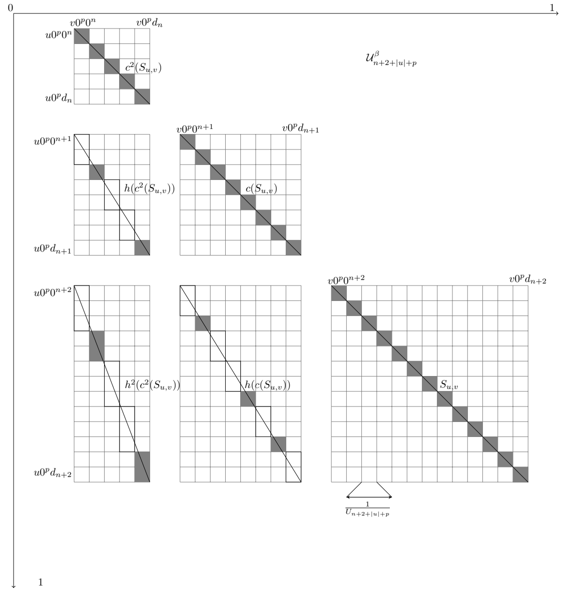


Figure 13: The situation occurring in the proof of Lemma 39, where we choose β to be the golden ratio.

from a point in $\mathcal{U}^{\beta}_{n+i+|u|+p}$ when n is sufficiently large. Indeed, either the point falls into a gray region from Figure 13, or not. In the first case, the point is at distance at most $U_\beta(j)/U_\beta(n+i+|u|+p)$ from a square region in $\mathcal{U}^{\beta}_{n+i+|u|+p}$; see Figure 12. Observe that this square region is of the form (6) where w does not end with 0. Otherwise, the point falls into a (white) square region of the form

$$\left(\frac{\mathrm{val}_{U_\beta}(v0^pw)}{U_\beta(n+i+|u|+p)}, \frac{\mathrm{val}_{U_\beta}(u0^pw'z)}{U_\beta(n+i+|u|+p)}\right) + \frac{Q}{U_\beta(n+i+|u|+p)}, \text{ with } |w| = |w'| = n, |z| = j.$$

Since n is large enough, there exists a word w'' not ending with 0 with $|w''| = n$, which is within a distance of $2 \cdot (C_\beta + 2)$ of w and w'. Then, applying the argument from the previous case proves the statement.

In particular, the result holds for the point $h^j(c^i((x_0, y_0)))$ belonging to $h^j(c^i(S_{u,v}))$. Hence, for all sufficiently large n, $d(h^j(c^i((x_0, y_0))), \mathcal{U}^{\beta}_n) < \epsilon/2$, and the conclusion follows. □

Corollary 40. *Let $(u, v) \in L_{U_\beta} \times L_{U_\beta}$ satisfying $(\star)$ and let $0 \le j \le i$. For every point (f, g) of the segment $h^j(c^i(S_{u,v}))$, there exists a sequence $((f_n, g_n))_{n\ge0}$ converging to (f, g) and such that $(f_n, g_n) \in \mathcal{U}^{\beta}_n$ for all n.*

Proof. Let (f, g) be a point of the segment $h^j(c^i(S_{u,v}))$. From the proof of Lemma 39, we have

$$d((f, g), \mathcal{U}^{\beta}_m) \le \frac{2 \cdot (C_\beta + 2) \cdot U_\beta(j)}{U_\beta(m)}$$

for all sufficiently large m. Consequently, there exists a sequence $((f_n, g_n))_{n\ge0}$ converging to (f, g) and such that $(f_n, g_n) \in \mathcal{U}^{\beta}_n$ for all n. □

The proof of the following main theorem is the same as the one of [9, Theorem 29], and uses Lemmas 34 and 39, and Corollary 40.

Theorem 41. *The sequence $(\mathcal{U}^{\beta}_n)_{n\ge0}$ converges to $\mathcal{L}^{\beta}$.*

As a final comment, let us mention that the extension to any prime number holds true and one simply has to adapt all the results, as in [9, Section 5].

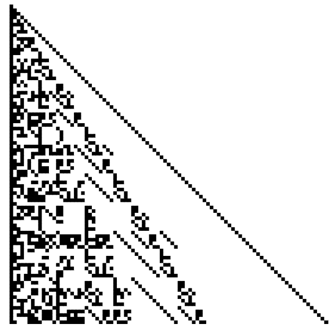


Figure 14: The set $\mathcal{U}_9^{\varphi}$.

6 Appendix

Example 42. We have represented the set $\mathcal{U}_9^{\varphi}$ in Figure 14.

