Spur Gear Vibration Mitigation by Means of Energy Pumping

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ABSTRACT
The dynamic behaviour of a spur gear pair can be studied in terms of transmission error considering a single degree of freedom system. Thus, a mechanical system exhibiting combined parametric excitation and clearance type nonlinearity is examined by means of numerical integrations and continuation methods in an effort to explain its complex behaviour, as it is commonly observed in the steady state forced response of rotating machines. The specific case of a preloaded mechanical oscillator having a periodically time varying stiffness function and subject to a symmetric backlash condition is considered. Even if such an oscillator represents the simplest model able to analyze a single spur gear pair, it exhibits a complex dynamic scenario, namely jumps, superharmonics, subharmonics resonances and dynamic bifurcations. In order to reduce the vibration of the system, a nonlinear absorber is applied. Unlike common linear and weakly nonlinear systems, systems with strongly nonlinear elements are able to react efficiently on the amplitude characteristics of the external forcing in a wide range of frequencies. A strongly and essentially nonlinear, lightweight, with cubic stiffness oscillator is then attached to the main nonlinear system under periodic parametric forcing and the feasibility of a possible application of this nonlinear energy sink (NES) for vibration absorption and mitigation is checked.

NOMENCLATURE

b	backlash
c	damping coefficient of the primary system
c_2	damping coefficient of the absorber
d_{gi}	base diameter of wheel i
f_1, f_2	smoothing functions
F_e	equivalent force
I_{gi}	rotary inertia of wheel i
I_3	rotary inertia of the absorber
I_e	equivalent inertia
$k(t)$	mesh stiffness
$k_{bs}(t)$	back side contact mesh stiffness
k_{LIN}	stiffness of the TMD
k_{NL}	stiffness of the NES
m_e	equivalent mass
STE	static transmission error
T_{gi}	torque on wheel i
T_e	equivalent torque
$x(t)$	dynamic transmission error
ζ	damping ratio of the primary system
θ	relative rotation between absorber and driven gear
θ_{gi}	angular position of wheel i
λ	smoothing shape parameter
ω	mesh frequency
ω_n	natural frequency of the primary system

1 INTRODUCTION

The reduction of vibration and noise is one of the main issue in spur gear design. Literature offers many approaches to evaluate the dynamic behaviour of such systems and their design optimization.
A strong interaction between noise and dynamic transmission error has been clearly proved; several experiments on gear systems have shown that nonlinear phenomena occur when the dynamic transmission error is present: multiple coexisting stable motions, superharmonic resonances, fold bifurcations, long period subharmonic and chaotic motions. [9]
In the last twenty years most of the spur gears dynamic models of spur gears were focused on non-linear aspects. Kahraman and Singh considered the effect of backlash and time varying mesh stiffness using harmonic balance method and digital simulation [10]. A similar model was developed by Theodossiades and Natsiavas who predicted chaotic behaviour by means of numerical integration: intermittent chaos and boundary crises [16]. Cai and Hayashi proposed a linear approximation for a pair of spur gears and compared the analytical solution with the numerical obtained by the nonlinear equation [3]. Amabili and Rivola obtained a continuous closed form solution for any rotational speed and computed the transition curves, stable and unstable regions, by means of the Hill infinite determinant [1]. Parker et al. studied the nonlinear dynamic response of a spur gear pair using a semianalytical approach and two different single degree of freedom models [14].
The present paper addresses the vibration mitigation of a spur gears pair. To this end, a single degree of freedom oscillator with clearance-type nonlinearity, periodic stiffness and constant viscous damping is considered. Such an oscillator represents the simplest model able to analyze a single spur gear pair, neglecting: bearings, shafts stiffness and multi mesh interactions. A test case considered in the following sections represents an actual gear pair that belongs to a gear box of an agricultural vehicle; such a gear pair gave rise to noise problems.
A shooting technique, based on direct numerical integration, is used to compute the steady state response. Besides, continuation of periodic solutions is carried out to obtain the complete dynamic behaviour by detecting instability regions and bifurcation points.
The feasibility of vibration mitigation by means of torsional dynamic absorbers is then checked, comparing a linear absorber, termed linear Tuned Mass Damper (TMD), and a purely nonlinear attachment, termed Nonlinear Energy Sink (NES).

