Nonlinear System Identification in Structural Dynamics: Current Status and Future Directions

G. Kerschen$^{(1)}$**, K. Worden**$^{(2)}$**, A.F. Vakakis**$^{(3)}$**, J.C. Golinval**$^{(1)}$

(1) Aerospace and Mechanical Engineering Department, University of Liège, Belgium
E-mail: g.kerschen,jc.golinval@ulg.ac.be

(2) Dynamics Research Group, University of Sheffield, U.K.
E-mail: k.worden@sheffield.ac.uk

(3) Division of Mechanics, National Technical University of Athens, Greece
Department of Mechanical and Industrial Engineering (adjunct), University of Illinois at Urbana-Champaign, U.S.A.
E-mail: vakakis@central.ntua.gr, avakakis@uiuc.edu

ABSTRACT

Nonlinear system identification aims at developing high-fidelity mathematical models in the presence of nonlinearity from input and output measurements performed on the real structure. The present paper is a discussion of the recent developments in this research field. Three of the latest approaches are presented, and application examples are considered to illustrate their fundamental concepts, advantages and limitations. Another objective of this paper is to identify future research needs, which would make the identification of structures with a high modal density in a broad frequency range viable.

1 INTRODUCTION

The demand for enhanced and reliable performance of vibrating structures in terms of weight, comfort, safety, noise and durability is ever increasing while, at the same time, there is a demand for shorter design cycles, longer operating life, minimization of inspection and repair needs, and reduced costs. With the advent of powerful computers, it has become less expensive both in terms of cost and time to perform numerical simulations, than to run a sophisticated experiment. The consequence has been a considerable shift toward computer-aided design and numerical experiments, where structural models are employed to simulate experiments, and to perform accurate and reliable predictions of the structure's future behavior.

Even if we are entering the age of virtual prototyping, experimental testing and system identification still play a key role because they help the structural dynamicist to reconcile numerical predictions with experimental investigations. The term 'system identification' is sometimes used in a broader context in the technical literature and may also refer to the extraction of information about the structural behavior directly from experimental data, i.e., without necessarily requesting a model (e.g., identification of the number of active modes or the presence of natural frequencies within a certain frequency range). In the present paper, system identification refers to the development (or the improvement) of structural models from input and output measurements performed on the real structure using vibration sensing devices.

Linear system identification is a discipline that has evolved considerably during the last thirty years. Modal parameter estimation — termed modal analysis — is indubitably the most popular approach to performing linear system identification in structural dynamics. The popularity of modal analysis stems from its great generality; modal parameters can describe the behavior of a system for any input type and any range of the input. Numerous approaches have been developed for this purpose [1, 2]. It is important to note that modal identification of highly damped structures or complex industrial structures with high modal density and large modal overlap are now within reach.

The focus in this overview paper is on structural system identification in the presence of nonlinearity. Nonlinearity is generic in Nature, and linear behavior is an exception. In structural dynamics, typical sources of nonlinearities are:

- Geometric nonlinearity results when a structure undergoes large displacements and arises from the potential energy. Large deformations of flexible elastic continua such as beams, plates and shells are also responsible for geometric nonlinearities.
- Inertia nonlinearity derives from nonlinear terms containing velocities and/or accelerations in the equations of motion, and takes its source in the kinetic energy of the system (e.g., convective acceleration terms in a continuum and Coriolis accelerations in motions of bodies moving relative to rotating frames).
- A nonlinear material behavior may be observed when the constitutive law relating stresses and strains is nonlinear. This is often the case in foams [3] and in resilient mounting systems such as rubber isolators [4].
- Damping dissipation is essentially a nonlinear and still not fully modeled and understood phenomenon. The modal damping assumption is not necessarily the most appropriate representation of the physical reality, and its widespread use is to be attributed to its mathematical convenience. Dry friction effects (bodies in contact, sliding with respect to each other) and hysteretic damping are examples of nonlinear damping [5]. It is important to note that dry friction affects the dynamics especially for small-amplitude motion, which is contrary to what might be expected by conventional wisdom.
- Nonlinearity may also result due to boundary conditions (for example, free surfaces in fluids, vibro-impacts due to loose joints or contacts with rigid constraints, clearances, imperfectly bonded elastic bodies), or certain external nonlinear body forces (e.g., magnetoelastic, electrodynamic or hydrodynamic forces). Clearance and vibro-impact nonlinearity possesses nonsmooth force-deflection characteristic and generally requires a special treatment compared with other types of nonlinearities [6].

Many practical examples of nonlinear dynamic behavior have been reported in the engineering literature. In the automotive industry, brake squeal which is a self-excited vibration of the brake rotor related to the friction variation between the pads and the rotor is an irritating but non-life-threatening example of an undesirable effect of nonlinearity. Many automobiles have viscoelastic engine mounts which show marked nonlinear behavior: dependence on amplitude, frequency and preload. In an aircraft, besides nonlinear fluid-structure interaction, typical nonlinearities include backlash and friction in control surfaces and joints, hardening nonlinearities in the engine-to-pylon connection, and saturation effects in hydraulic actuators. In mechatronic systems, sources of nonlinearities are friction in bearings and guideways, as well as backlash and clearances in robot joints. In civil engineering, many demountable structures such as grandstands at concerts and sporting events are prone to substantial structural nonlinearity as a result of looseness of joints. This creates both clearances and friction and may invalidate any linear model-based simulations of the behavior created by crowd movement. Nonlinearity may also arise in a damaged structure: fatigue cracks, rivets and bolts that subsequently open and close under dynamic loading or internal parts impacting upon each other.

With continual interest to expand the performance envelope of structures at ever increasing speeds, there is the need for designing lighter, more flexible, and consequently, more nonlinear structural elements. It follows that the demand to utilize nonlinear (or even strongly nonlinear) structural components is increasingly present in engineering applications. Therefore, it is rather paradoxical to observe that very often linear behavior is taken for granted in structural dynamics. Why is it so ? It should be recognized that at sufficiently small-amplitude motions, linear theory may be accurate for modeling, although it is not always the case (e.g., dry friction). However, the main reason is that nonlinear dynamical systems theory is far less established than its linear counterpart. Indeed, the basic principles that apply to a linear system and that form the basis of modal analysis are no longer valid in the presence of nonlinearity. In addition, even weak nonlinear systems can exhibit extremely interesting and complex phenomena which linear systems cannot. These phenomena include jumps, bifurcations, saturation, subharmonic, superharmonic and internal resonances, resonance captures, limit cycles, modal interactions and chaos. Readers who look for an introduction to nonlinear oscillations may consult [7−10]. More mathematically inclined readers may refer to [11, 12]. A tutorial which emphasizes the important differences between linear and nonlinear dynamics is available in [13].

This is not to say that nonlinear systems have not received considerable attention during the last decades. Even if, for years, one way to study nonlinear systems was the linearization approach [14, 15], many efforts have been spent in order to develop theories for the investigation of nonlinear systems in structural dynamics. A nonlinear extension of the concept of mode shapes was proposed in [16, 17] and further investigated in [18−20]. Weakly nonlinear systems were thoroughly analyzed using perturbation theory [7]. Perturbation methods include for instance the method of averaging, the Lindstedt-Poincaré technique and the method of multiple scales and aim at obtaining asymptotically uniform approximations of the solutions. During the last decade or so, one has witnessed a transition from weakly nonlinear structures to strongly nonlinear structures (by strongly nonlinear systems, a system for which the nonlinear terms are the same order as the linear terms is meant) thanks to the extension of classical perturbation techniques.

Focusing now on the development (or the improvement) of structural models from experimental measurements in the presence of nonlinearity, i.e., nonlinear system identification, one is forced to admit that there is no general analysis method that can be applied to all systems in all instances, as it is the case for modal analysis in linear structural dynamics. In addition, many techniques which are capable of dealing with systems with low dimensionality collapse if they are faced with system with high modal density. Two reasons for this failure are the inapplicability of various concepts of linear theory and the highly 'individualistic' nature of nonlinear systems. A third reason is that the functional $S[\bullet]$ which maps the input $x(t)$ to the output $y(t)$, $y(t) = S[x(t)]$, is not known beforehand. For instance, the ubiquitous Duffing oscillator, the equation of motion of which is $m\ddot{y}(t)+c\dot{y}(t)+ky(t)+k_3y^3(t) = x(t)$, represents a typical example of polynomial form of restoring force nonlinearity, whereas hysteretic damping is an example of nonpolynomial form of nonlinearity. This represents a major difficulty compared with linear system identification for which the structure of the functional is well defined.

Even if there is a difference between the way one did nonlinear system identification 'historically' and the way one would do it now, the identification process may be regarded as a progression through three steps, namely detection, characterization and parameter estimation, as outlined in Figure 1. Once nonlinear behavior has been detected, a nonlinear system is said to be characterized after the location, type and functional form of all the nonlinearities throughout the system are determined. The parameters of the selected model are then estimated using linear least-squares fitting or nonlinear optimization algorithms depending upon the method considered.

Nonlinear system identification is an integral part of the verification and validation (V&V) process. According to [21], verification refers to solving the equations correctly, i.e., performing the computations in a mathematically correct manner, whereas validation refers to solving the correct equations, i.e., formulating a mathematical model and selecting the coefficients such that physical phenomenon of interest is described to an adequate level of fidelity. The discussion of verification and validation is beyond the scope of this overview paper; the reader may consult for instance [21−23].