Example 43. Let us consider the case when $\beta = \varphi$ is the golden ratio. We have represented in Figure 15 $\mathcal{U}_{9,2}^{\varphi}$ when considering binomial coefficients congruent to 2 modulo 3 (instead of odd coefficients) and an approximation of the limit set $\mathcal{L}^{\varphi}$ proceeding as in Example 33.

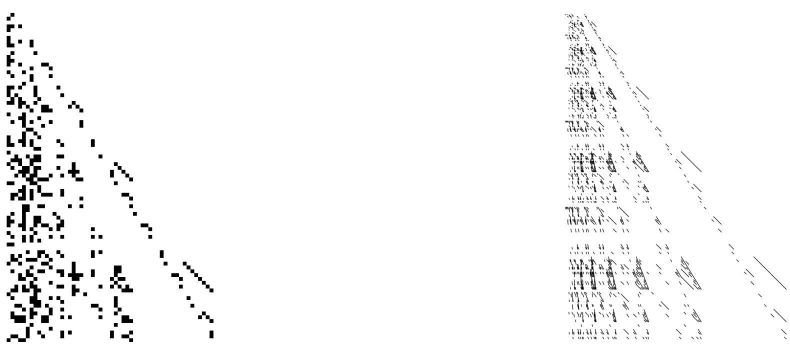


Figure 15: The set $\mathcal{U}_{9,2}^{\varphi}$ (on the left) and an approximation of the corresponding limit set $\mathcal{L}^{\varphi}$ (on the right).

In this last example, we give an approximation of the limit object $\mathcal{L}^{\beta}$ for several different values of β. A real number $\beta > 1$ is a *Pisot number* if it is an algebraic integer whose conjugates have modulus less than 1.

Example 44. Let us define several Parry numbers. Let $\beta_1 \approx 2.47098$ be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - X^2 - 1$, which is a Parry and Pisot number; see Example 18. Let $\beta_2 \approx 1.38028$ be the dominant root of the polynomial $P(X) = X^4 - X^3 - 1$, which is a Parry and Pisot number; see Example 38. Let $\beta_3 \approx 2.80399$ be the dominant root of the polynomial $P(X) = X^4 - 2X^3 - 2X^2 - 2$. We can show that β_3 is a Parry number, but not a Pisot number. Let $\beta_4 \approx 1.32472$ be the dominant root of the polynomial $P(X) = X^5 - X^4 - 1$. We can show that β_4 is a Parry number and also the smallest Pisot number. In Figure 16, we depict an approximation of $\mathcal{L}^{\beta}$ for β in $\{\varphi^2, \beta_1, \dots, \beta_4\}$.

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References

[1] H. Belbachir, L. Németh, and L. Szalay, Hyperbolic Pascal triangles, *Appl. Math. Comput.* **273** (2016), 453–464.

[2] H. Belbachir and L. Szalay, On the arithmetic triangles, *Šiauliai Math. Semin.* **9** (2014), no. 17, 15–26.

[3] V. Berthé, M. Rigo (Eds.), *Combinatorics, automata and number theory*, Encycl. of Math. and its Appl. **135**, Cambridge University Press, 2010.

[4] A. Bertrand-Mathis, Comment écrire les nombres entiers dans une base qui n'est pas entière, *Acta Math. Hungar* 54 (1989), 237–241.

[5] É. Charlier, N. Rampersad, M. Rigo, L. Waxweiler, The minimal automaton recognizing $m\mathbb{N}$ in a linear numeration system, *Integers* **11B** (2011), Paper No. A4, 24 pp.

[6] K. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, New York, 1985.

[7] F. von Haeseler, H.-O. Peitgen, and G. Skordev, Pascal's triangle, dynamical systems and attractors, *Ergod. Th. & Dynam. Sys.* **12** (1992), 479–486.

[8] É. Janvresse, T. de la Rue, and Y. Velenik, Self-similar corrections to the ergodic theorem for the Pascal-adic transformation, *Stoch. Dyn.* **5** (2005), no. 1, 1–25.

[9] J. Leroy, M. Rigo, and M. Stipulanti, Generalized Pascal triangle for binomial coefficients of words, *Adv. in Appl. Math.* **80** (2016), 24–47.

[10] J. Leroy, M. Rigo, and M. Stipulanti, Counting the number of non-zero coefficients in rows of generalized Pascal triangles, *Discrete Math.* **340** (2017), 862–881.

[11] J. Leroy, M. Rigo, and M. Stipulanti, Behavior of digital sequences through exotic numeration systems, *Electron. J. Combin.* **24** (2017), no. 1, Paper 1.44, 36 pp.

[12] J. Leroy, M. Rigo, and M. Stipulanti, Counting Subword Occurrences in Base-b Expansions, *Integers* **18A** (2018), Paper No. A13, 32 pp.