2 DYNAMIC MODEL OF A SPUR GEAR PAIR

The dynamic model used in this paper to represent a spur gear pair is the simple lumped mass system showed in Fig. 1. Such a model considers spur gears as rigid disks, coupled along the line of action through a time varying mesh stiffness *k(t)* and a constant mesh damping *c*; θ_{g1} is the angular position of the driver wheel (pinion), θ_{g2} is the angular position of the driven wheel (gear); T_{g1} is the driving torque, T_{g2} is the breaking torque; $I_{g1}$and I_{g2} are the rotary inertias; d_{g1} and d_{g2} are the base diameters.

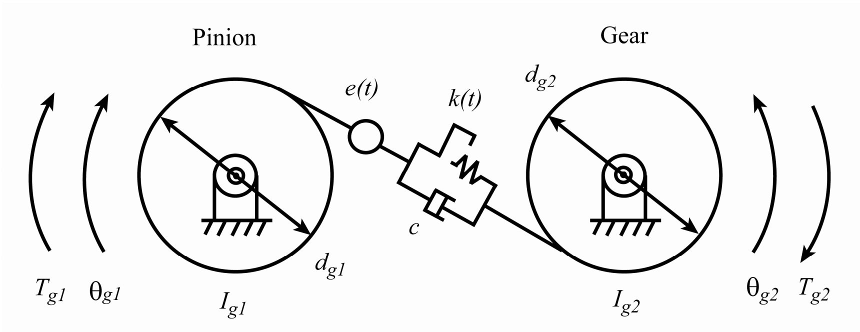

Figure 1 – Dynamic model of a spur gear pair

According to literature [10], as long as manufacturing errors are neglected, the relative dynamics of gears along the line of action can be represented by the following equation of motion:

$$m_e\ddot{x}(t) + c\dot{x}(t) + k(t)f_1(x(t)) + k_{bs}(t)f_2(x(t)) = F_e \tag{2.1}$$

where m_e is the equivalent mass:

$$m_e = \left(\frac{d_{g1}^2}{4I_{g1}} + \frac{d_{g2}^2}{4I_{g2}}\right) \tag{2.2}$$

F_e is the equivalent applied preload:

$$F_e = m_e\left(\frac{d_{g1}T_{g1}}{2I_{g1}} + \frac{d_{g2}T_{g2}}{2I_{g2}}\right) \tag{2.3}$$

In the following T_{g1}, T_{g2}, and therefore F_e are assumed to be constant.
The dynamic transmission error *x(t)* along the line of action is defined as:

$$x(t) = \frac{d_{g1}}{2}\theta_{g1}(t) - \frac{d_{g2}}{2}\theta_{g2}(t) \tag{2.4}$$

where $k_{bs}(t)$ is the back side contact stiffness.
Smoothing backlash functions are considered in order to simulate clearances:

$$\begin{aligned} f_1(x) &= \frac{1}{2}\{[x(t)-b][1+\tanh(\lambda(x(t)-b))]\} \\ f_2(x) &= \frac{1}{2}\{[x(t)+b][1+\tanh(-\lambda(x(t)-b))]\} \end{aligned} \tag{2.5}$$

where *2b* is the backlash along the line of action and λ is the shape parameter.
The gear pair mesh stiffness plays the role of a parametric excitation and along the line of action is given by

$$k(t) = \frac{2T_{g1}}{d_{g1}STE(t)} = \frac{4T_{g1}}{d_{g1}^2\delta(t)} \tag{2.6}$$

with

$$\delta(t) = \theta_1(t) - \frac{d_{g1}}{d_{g2}}\theta_2(t) \quad \text{and} \quad STE(t) = \frac{d_{g1}\delta(t)}{2} \tag{2.7}$$

where *δ(t)* is the difference between the nominal position of the pinion given by the exact kinematics and the actual position influenced by the teeth flexibility; *STE(t)* is the static transmission error along the line of action and is time dependent because, while meshing, the reciprocal position of wheels, the contact point and the number of teeth in contact can change. Parker et al. described a methodology to compute $\theta_1(t)$ and $\theta_2(t)$, referred as the rotational DOFs of the pinion and the gear, for small ''rigid-body'' motions [14]; this approach is followed here for static analyses to evaluate the mesh stiffness.
Since no manufacturing errors are included, i.e. *e(t)=0*, the mesh stiffness is periodic within a mesh cycle and therefore it can be expanded in terms of Fourier series:

$$k(t) = k_0 + \sum_{j=1}^{N} k_j \cos(j\omega t - \varphi_j) \tag{2.8}$$

where ω is the mesh circular frequency. Amplitudes k_j and phases φ_j are obtained using the Discrete Fourier Transform (DFT). The number of samples *n* is related to the number of harmonics $N = (n-1)/2$; in the following, $n = 15$ is considered to ensure enough accuracy in the expansion.
Similarly, we have:

$$k_{bs}(t) = k_0 + \sum_{j=1}^{N} k_j \cos(-(j\omega t - \varphi_j + j\frac{s_{ts,1}}{d_{g1}})) \tag{2.9}$$

where $s_{ts,1}$ is the thickness of the pinion tooth at the pitch operating diameter [2].
Nondimensionalized form of equation (2.1) is obtained by considering the following manipulation:

$$\omega_n = \sqrt{\frac{k_0}{m_e}}; \qquad \zeta = \frac{c}{2m_e\omega_n}; \qquad \tau = \omega_n t; \qquad \bar{F}_e = \frac{F_e}{bm_e\omega_n^2}; \qquad \bar{x} = \frac{x}{b}; \qquad \bar{k}_j = \frac{k_j}{m_e\omega_n^2} \tag{2.10}$$

and it comes:

$$\bar{x}''(\tau)+2\zeta\bar{x}'(\tau)+\bar{k}(\tau)\bar{f}_1(\bar{x}(\tau))+\bar{k}_{bs}(\tau)\bar{f}_2(\bar{x}(\tau))=\bar{F}_e \quad (2.11)$$

3 BIFURCATION ANALISYS

The complete characterization of the dynamical behavior of system (2.11) is performed by considering a bifurcation analysis carried out by means of the continuation software MatCont [5].
A shooting method for finding periodic solutions of the system is used, based on direct numerical time integration and the Newton-Raphson algorithm.
According to this method, widely discussed in literature [13], the initial value problem is converted into a two-point boundary-value problem.
Starting from the state space formulation of the nonautonomous system :

$$\dot{\mathbf{x}}=f(\mathbf{x},t) \quad (3.1)$$

the shooting method allows to find out periodic solutions of the system by solving a two-point boundary value problem, defined by a periodicity condition, commonly formulated as follows:

$$\mathbf{F}(\mathbf{x}_0,T)=\mathbf{x}(T,\mathbf{x}_0)-\mathbf{x}_0=0 \quad (3.2)$$

where T is the period time of the periodic solution and $\mathbf{x}_0$ the initial condition vector. The system (3.2) is solved with a Newton – Raphson iterative method.
Once the periodic solution $\mathbf{x}(T,\mathbf{x}_0)$ is determined, the stability analysis can be performed by studying the Floquet multipliers, that correspond to the $2n$ eigenvalues of the system Monodromy matrix evaluated at the period T.
To compute a complete frequency response of the system, including bifurcation points, a continuation of periodic solutions is carried out, spanning the excitation frequency broadband. The family of the existing periodic orbits for sweeping frequency corresponds to a curve in the $2n+1$ dimensional space $\mathbf{x}(T,\mathbf{x}_0)$ of initial conditions and forcing period. In the present work, this curve is determined using the *Moore-Penrose continuation* method [5].
At each step of the continuation the presence of bifurcations is checked by using test functions that vanish when a bifurcation point is reached (for details, see [11])
Knowing the initial conditions of the periodic orbit for a given excitation frequency, a time integration over one period is computed and the amplitude over this time span is extracted to plot a Frequency – Amplitude diagram.
Table 1 summarizes the geometrical and physical parameters of an actual spur gear pair selected as test case.

Data	Pinion	Gear
Base radius $d_{g1}/2$, $d_{g2}/2$ [mm]	39.467	60.610
Inertias I_{g1}, I_{g2} [kg m^2]	0.0008	0.0048
Applied Torques T_{g1}, T_{g2} [Nm]	470	0.0048
Backlash 2*b* on line of action [mm]	0.312	
Backside stiffness phase [rad]	1.594	
Damping ratio (ζ)	0.03	