2 NONLINEAR SYSTEM IDENTIFICATION IN STRUCTURAL DYNAMICS: CURRENT STATUS

Nonlinear structural dynamics has been studied for a relatively long time, but the first contributions to the identification of nonlinear structural models date back to the 1970s [24, 25]. Since then, numerous methods have been developed because of the highly individualistic nature of nonlinear systems. A large number of these methods were targeted to single-degree-of-freedom (SDOF) systems, but significant progress in the identification of multi-degree-of-freedom (MDOF) lumped parameter systems has been realized during the last ten years. To date, continuous structures with localized nonlinearity are within reach. Part of the reason for this shift in emphasis is the increasing attention that this research field has attracted, especially in recent years. We also note that the first textbook on the subject was written by Worden and Tomlinson [26].

The present paper is a discussion of the recent developments in this research field. For a review of the past developments, the reader is referred to the companion paper [27] or to the more extensive overview [13]. In particular, this paper aims at discussing three techniques that show promise in this research field. One of their common features is that they are inherently capable of dealing with MDOF systems. Numerical and/or experimental examples are also presented to illustrate their basic concepts, assets and limitations.

2.1 A frequency-domain method: the conditioned reverse path method

Spectral methods based on the reverse path analysis were developed and utilized for identification of SDOF nonlinear systems in [28−34]. The concept of reverse path is discussed at length in [35], and for its historical evolution, the reader may refer to the extensive literature review provided by Bendat [36]. A generalization of reverse path spectral methods for identification of MDOF systems was first proposed by Rice and Fitzpatrick [37]. This method determines the nonlinear coefficients together with a physical model of the underlying linear structure and requires excitation signals at each response location. A second alternative referred to as the conditioned reverse path (CRP) method was developed in [38] and is exposed in this section. It estimates the nonlinear coefficients together with a FRF-based model of the underlying linear structure and does not ask for a

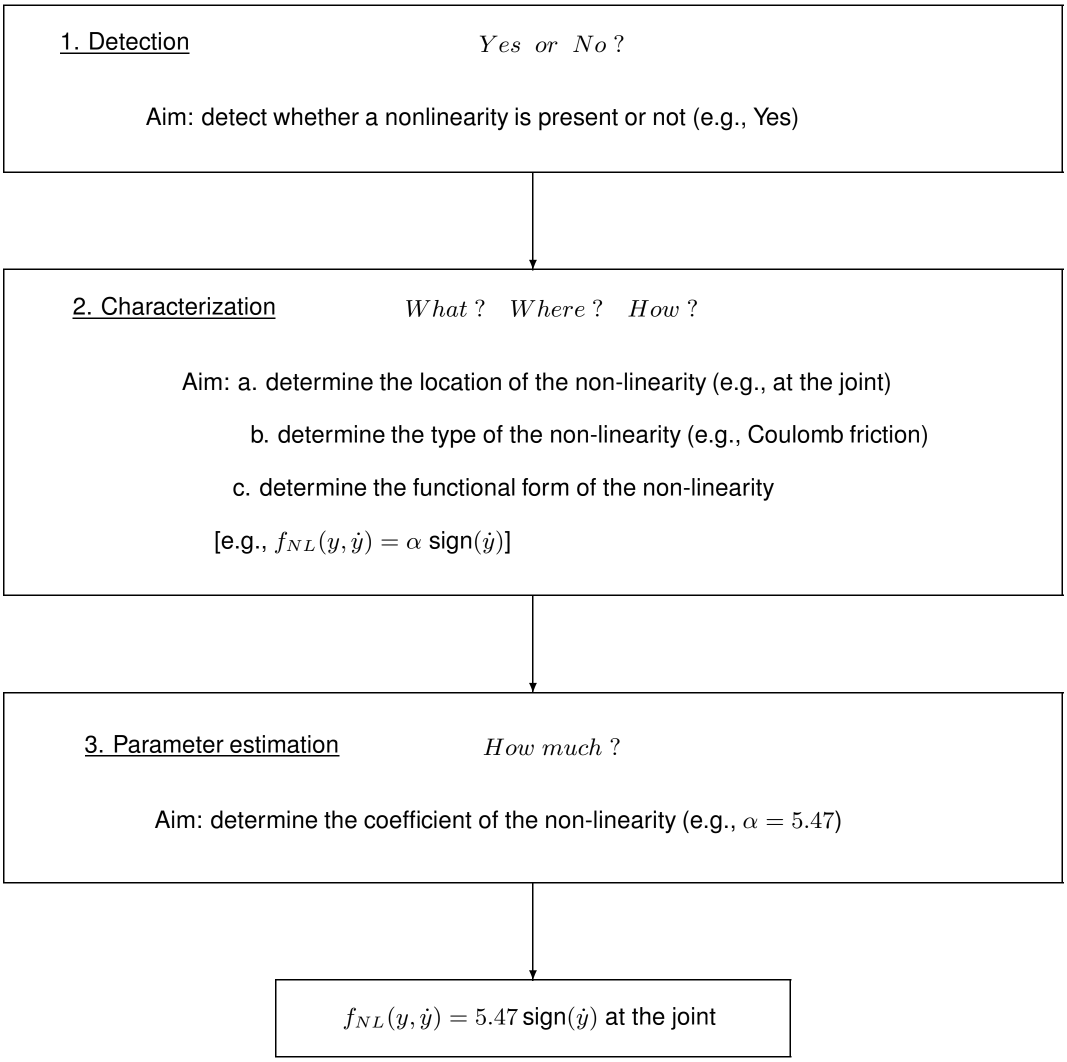

Figure 1: Identification process.

particular excitation pattern [we note that a physical model of the underlying structure can also be built using structural model updating techniques as discussed in [39]]. A detailed discussion of the fundamental differences between the two techniques is given in [40−42]. The CRP method was compared to the RFS method using numerical examples in [43] whereas it was used for identification of experimental systems in [44−47].

Another interesting method in this context is the nonlinear identification through feedback of the output (NIFO) method introduced in [48]. As for the CRP method, the central issue is to eliminate the distortions caused by the presence of nonlinearities in FRFs. It exploits the spatial information and treats the nonlinear forces as internal feedback forces in the underlying linear model of the system.

2.1.1 THEORY

Frequency-domain modal parameter estimation techniques are extensively used to identify the properties of linear systems. They extract modal parameters from H_1 and H_2 estimated FRFs [1]

$$H_1(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)}, \quad H_2(\omega) = \frac{S_{yy}(\omega)}{S_{yx}(\omega)} \tag{1}$$

where $S_{yy}(\omega)$, $S_{xx}(\omega)$ and $S_{yx}(\omega)$ contain the PSD of the response (e.g., acceleration signal), the PSD of the applied force and the cross-PSD between the response and the applied force, respectively. In the presence of nonlinear forces, the H_1 and H_2 estimators cannot be used because nonlinearities corrupt the underlying linear characteristics of the response.

The CRP method was therefore introduced to accommodate the presence of nonlinearity. It employs spectral conditioning techniques to remove the effects of nonlinearities before computing the FRFs of the underlying linear system. The key idea of the formulation is the separation of the nonlinear part of the system response from the linear part and the construction of uncorrelated response components in the frequency domain. The nonlinear coefficients are estimated during the second phase of the method.

Estimation of the underlying system properties

The vibrations of a nonlinear system are governed by the following equation

$$\mathbf{M}\ddot{\mathbf{y}}(t) + \mathbf{C}\dot{\mathbf{y}}(t) + \mathbf{K}\mathbf{y}(t) + \sum_{j=1}^{n} \mathbf{A}_j \mathbf{z}_j(t) = \mathbf{x}(t) \tag{2}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the structural matrices; $\mathbf{y}(t)$ is the vector of displacement coordinates; $\mathbf{z}_j(t)$ is a nonlinear function vector; $\mathbf{A}_j$ contains the coefficients of the term $\mathbf{z}_j(t)$; $\mathbf{x}(t)$ is the applied force vector. For example, in the case of a grounded cubic stiffness at the ith DOF, the nonlinear function vector is

$$\mathbf{z}(t) = [0 \; ... \; y_i(t)^3 \; ...0]^T \tag{3}$$

In the frequency domain, equation (2) becomes

$$\mathbf{B}(\omega)\mathbf{Y}(\omega) + \sum_{j=1}^{n} \mathbf{A}_j \mathbf{Z}_j(\omega) = \mathbf{X}(\omega) \tag{4}$$

where $\mathbf{Y}(\omega), \mathbf{Z}_j(\omega)$ and $\mathbf{X}(\omega)$ are the Fourier transform of $\mathbf{y}(t), \mathbf{z}_j(t)$ and $\mathbf{x}(t)$, respectively; $\mathbf{B}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ is the linear dynamic stiffness matrix.