[13] M. Lothaire, *Combinatorics on Words*, Cambridge Mathematical Library, Cambridge University Press, 1997.

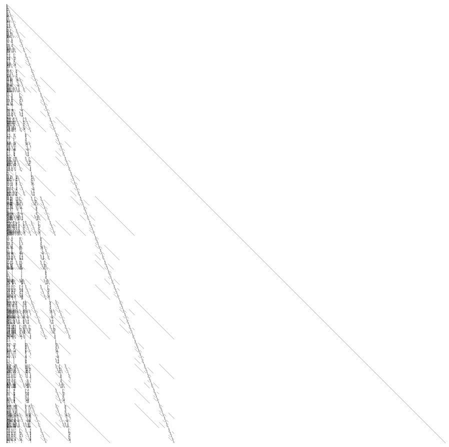
[14] M. Lothaire, *Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, vol. 90, 2002.

[15] W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.

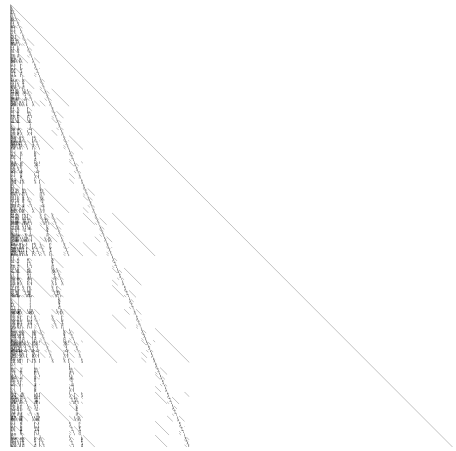
[16] M. Rigo, *Formal languages, automata and numeration systems. 1. Introduction to combinatorics on words*, ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2014.

[17] M. Rigo, *Formal languages, automata and numeration systems. 2. Applications to recognizability and decidability*, ISTE, London; John Wiley & Sons, Inc., Hoboken, NJ, 2014.

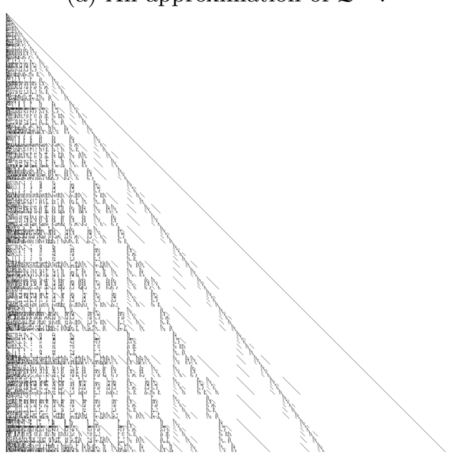
[18] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Published electronically at `http://oeis.org`, 2018.



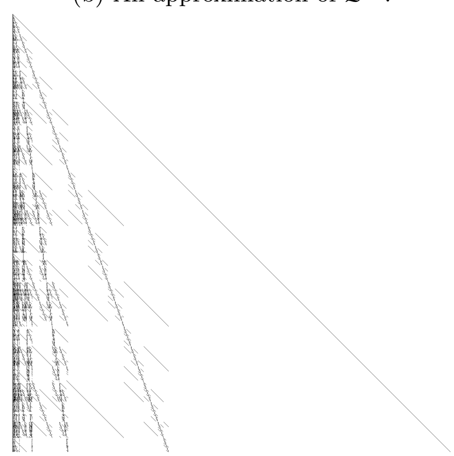
(a) An approximation of $\mathcal{L}^{\varphi^2}$.



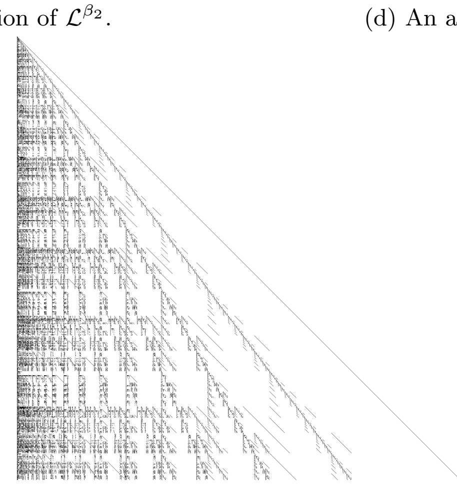
(b) An approximation of $\mathcal{L}^{\beta_1}$.



(c) An approximation of $\mathcal{L}^{\beta_2}$.



(d) An approximation of $\mathcal{L}^{\beta_3}$.



(e) An approximation of $\mathcal{L}^{\beta_4}$.

Figure 16: An approximation of the limit object $\mathcal{L}^{\beta}$ for different values of β.