Table 1 – Test case data

In this analysis, the frequency of the parametric excitation is varied from $0.1\omega_n$ to $2.5\omega_n$ in order to include superharmonic motions and the subharmonic parametric resonance.
The Frequency – Amplitude diagram, depicted in Figure 2, provides a complete scenario of the dynamics of the system. Stable and unstable branches of periodic solutions are plotted using solid and dotted line, respectively, whereas bifurcation points are denoted by red stars.
The system (2.11) shows a complex dynamic behavior and some particular nonlinear phenomena are observed. At low frequencies, superharmonic resonances appear, when $\omega = \omega_n/j$ (with $j = 1,..,N$), due to the harmonics of the mesh stiffness Fourier expansion.
The system also exhibits a fundamental resonance with softening behavior due to loss of contact, induced by the presence of backlash. Close to ω_n the vibration amplitude reaches high levels and the system experiences a loss of stability through limit point bifurcations (labelled LP in Figure 1).

It is also well known that parametrically excited systems can undergo a parametric instability that gives rise to *2T* response. The parametric instability takes place close to $\omega = 2\omega_n$ trough period doubling bifurcation (labelled PD in Figure 1) and leads to a branch of stable period doubled solutions with high amplitude levels. Similarly to the fundamental, the parametric resonance presents a softening behavior as well as fold bifurcations.

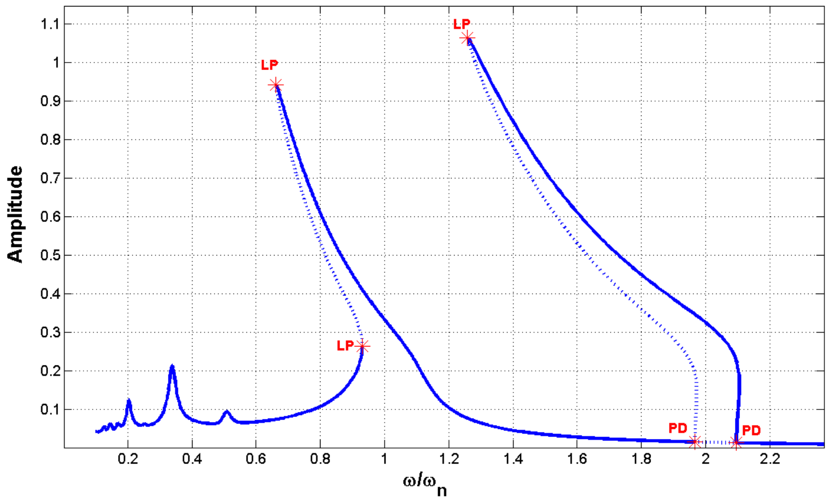

Figure 2 – FRF of the primary system (— stable, ···· unstable)

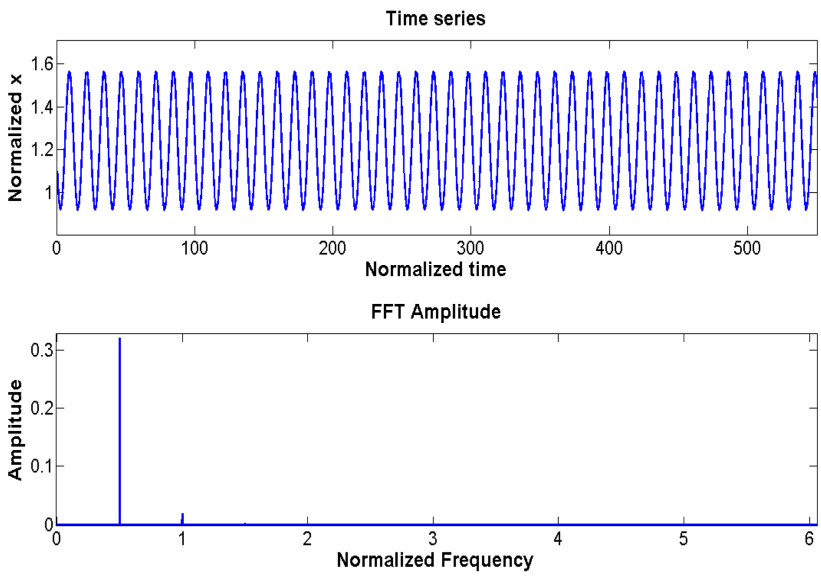

Figure 3 – Time series and FFT of stable steady state response at $2\omega_n$

Figure 3 shows the time series and the Fourier spectrum of the stable steady state response at $\omega = 2\omega_n$. The spectrum reflects that in the period doubling frequency range, the steady state response of the system has frequency equal to one half the mesh frequency ω.