Without loss of generality, let us assume that a single nonlinear term $\mathbf{Z}_1$ is present. The spectrum of the measured responses $\mathbf{Y}$ can be decomposed into a component $\mathbf{Y}_{(+1)}$ correlated with the spectrum of the nonlinear vector $\mathbf{Z}_1$ through a frequency response matrix $\mathbf{L}_{1Y}$, and a component $\mathbf{Y}_{(-1)}$ uncorrelated with the spectrum of the nonlinear vector; i.e., $\mathbf{Y} = \mathbf{Y}_{(+1)} + \mathbf{Y}_{(-1)}$. In what follows, the minus (plus) sign signifies uncorrelated (correlated) with. Likewise, the spectrum of the external force $\mathbf{X}$ can be decomposed into a component $\mathbf{X}_{(+1)}$ correlated with the spectrum of the nonlinear vector $\mathbf{Z}_1$ through a frequency response matrix $\mathbf{L}_{1X}$, and a component $\mathbf{X}_{(-1)}$ uncorrelated with the spectrum of the nonlinear vector; i.e., $\mathbf{X} = \mathbf{X}_{(+1)} + \mathbf{X}_{(-1)}$. Since both vectors $\mathbf{Y}_{(-1)}$ and $\mathbf{X}_{(-1)}$ are uncorrelated with the nonlinear vector, they correspond to the response of the underlying linear

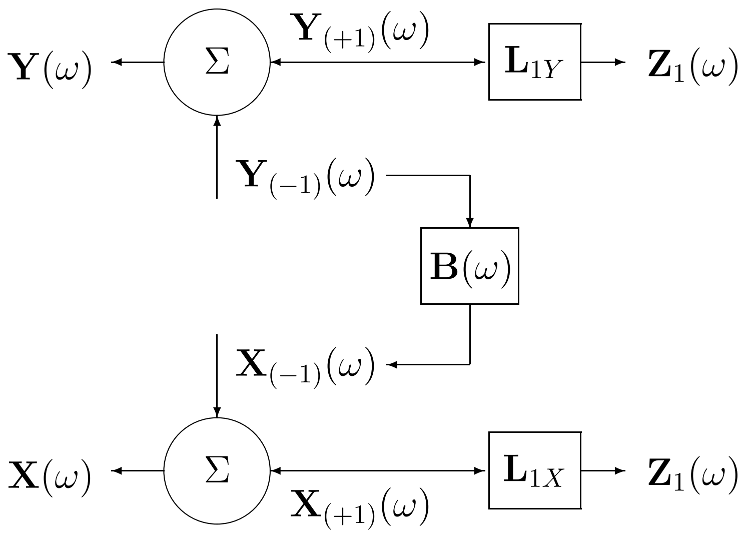

Figure 2: Decomposition of the force and response spectra in the presence of a single nonlinearity .

system and the force applied to this system, respectively; as a result, the path between them is the linear dynamic stiffness matrix $\mathbf{B}$

$$\mathbf{X}_{(-1)}(\omega) = \mathbf{B}(\omega)\mathbf{Y}_{(-1)}(\omega) \tag{5}$$

The whole procedure is presented in diagram form in Figure 2.

The generalization to multiple nonlinearities is straightforward. In this case, the spectra of the response and the force need to be uncorrelated with all n nonlinear function vectors

$$\begin{cases} \mathbf{Y}_{(-1:n)} = \mathbf{Y} - \sum_{j=1}^{n} \mathbf{Y}_{(+j)} = \mathbf{Y} - \sum_{j=1}^{n} \mathbf{L}_{jY}\mathbf{Z}_{j(-1:j-1)} \\ \mathbf{X}_{(-1:n)} = \mathbf{X} - \sum_{j=1}^{n} \mathbf{L}_{jX}\mathbf{Z}_{j(-1:j-1)} \end{cases} \tag{6}$$

$\mathbf{Y}_{(-1:n)}$ and $\mathbf{X}_{(-1:n)}$ are both uncorrelated with the nonlinear function vectors; the path between them is the linear dynamic stiffness matrix $\mathbf{B}$

$$\mathbf{X}_{(-1:n)}(\omega) = \mathbf{B}(\omega)\mathbf{Y}_{(-1:n)}(\omega) \tag{7}$$

By transposing equation (7), premultiplying by the complex conjugate of $\mathbf{Y}$ (i.e., $\mathbf{Y}^*$) taking the expectation $E[\bullet]$ and multiplying by $2/T$, the underlying linear system can be identified without corruption from the nonlinear terms

$$\begin{aligned} \mathbf{S}_{yx(-1:n)} &= \frac{2}{T}E[\mathbf{Y}^*\mathbf{X}_{(-1:n)}^T] = \frac{2}{T}E[\mathbf{Y}^*(\mathbf{B}\mathbf{Y}_{(-1:n)})^T] \\ &= \frac{2}{T}E[\mathbf{Y}^*\mathbf{Y}_{(-1:n)}^T\mathbf{B}^T] = \mathbf{S}_{yy(-1:n)}\mathbf{B}^T \end{aligned} \tag{8}$$

where $\mathbf{S}_{yx(-1:n)}$ and $\mathbf{S}_{yy(-1:n)}$ are conditioned PSD matrices. Calculation of these matrices is laborious and involves a recursive algorithm. For the sake of conciseness, only the final formulae are given herein. In [49], it is shown that

$$\mathbf{S}_{ij(-1:r)} = \mathbf{S}_{ij(-1:r-1)} - \mathbf{S}_{ir(-1:r-1)}\mathbf{L}_{rj}^T \tag{9}$$

where

$$\mathbf{L}_{rj}^T = \mathbf{S}_{rr(-1:r-1)}^{-1}\mathbf{S}_{rj(-1:r-1)} \tag{10}$$

It follows from equation (8) that the dynamic compliance matrix $\mathbf{H}$ which contains the FRFs of the underlying linear system takes the form

$$H_{c2} : \mathbf{H}^T = \mathbf{S}_{yx(-1:n)}^{-1}\mathbf{S}_{yy(-1:n)} \tag{11}$$

This expression is known as the conditioned H_{c2} estimate. If relation (7) is multiplied by the complex conjugate of $\mathbf{X}$ instead of $\mathbf{Y}$, the conditioned H_{c1} estimate is obtained

$$H_{c1} : \mathbf{H}^T = \mathbf{S}_{xx(-1:n)}^{-1} \mathbf{S}_{xy(-1:n)} \tag{12}$$

When FRFs of linear systems are estimated, H_1 always produces better estimates when there is measurement noise on the outputs, and H_2 produces better estimates when the noise is on the input measurements. Intuition may lead us to expect the H_{c2} estimate to perform better than the H_{c1} estimate in the presence of uncorrelated noise only in the excitation. Likewise, the H_{c1} estimate is expected to perform better than the H_{c2} estimate in the presence of uncorrelated noise only in the response. However, experience shows that the H_{c2} estimate gives more accurate estimation of the FRFs of the underlying linear system in both situations. This may be a result of the conditioning required to calculate these estimates.

Estimation of the nonlinear coefficients

Once the linear dynamic compliance $\mathbf{H}$ has been computed by solving equation (11) or (12) at each frequency, the nonlinear coefficients $\mathbf{A}_j$ can be estimated. By applying to equation (4) the same procedure as the one used for obtaining equation (8) from equation (7), the following relationship is obtained

$$\mathbf{S}_{ix(-1:i-1)} = \mathbf{S}_{iy(-1:i-1)} \mathbf{B}^T + \sum_{j=1}^{n} \mathbf{S}_{ij(-1:i-1)} \mathbf{A}_j^T \tag{13}$$

It should be noted that $\mathbf{S}_{ij(-1:i-1)} = E\left[\mathbf{Z}_{i(-1:i-1)}^* \mathbf{Z}_j^T\right] = \mathbf{0}$ for $j < i$ since $\mathbf{Z}_{i(-1:i-1)}^*$ is uncorrelated with the spectrum of the nonlinear function vectors $\mathbf{Z}_1$ through $\mathbf{Z}_{i-1}$. If equation (13) is premultiplied by $\mathbf{S}_{ii(-1:i-1)}^{-1}$, the first term in the summation is $\mathbf{A}_i^T$. Equation (13) is then transformed into

$$\mathbf{A}_i^T = \mathbf{S}_{ii(-1:i-1)}^{-1} \left(\mathbf{S}_{ix(-1:i-1)} - \mathbf{S}_{iy(-1:i-1)} \mathbf{B}^T - \sum_{j=i+1}^{n} \mathbf{S}_{ij(-1:i-1)} \mathbf{A}_j^T \right) \tag{14}$$

Because the expression of the linear dynamic compliance has been computed, equation (14) is rewritten in a more suitable form

$$\mathbf{A}_i^T \mathbf{H}^T = \mathbf{S}_{ii(-1:i-1)}^{-1} (\mathbf{S}_{ix(-1:i-1)} \mathbf{H}^T - \mathbf{S}_{iy(-1:i-1)} - \sum_{j=i+1}^{n} \mathbf{S}_{ij(-1:i-1)} \mathbf{A}_j^T \mathbf{H}^T) \tag{15}$$

The identification process starts with the computation of $\mathbf{A}_n$ working backwards to $\mathbf{A}_1$. As for the reverse path method, the nonlinear coefficients are imaginary and frequency dependent. The imaginary parts, without any physical meaning, should be negligible when compared to the real parts. On the other hand, by performing a spectral mean, the actual value of the coefficients should be retrieved.

Coherence functions

The ordinary coherence function can be used to detect any departure from linearity or to detect the presence of uncorrelated noise on one or both of the excitation and response signals.