From a practical viewpoint, it is to stress that the system under investigation exhibits undesired high levels of vibration and coexistence of multiple stable solutions. Such features turn out to be dangerous and represent some of the most important causes of radiated noise by a gearbox.

In the following section the same procedure is used to address the dynamics of the system coupled to a torsional vibration absorbers.

4 COUPLING WITH VIBRATION ABSORBERS:

The TMD is currently the most popular solution for passive vibration isolation and consists in an additional linear SDOF oscillator coupled to the primary system for the purpose of attenuating oscillation over a narrow frequency range centered at the natural frequency of the absorber [6]. The effective bandwidth of the TMD depends on its damping and a trade-off exists between attenuation efficiency and bandwidth [4]. However a linear absorber poses problems when the external forcing frequency is not fixed and the system is submitted to mistuning.

Recently it has been demonstrated that various systems consisting of linear primary structure and strongly nonlinear attachments (NES) show localization and irreversible energy pumping (Targeted Energy Transfer) to prescribed parts of the structure dependent on initial conditions and external forcing [12, 7, 17]. Addition of a relatively small and spatially localized attachment leads to essential changes in the properties of the whole system. Unlike common linear systems, systems with strongly nonlinear elements are able to react efficiently on the amplitude characteristics of the external forcing in a wide range of frequencies [15]. It has been demonstrated that in close vicinity of the main resonance the linear system with NES can exhibit quasiperiodic, rather than steady-state response, leading to qualitatively different dynamical behaviour (repetitive Targeted Energy Transfer).

In the present section the application of an NES to the nonlinear primary system is considered. It is to mention that in this study, unlike previous works, the NES is applied to a nonlinear primary structure subjected to periodic forcing. The objective lies in studying an expected frequency broadband vibration mitigation. As a comparison, the system coupled to a TMD is also investigated.

The absorber consists in a rotational wheel of inertia I_3 added on the shaft of the driven gear and coupled by a spring and a viscous dashpot.

The equations of motion of the new system under investigation are given by the following expression:

$$\begin{cases} m_e\ddot{x}(t) + c_1\dot{x}(t) + k(t)f_1(x(t)) + k_{bs}(t)f_2(x(t)) - \alpha c_2\dot{\theta}(t) - \alpha g(\theta(t)) = F_e \\ J_e\ddot{\theta}(t) - \beta c_1\dot{x}(t) - \beta\left[k(t)f_1(x(t)) + k_{bs}(t)f_2(x(t))\right] + c_2\dot{\theta}(t) + g(\theta(t)) = -T_e \end{cases} \tag{4.1}$$

where

$$I_e = \left(\frac{1}{I_{g2}} + \frac{1}{I_3}\right)^{-1}; \qquad \alpha = \frac{m_e d_{g2}}{2J_{g2}}; \qquad \beta = \frac{J_e d_{g2}}{2J_2}; \qquad T_e = \frac{I_e T_{g2}}{I_{g2}} \tag{4.2}$$

θ(t) is the relative rotation between the driven gear and the absorber:

$$\theta = \theta_{g2} - \theta_3 \tag{4.3}$$

and *g(θ(t))* is the elastic reaction provided by the absorber:

$$g(\theta(t)) = k_{LIN}\theta(t) \text{ for the TMD} \tag{4.4a}$$

$$g(\theta(t)) = k_{NL}\theta(t)^3 \text{ for the NES} \tag{4.4b}$$

The stiffness of the linear TMD is tuned on the natural frequency of the system (1.1), i.e. such that the relation:

$$k_{LIN} = \omega_n^2 I_e \tag{4.5}$$

is fulfilled.

The TMD damping coefficient for is set in order to have the same damping ratio as the primary system, namely

$$c_2 = 2\zeta\sqrt{I_e k_{LIN}} \tag{4.6}$$

The same value of damping coefficient is chosen for the NES, whereas the cubic stiffness is parametrically chosen, the objective being the occurrence of quasiperiodic regimes of motion.

Table 2 collects the values of the parameters used to carry out the computation on system (4.1).

I_3 [kg m^2]	$0.4I_{g2}$
c_2 [Nm·s/rad]	2.5
k_{LIN} [Nm/rad]	$1.2\text{x}10^6$
k_{NL} [Nm/rad^3]	$5\text{x}10^{12}$

Table 2 – Absorbers data

The inertia of the absorber is fixed to 40% of I_{g2}. In practice, this value may seem to be large; however, the focus of the study is set on the new dynamics created by the introduction of an NES on a nonlinear primary structure and not on the optimization of the NES parameters.