For a multiple input model with correlated inputs, the sum of ordinary coherences between the inputs and the output may be greater than unity. To address this problem, the ordinary coherence function has been superseded by the cumulative coherence function γ_{Mi}^2

$$\gamma_{Mi}^2(\omega) = \gamma_{y_i x(-1:n)}^2(\omega) + \gamma_{zx}^2(\omega) = \gamma_{y_i x(-1:n)}^2(\omega) + \sum_{j=1}^{n} \gamma_{jx(-1:j-1)}^2(\omega) \tag{16}$$

$\gamma_{y_i x(-1:n)}^2$ is the ordinary coherence function between the ith element of $Y_{(-1:n)}$ and excitation X

$$\gamma_{y_i x(-1:n)}^2 = \frac{\left|S_{y_i x(-1:n)}\right|^2}{S_{y_i y_i(-1:n)} S_{xx}} \tag{17}$$

It indicates the contribution from the linear spectral component of the response of the ith signal. $\gamma_{jx(-1:j-1)}^2$ is the ordinary coherence function between the conditioned spectrum $Z_{j(-1:j-1)}$ and excitation X

$$\gamma_{jx(-1:j-1)}^2 = \frac{\left|S_{jx(-1:j-1)}\right|^2}{S_{jj(-1:j-1)} S_{xx}} \tag{18}$$

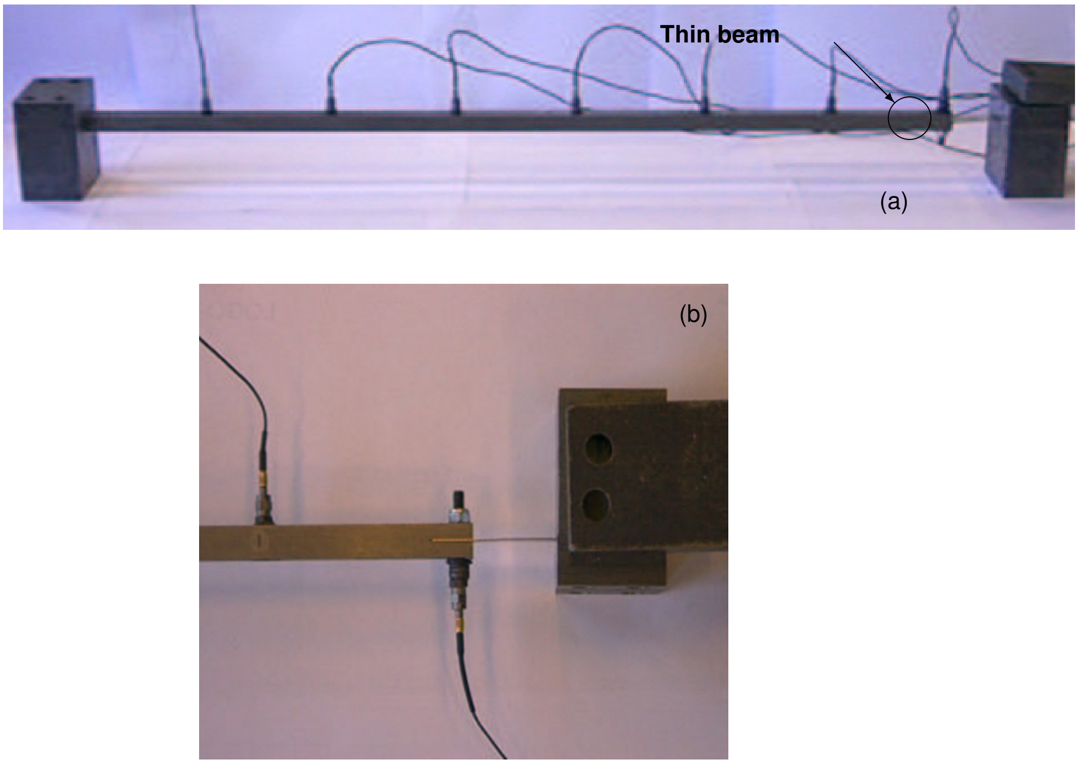

Figure 3: Cantilever beam connected to a thin, short beam (ECL benchmark; COST Action F3): (a) experimental fixture; (b) close-up of the connection.

and $\sum_{j=1}^{n} \gamma^2_{jx(-1:j-1)}$ indicates the contribution from the nonlinearities.

The cumulative coherence function is always between 0 and 1 and may be considered as a measure of the model accuracy; it is a valuable tool for the selection of an appropriate functional form for the nonlinearity.

2.1.2 APPLICATION EXAMPLE

The CRP method was applied to the experimental structure depicted in Figure 3 in [45]. A cantilever beam is connected at its right end to a thin, short beam that exhibits a geometric nonlinearity when large deflections occur. The identification was carried within the range 0-500 Hz in which three structural modes exist. For more details about this experiment, the reader is invited to consult [45]. This structure was also investigated within the framework of the European COST Action F3 [50].

Figures 4, 5 and 6 summarize the results obtained. Figure 4 represents three different FRFs in the vicinity of the first two resonances: (a) the FRF measured using the classical H_2 estimate at low level of excitation (i.e., 1.4 Nrms) for which the geometric nonlinearity is not activated; it should therefore correspond to the FRF of the underlying linear system; (b) the FRF measured using the classical H_2 estimate at high level of excitation (i.e., 22 Nrms); (c) the FRF measured using the H_{c2} estimate at high level of excitation (i.e., 22 Nrms). It can clearly be observed that the FRF measured using H_2 estimate at 22 Nrms is contaminated by the presence of the geometric nonlinearity whereas the FRF measured using H_{c2} estimate at 22 Nrms is a very accurate estimation of the FRF of the underlying linear system. The accuracy of the identification is confirmed in Figure 5; overall, the cumulative coherence is close to 1. Figure 6 represents the real part of the nonlinear coefficient A, and its spectral mean performed within the range 10-250 Hz is equal to $1.96\,10^9 + i\,1.55\,10^7 \;\; N/m^{2.8}$. As expected, the imaginary part of the coefficient is two orders of magnitude below the real part and can be safely neglected.

A final remark concerns the functional form of the nonlinearity. Although a cubic nonlinearity was expected due to the presence of a geometric nonlinearity, the model $f(y_c) = A\,|y_c|^{\alpha}\,\text{sign}(y_c)$ where y_c is the response at the bolted connection between the

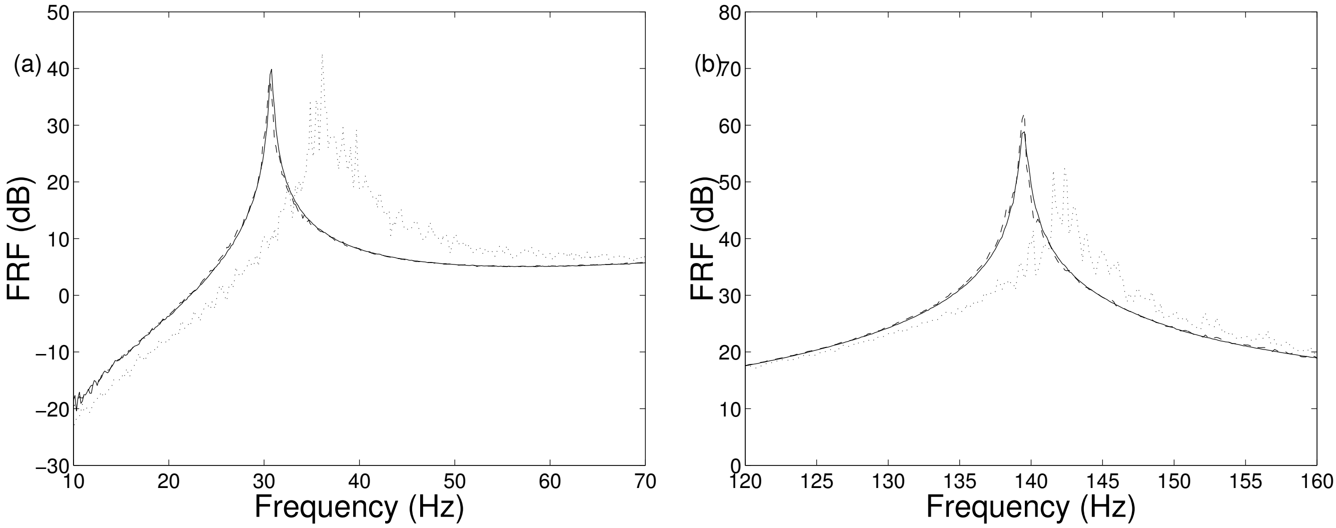

Figure 4: Magnitude of H_{73} (ECL benchmark): (a) first resonance; (b) second resonance. (———, FRF measured using H_2 estimate at 1.4 Nrms (the geometric nonlinearity is not activated); ······, FRF measured using H_2 estimate at 22 Nrms; - - -, FRF measured using H_{c2} estimate at 22 Nrms).

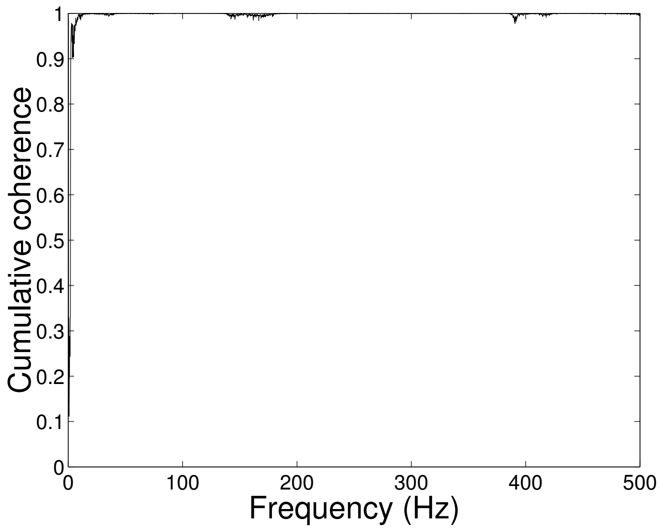

Figure 5: Cumulative coherence γ^2_{M7} (22 Nrms).

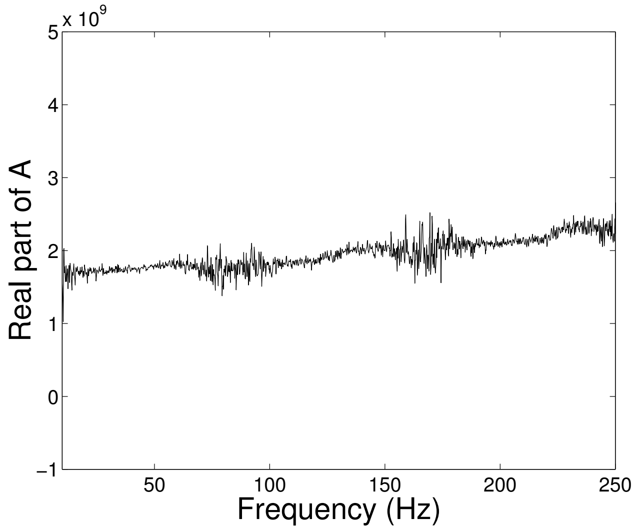

Figure 6: Real part of the nonlinear coefficient (22 Nrms).

two beams was considered during the identification for greater flexibility. The exponent α was determined by maximizing the spectral mean of the cumulative coherence function and was found to be 2.8.

2.1.3 ASSESSMENT

Although it is difficult to draw general conclusions from a single example, it turns out that the CRP method is a very appealing and accurate method for parameter estimation of nonlinear structural models. In addition, the cumulative coherence is a valuable tool for the characterization of the nonlinearity. The formulation of the method is such that it targets identification of MDOF systems, which enabled the identification of a numerical model with 240 DOFs and two localized nonlinearities in [39].

An extension of the method to the identification of physical models instead of FRF-based models is discussed in [39]. In this study, a finite element model of the underlying linear structure is built from the knowledge of the geometrical and mechanical properties of the structure and is updated using linear model updating techniques based upon FRFs [51−53].

A possible drawback of the method is that it requires the measurements of the structural response at the location of the nonlinearity, which is not always feasible in practice. Also, it is not yet clear how the method would perform in the presence of several nonlinearities, which is typical of a structure with a large number of discrete joints. Finally, future research should investigate how the method could deal with distributed nonlinearities and hysteretic systems modeled using internal state variables (e.g, the Bouc-Wen model).

2.2 A modal method: the nonlinear resonant decay method

2.2.1 THEORY

Classical force appropriation methods [54, 55] are used in the identification of linear systems to determine the multi-point force vector that induces single-mode behavior, thus allowing each normal mode to be identified in isolation. For a proportionally damped linear structure, the final model consists of a set of uncoupled SDOF oscillators in modal space.

An extension of the force appropriation approach to the identification of nonproportionally damped linear systems, termed the resonant decay (RD) method, is presented in [56]. An appropriated force pattern with a single sine wave is applied as a ‘burst’ to excite a given mode of interest. Once the excitation ceases, the free decay of the system includes a response from any modes coupled by damping forces to the mode being excited. A curve fit to a limited subset of modes can then be performed to yield any significant damping terms which couple the corresponding SDOF oscillators.

A generalization of this methodology for identification of nonlinear systems is described in this section. For the analysis of large nonlinear structures with high modal density in a broad frequency range, an enormous number of parameters is to be identified because the nonlinear modal restoring forces $\mathbf{f}_m(\mathbf{u}, \dot{\mathbf{u}})$ are potentially functions of the many modal displacements $u_i(t)$ and/or velocities $\dot{u}_i(t)$ (in other words, the nonlinearity may be responsible for many terms coupling the SDOF oscillators); this renders parameter estimation intractable.

The method developed in [57] offers a practical solution to this critical issue by proposing a multi-stage identification of the linear modal space-based model in which the initial estimation problem is replaced by a sequence of low-dimensional problems. At this point, we note that the selective sensitivity approach developed in [58] also proposes to identify the entire system via a sequence of low-dimensional estimation problems through the use of selective excitation. In [57], the scale of the identification problem is reduced by classifying the modes[1] into different categories: (i) linear proportionally damped modes, well separated in frequency; (ii) linear proportionally damped modes, very close in frequency; (iii) linear nonproportionally damped modes; (iv) modes influenced by nonlinear effects with no significant nonlinear coupling to other modes; and (v) modes influenced by nonlinear effects with significant nonlinear coupling to other modes. The set of uncoupled SDOF oscillators in modal space is therefore enhanced by the inclusion of modal damping cross-coupling terms for nonproportionally damped modes, 'direct' nonlinear terms $f_m(u_j, \dot{u}_j)$ if the jth mode behaves nonlinearly and nonlinear cross-coupling terms $f_m(u_i, \dot{u}_i, u_j, \dot{u}_j)$ if the ith and jth modes are nonlinearly coupled.

Modes of type (i) may be identified using classical curve-fitting methods. Modes of type (ii) may benefit from identification using force appropriation. Force appropriation and the RD method are suitable for modes of type (iii). Anticipating that only a relatively small portion of modes will actually behave in a nonlinear fashion for most structures (this assumption implies that the method targets weakly nonlinear systems), two methodologies which enable the treatment of modes affected by nonlinearity [i.e., modes of type (iv) and (v)] individually or in small groups were developed:

- The FANS method [59] extends the classical linear force appropriation approach to nonlinear systems through the use of a force pattern that includes higher harmonic terms. The parameters are optimized such that the nonlinear coupling terms are counteracted, which prevents any response other than the mode of interest. The direct linear and nonlinear terms for that mode may be estimated using a classical SDOF RFS identification.
- The nonlinear resonant decay (NLRD) method [57] is an extension of the RD method to nonlinear systems and enables small groups of modes to be excited. A classical appropriated force pattern with a single sine wave is applied as a 'burst' to excite a given mode of interest 'approximately'. If the mode is uncoupled nonlinearly, then it should dominate the response in the steady-state phase. If it is nonlinearly coupled, other modes may also exhibit a significant response. During the decay, the presence of linear damping couplings as well as nonlinear couplings between the modes is apparent. A 'low-order' regression analysis in modal space using the RFS method is then carried out for identification of direct and cross-coupling terms.

2.2.2 APPLICATION EXAMPLE

The NLRD method is applied in [57] to a 5-DOF spring-mass system clamped at both extremities and designed to be symmetric in its linear components. The system has a cubic stiffness nonlinearity between the second and fourth DOFs. The system is linear in modes 1, 3 and 5; modes 2 and 4 are nonlinear and coupled together. In order to illustrate the burst principle, a burst is applied to excite mode 5 as shown in Figure 7[2]. Because mode 5 behaves linearly and the correct appropriated force vector is used, no modal force is input to the other modes, and only mode 5 responds. Consider now a burst applied to mode 4 as shown in Figure 8. There is only a modal force for mode 4 but now mode 2 responds due to the nonlinear coupling. Modes 1, 3 and 5 are not excited because of the force appropriation. A curve fitting can then be carried out for mode 4 using only the modal responses associated with modes 2 and 4; the scale of the identification has been effectively reduced.

[1] It is emphasized that a mode refers to the mode of the underlying linear system; the discussion does not refer to the nonlinear normal modes

[2] The results in Figures 7 and 8 were obtained by Dr. Jan Wright and co-workers — the authors are very grateful for permission to use them.