Figure 4 illustrates the steady state response of the system with the TMD (black line) compared to the frequency response of the system (1.1) computed in the previous section (blue line).

As expected, the TMD turns out to be very efficient in the neighbourhood of the natural frequency, but two different peaks appear around the tuning frequency. The same behavior can be observed close to the superharmonic resonances.

An important feature is that the parametric instability can be completely avoided, so that the subharmonic resonance does not occur anymore. Conversely, the presence of a softening peak with loss of stability through fold bifurcations clearly limits its efficiency and suggests the investigation of the application of a purely nonlinear attachment.

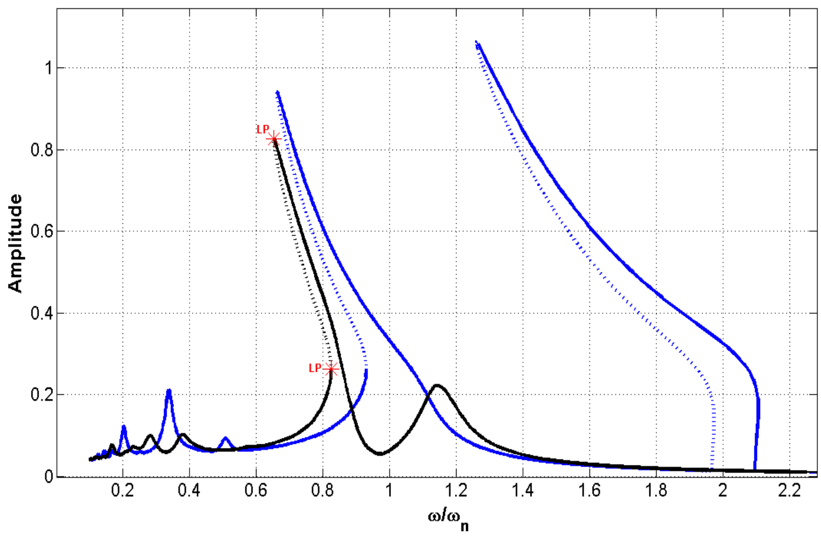

Figure 4 – FRF of system with the TMD (— stable, ···· unstable) compared to the primary system (— stable, ···· unstable)

In Figure 5 the continuation of periodic solutions of the system coupled with the NES is presented and overlapped to the response of the system without any attachment.

The addition of the NES results in a more complicated dynamics of the system (4.1). Between $0.95\omega_n$ and $1.1\omega_n$, the system possesses quasiperiodic regimes reached through a Neimark-Sacker bifurcation (labelled NS in Figure 5) that allow to cut the peak corresponding to the main resonance via energy pumping mechanism from the primary system to the nonlinear absorber. In [8], it is demonstrated that for a linear oscillator coupled with a NES quasiperiodic beating regimes with strongly modulated response can provide more efficient energy suppression than a steady-state response. In the present application the same mechanism seems to be activated, providing a broadband mitigation.

It is to stress that the parametric instability with flip bifurcation is still present but even around $2\omega_n$ quasiperiodic attractors exist and targeted energy transfer is achieved.

Quasiperiodic motions with strong modulation at ω_n and $2\omega_n$ are depicted and compared to the steady state response of the primary system without any absorber in Figure 6 and 7.

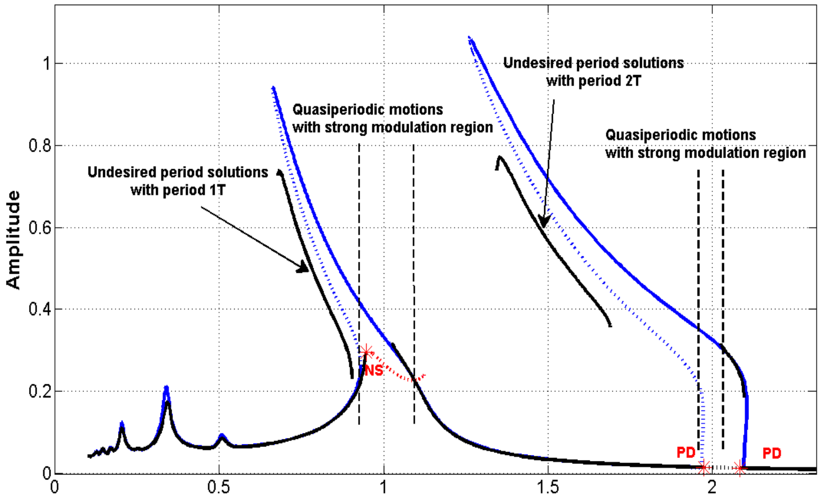

Figure 4 – FRF of system with the NES (— stable, ···· unstable, ···· quasiperiodic) compared to the primary system (— stable, ···· unstable)