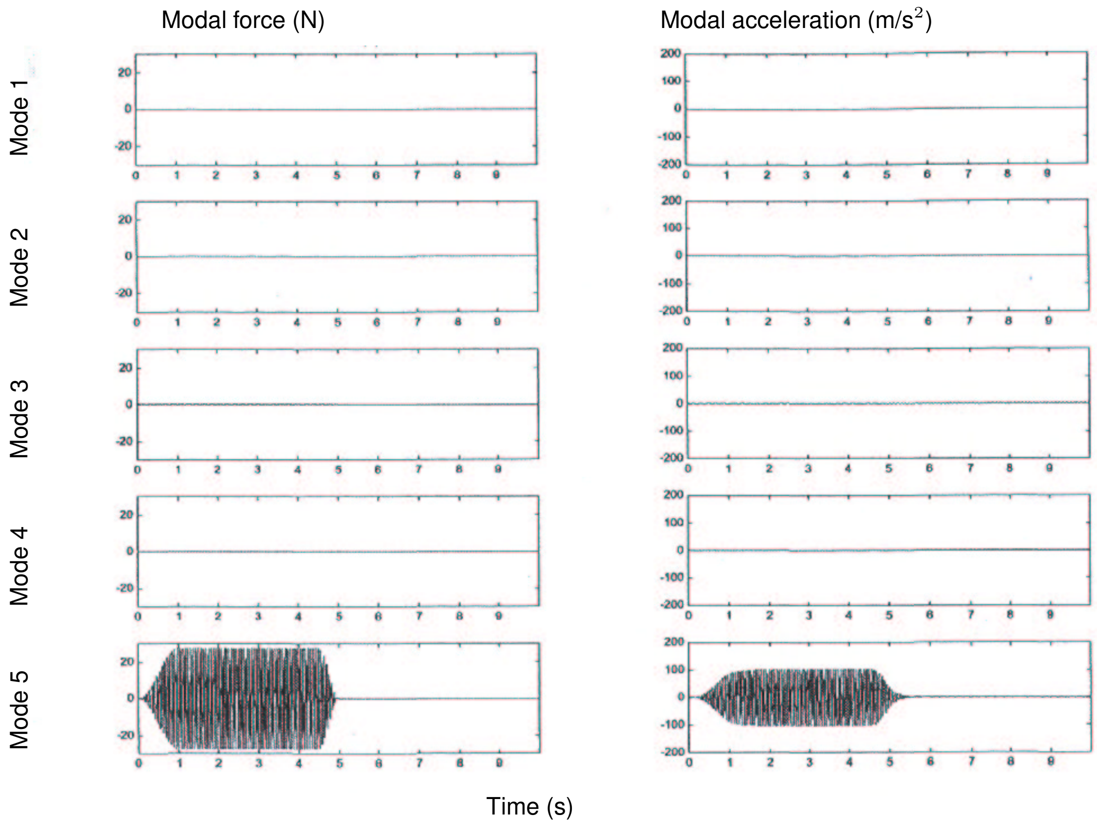

Figure 7: Modal forces and responses to burst excitation of mode 5 using perfect appropriation [(Wright *et al.*, 2001)].

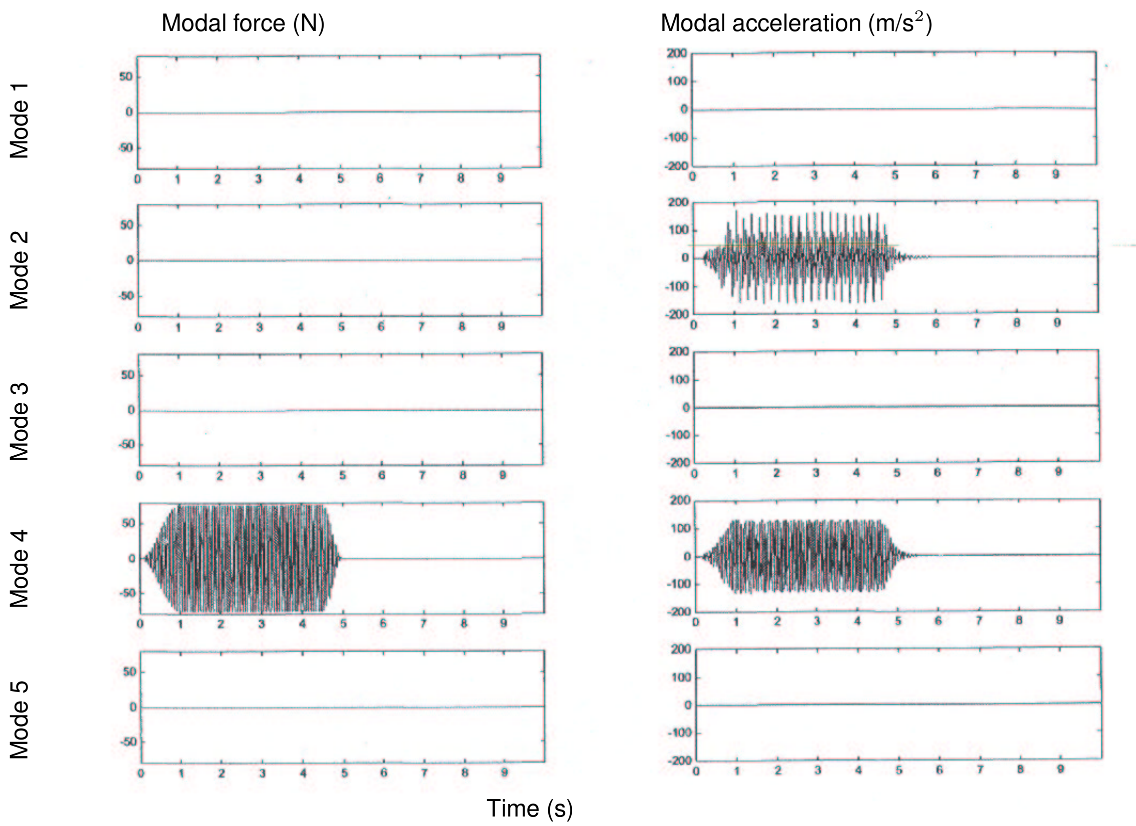

Figure 8: Modal forces and responses to burst excitation of mode 4 using perfect appropriation [(Wright *et al.*, 2001)].

2.2.3 ASSESSMENT

Although this nonlinear system identification technique has not yet been applied to large continuous structures, the authors believe that it paves the way for the analysis of practical systems with high modal density. Because modes are treated individually or in small groups, the method has the inherent ability to 'split' the original and complex identification problem into a sequence of much simpler and smaller problems. One may also account for nonproportional damping, which is another interesting feature of the method.

Imperfection force appropriation and modal matrix may reduce the accuracy of the identification as discussed in [57]. As a result, the number, location and pattern of excitation sources should be determined in a judicious manner in order for this process to be successful; shaker-structure interaction may also be an issue for light-weight structures.

2.3 A finite element method: structural model updating

For the investigation of more complex structures in a wider frequency range, resorting to models with many DOFs is inevitable. However, the estimation of all the model parameters from experimental measurements may quickly become intractable. A solution to this problem is to use structural modeling techniques which compute the model parameters based on the knowledge of the geometrical and mechanical properties of the structure.

Despite the high sophistication of structural modeling, practical applications often reveal considerable discrepancies between the model predictions and experimental results, due to three sources of errors, namely modeling errors (e.g, imperfect boundary conditions or assumption of proportional damping), parameter errors (e.g., inaccuracy of Young's modulus) and testing errors (e.g, noise during the measurement process). There is thus the need to improve structural models through the comparison with vibration measurements performed on the real structure; this is referred to as structural model updating.

Very often, the initial model is created using the finite element method (see, e.g., [60]), and structural model updating is termed finite element model updating. Finite element model updating was first introduced in the 1970s for linear structures [61, 62]. For a detailed description of this field of research and the issues commonly encountered (e.g., model matching step and error localization), the reader is invited to consult [63−65].

The literature on methods that propose to update nonlinear dynamic models is rather sparse. In [66], parameters of nonlinear elements are updated by fitting simulated time history functions and the corresponding measurement data. The problem of estimating the initial values as well as the problem of increasing error between simulated and measured time history functions is overcome by using the method of modal state observers. Kapania and Park [67] proposed to compute the sensitivity of the transient response with respect to the design parameters using the time finite element method. The minimum model error estimation algorithm is exploited in [68] to produce accurate models of nonlinear systems. In this algorithm, a two-point BVP is solved in order to obtain estimates of the optimal trajectories together with the model error. In [69, 70], model updating is realized through the minimization of an objective function based on the difference between the measured and predicted time series. The optimization is achieved using the differential evolution algorithm which belongs to the class of genetic algorithms. The formulation proposed by Meyer and co-authors [71, 72] involves a linearization of the nonlinear equilibrium equations of the structure using the harmonic balance method. Updating of the finite element model is carried out by minimizing the deviations between measured and predicted displacement responses in the frequency domain. In [73], model updating is performed in the presence of incomplete noisy response measurements. A stochastic model is used for the uncertain input, and a Bayesian probabilistic approach is used to quantify the uncertainties in the model parameters. In [39], a two-step methodology which decouples the estimation of the linear and nonlinear parameters of the finite element model is proposed. This methodology takes advantage of the CRP method and is applied to a numerical application consisting of an aeroplane-like structure.

Due to the inapplicability of modal analysis, test-analysis correlation which is inherent to structural model updating is a difficult task in the presence of nonlinearity. Several efforts have been made in order to define features (i.e., variables or quantities identified from the structural response that give useful insight into the dynamics of interest) that facilitate correlation. In the case of pyroshock response, NASA has proposed criteria such as peak amplitude, temporal moments and shock response spectra as appropriate features of the response signal [74]. In [75, 76], the peak response and time of arrival are defined as features in order to study the transient dynamics of a viscoelastic material. In [77], the envelope of transient acceleration responses is considered as the best information to identify joint parameters associated with adjusted Iwan beam elements. The proper orthogonal (POD) method, also known as Karhunen-Loève transform or principal component analysis, has been investigated in several studies [78−81]. Specifically, the modes extracted from the decomposition, the proper orthogonal modes (POMs), have been shown to be interesting features for the purpose of test-analysis correlation. In [82, 83], the POMs together with the wavelet transform of their amplitude modulations are considered for finite element model updating. Although it is frequently applied to nonlinear problems, it should be borne in mind that the POD only gives the optimal approximating linear manifold in the configuration space represented by the data. This is the reason why finite element model updating was performed in [84, 85] using the features extracted from a nonlinear generalization of the POD, termed nonlinear principal component analysis [86]. In [87], the POD is coupled with neural network and genetic algorithms for approximation and calibration of nonlinear structural models.

A statistics-based model updating and validation philosophy is proposed in [75, 76]. The motivation for including statistical analysis is driven by the desire to account for the effects of environmental and experimental variability. The feature comparison is implemented using metrics such as Mahalanobis distance and Kullback-Leibler relative entropy function. In addition, the finite element model is replaced by an equivalent meta-model with a much smaller analytical form. This strategy aims at reducing the number of computer simulations required during optimization while maintaining the pertinent characteristics of the problem. The demonstration application consists in analyzing the response of a steel/polymer foam assembly during a drop test.

2.3.1 THEORY

The structural model updating process is presented in diagram form in Figure 9. It can be decomposed into four steps: (1) experimental measurements and structural modeling; (2) feature extraction and correlation study; (3) selection of the updating parameters and (4) minimization of the objective function. The success of model updating is conditional upon each step being properly carried out.

It is noted that the emphasis in the present section is put upon model updating using time-domain measurements.

Experimental measurements and structural modeling

Experiment design (e.g, selection of excitation sources, number and location of sensors) is a crucial step but it is not discussed herein. It is therefore assumed that vibration tests have been performed on the real structure; a matrix $\mathbf{Y}(t)$ containing m samples of the response (e.g., acceleration data) measured at n different locations on the structure is formed

$$\mathbf{Y}(t) = [\mathbf{y}(t_1)\ \cdots\ \mathbf{y}(t_m)] = \begin{bmatrix} y_1(t_1) & \cdots & y_1(t_m) \\ \cdots & \cdots & \cdots \\ y_n(t_1) & \cdots & y_n(t_m) \end{bmatrix} \tag{19}$$

From the knowledge of the geometrical and mechanical properties of the structure, a structural model can be created. By imposing in this model the same excitation conditions $\mathbf{x}(t)$ as for the real structure, the structural response can be predicted using time-integration algorithms; the matrix $\hat{\mathbf{Y}}(t)$ is obtained. At this stage, verification, i.e., ‘solving the equations correctly’ [21], is necessary, but its description would take us too far afield.

Feature extraction and correlation study

Matrix $\hat{\mathbf{Y}}(t)$ generally differs from $\mathbf{Y}(t)$ due to three sources of errors, namely modeling errors (e.g, imperfect boundary conditions or assumption of proportional damping), parameter errors (e.g., inaccuracy of Young’s modulus) and testing errors (e.g, noise during the measurement process). However, estimating the predictive capability of a structural model based only on its ability to match measured time series may be hazardous. The comparison between experimental features $\mathbf{f}_i$ and predicted features $\hat{\mathbf{f}}_i$ should be preferred. In linear dynamics, natural frequencies and mode shapes provide a sound basis for ascertaining whether the prediction of the model will adequately represent the overall dynamic response of the structure. Another well established technique is to use data in the frequency domain because the effort of experimental modal analysis is avoided, and averaging to reduce noise effects is straightforward.

When performing test-analysis correlation for nonlinear structures, the features commonly defined for linear structures do no longer provide an accurate characterization of the dynamics, as explained in the tutorial section. The definition of features that enhance the effect of nonlinearity on the structural behavior is therefore necessary. NNMs provide a valuable theoretical tool for understanding dynamic phenomena such as mode bifurcations and nonlinear mode localization but it is a little early to tell if they will be of substantial help for structural model updating. For this reason, other features have been considered in the technical

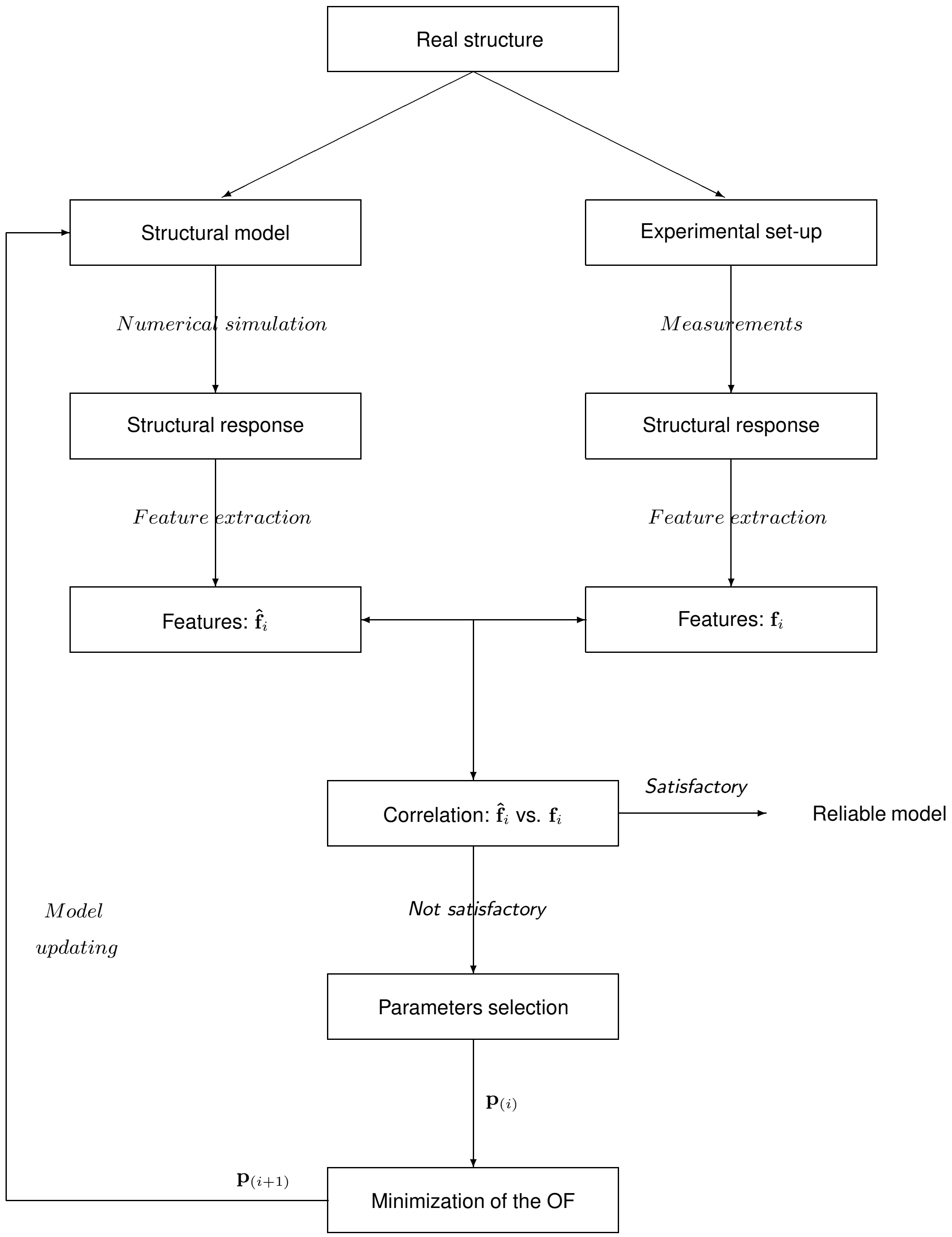

Figure 9: Model updating sequence of nonlinear systems (OF: objective function).

literature.

Selection of the updating parameters

If correlation is not satisfactory, the structural model is to be updated. The correction of the model begins with the selection of the updating parameters. Parameter selection is a difficult and critical step, and the success of the model updating process is conditional upon the ability to identify the adequate parameters. For this purpose, error localization techniques and sensitivity analysis may be useful [64, 65], but physical understanding of the structural behavior and engineering judgment play the key role (see for instance [88]).

Minimization of the objective function

New values of the updating parameters are computed through the minimization of an objective function J

$$\min_{\mathbf{p}} J = \|R(\mathbf{p})\|^2 \tag{20}$$

where vector $\mathbf{p}$ contains the updating parameters. The residue $R(\mathbf{p})$ may simply be the norm of the difference between the predicted and experimental features. The objective function is generally nonlinear with respect to the updating parameters, and it is necessary to use optimization algorithms to perform the minimization.

2.3.2 APPLICATION EXAMPLE

Structural model updating was applied to the experimental system depicted in Figure 3 in [83]. This structure was also investigated within the framework of the European COST Action F3 [50]. An impulsive force was imparted to the cantilever beam using an impact hammer, and the structural response was measured using seven accelerometers evenly spaced across the beam.

A structural model was created using the finite element method, and the effect of the geometric nonlinearity was modeled with a grounded spring at the connection between the cantilever beam and the short beam. The accelerations of the numerical model were computed using Newmark's method.