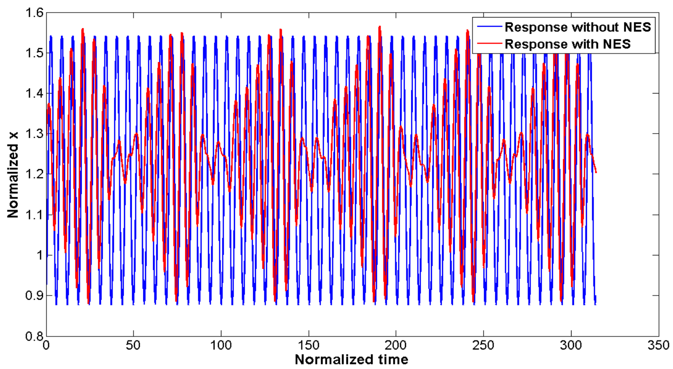

Figure 6 – Quasiperiodic motion brought by the NES at ω_n

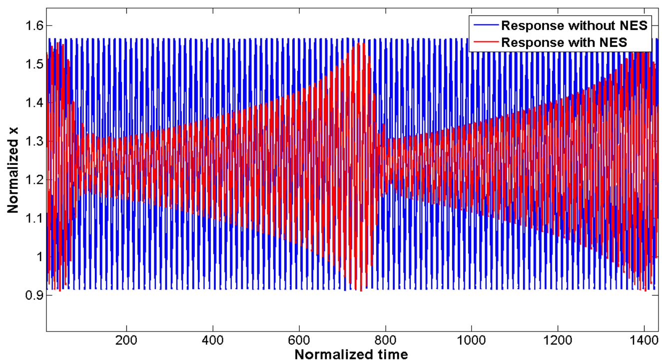

Figure 7 – Quasiperiodic motion brought by the NES at $2\omega_n$

The frequency response of the system with the NES also highlights that undesired high amplitude branch of periodic solutions coexist with lower amplitude steady state responses. This means that the solution can be attracted to undesired limit cycles depending on the initial conditions.

The bifurcation analysis carried out in this section reveals that it is possible to activate energy pumping mechanism by introducing a purely nonlinear absorber, but undesired branches of periodic solutions are present, so that further studies on the possibility to avoid this limit cycle oscillations are required.

5 CONCLUSIONS

The nonlinear dynamics of a spur gear pair is studied considering a parametrically excited and piecewise linear single degree of freedom system that includes time varying stiffness and backlash.
A bifurcation analysis is carried out by using a continuation method and highlights a complex dynamical scenario, i.e. nonlinear resonances with softening behaviour and subharmonic instabilities.
This paper investigated the possibility of passively mitigating vibrations in a such nonlinear primary system using an absorber characterized by an essential stiffness nonlinearity. The motivation for using a NES is, unlike linear absorbers, its absence of preferential natural frequency, which enables it to resonate with and extract energy from the primary system in an a priori frequency-independent fashion and eventually dissipate it with a dashpot.
The study of the system coupled with the NES shows that it’s possible to activate a mechanism of repetitive targeted energy transfer that results in a quasiperiodic regime of motion. Hence the use of an NES seems promising for vibration mitigation of nonlinear primary systems such as a spur gear pair, since passive targeted energy transfers from the system to the NES can reduce the amplitude of vibration. However, the increased complexity of the dynamical behaviour of a system with an attached NES suggests further studies. In particular, with the configuration considered in this work undesired limit cycle oscillations coexisting with low amplitude periodic solutions are observed. This point, which is still under investigation, suggests to perform a study on the basins of attraction of the coexisting solutions and to investigate the possibility of developing a tuning procedure of the NES that enables the complete avoidance or a remarkable reduction of the undesired attractors.

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