The correlation study was performed by comparing experimental and predicted POMs. Although the POMs do not have the theoretical foundations of the NNMs, they do provide a good characterization of the dynamics of a nonlinear system. Another advantage is that their computation is straightforward; it involves a singular value decomposition of the response matrix $\mathbf{Y}(t)$

$$\mathbf{Y} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \tag{21}$$

where each column of matrix $\mathbf{U}$ contains a POM. Matrix $\mathbf{\Sigma}$ gives information about the participation of the POMs in the system response whereas their amplitude modulations are contained in matrix $\mathbf{V}$. Insight into the frequency of oscillation of the POMs is available by applying the wavelet transform to matrix $\mathbf{V}$. For a detailed description of the POD, the reader is invited to consult [89], and an overview of the POD for dynamical characterization of nonlinear structures is available in [81]. Figure 10 shows that the first two POMs predicted by the initial finite element model are not in close agreement with those of the experimental structure. Because these two POMs account for more than 90% of the total energy contained in the system response, the model must be improved.

Several parameters were not known precisely in the initial model, especially the stiffness of the bolted connection between the two beams and the coefficient and exponent of the nonlinearity; they were thus selected as updating parameters. After optimization, the coefficient and exponent of the nonlinearity were $1.65\,10^9\ N/m^{2.8}$ and 2.8, respectively, which is in good concordance with the estimates given by the CRP method (see Section 2.1.2). There is now a satisfactory match between the experimental POMs and those predicted by the updated finite element model as shown in Figure 10. Figure 11 displays the wavelet transform of the amplitude modulation of the first POM; the dominant frequency component is around 50 Hz, but harmonics — a typical feature of nonlinear systems — can also clearly be observed. There is also a good agreement between the experimental and numerical results in Figure 11, which confirms that the updated model has a good predictive accuracy.

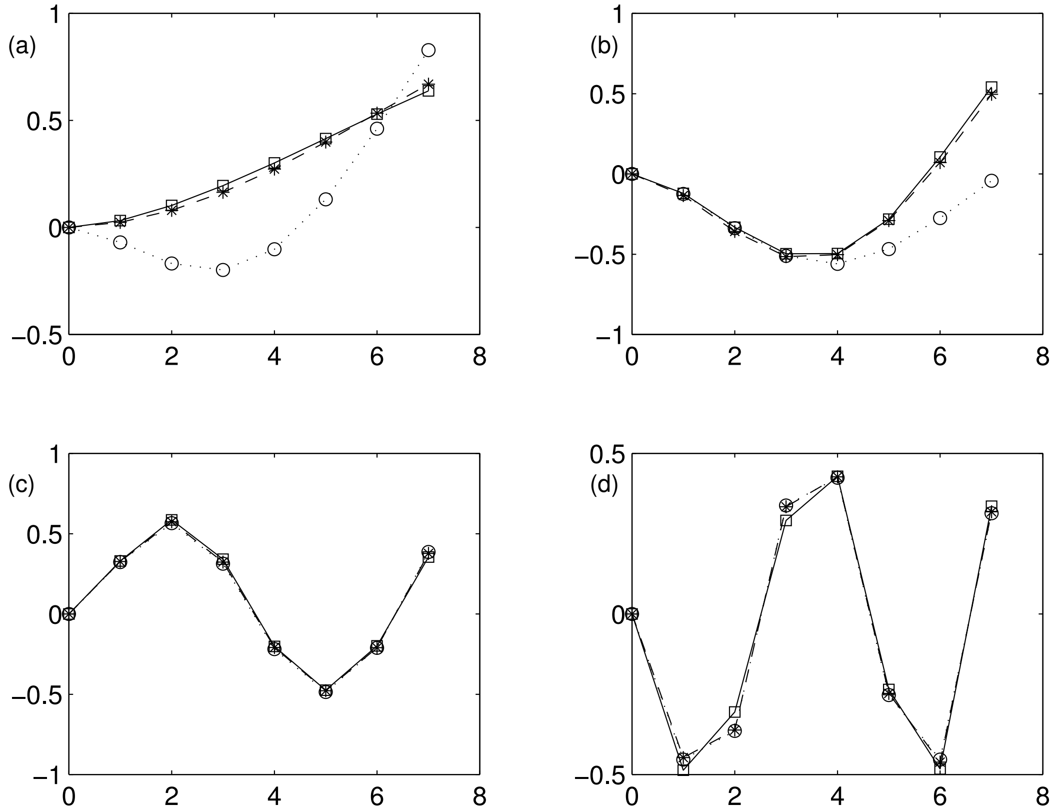

Figure 10: Proper orthogonal modes (POMs): (a) 1st POM; (b) 2nd POM; (c) 3rd POM; (d) 4th POM (——□——, Experimental POM; ···O···, initial finite element model; — ∗ —, updated finite element model).

2.3.3 ASSESSMENT

Structural model updating has the inherent ability to provide reliable models of more complex nonlinear structures. For instance, numerical examples with a few hundred DOFs are investigated in [71, 79, 84, 85], whereas a fully integrated experimental system is considered in [80].

However, several crucial issues remain largely unresolved, and there is much research to be done:

- There are no universal features applicable to all types of nonlinearities; test-analysis correlation is still a difficult process.
- It is generally assumed that the analyst has the ability to formulate an appropriate initial model and to identify precisely the source and location of the erroneous parameters; these are extremely challenging tasks when dealing with complex structures.
- Many of the error criteria formulations lead to objective functions with a highly nonlinear solution space; multiple parameter sets may potentially yield equally good reproduction of the experimental measurements, especially when limited measure-

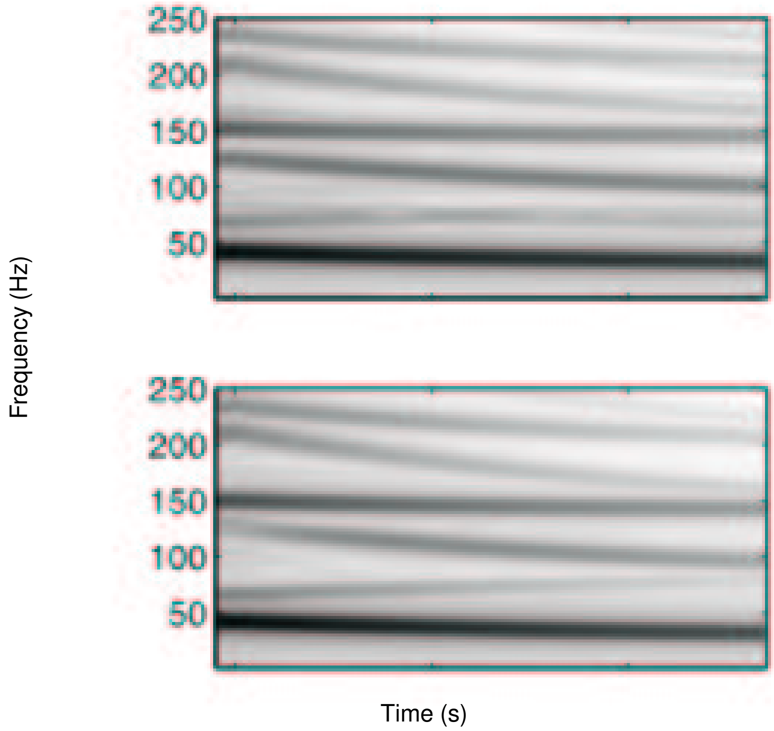

Figure 11: Wavelet transform of the amplitude modulation of the first POM. Top plot: experimental structure; bottom plot: updated finite element model.

ment data is available [We note that info-gap models may offer an elegant solution to this problem [90]]. In addition, the initial model cannot be assumed to be close to the 'actual' model because a priori knowledge about nonlinearity is often limited; the starting point of the optimization may be far away from the sought minimum. For all these reasons, objective function minimization may be challenging and time consuming.

3 SUMMARY AND FUTURE RESEARCH NEEDS

This paper reviews some of the recent developments in nonlinear system identification, the objective of which is to produce high fidelity models that may be used for purposes such as

– Virtual prototyping; this encompasses the selection of optimal system parameters in order to meet specific design goals, the prediction of the occurrence of undesirable instabilities and bifurcations (e.g., aeroelastic instabilities), the impact of structural modifications and the study of the effects of structural, environmental or other types of uncertainties on the robustness of operation;

– Development of diagnostic and prognostic tools that enable simple, accurate, economic, and preferably on-line detection of structural faults at an early stage of their developments before they become catastrophic for the operation of the system;

– Structural control, e.g, the control of mechatronic systems or of structural vibrations produced by earthquake or wind.

Because of the highly individualistic nature of nonlinear systems and because the basic principles that apply to linear systems and that form the basis of modal analysis are no longer valid in the presence of nonlinearity, one is forced to admit that there is no general analysis method that can be applied to all systems in all instances. As a result, numerous methods for nonlinear system identification have been developed during the last three decades. A large proportion of these methods were targeted to SDOF systems, but significant progress in the identification of MDOF lumped parameter systems has been realized recently. To date, continuous structures with localized nonlinearity are within reach.

For simple structures or approximate models of more complex structures, it is reasonable to estimate all the model parameters. However, for the analysis of structures with a large number of DOFs and with a high modal density in a broad frequency range, resorting to multi-parameter complex structural models is inevitable. This critical issue begins to be resolved by several recent approaches among which we can cite:

– Frequency-domain methods such as the CRP and NIFO methods have, in principle, the capability of identifying the dynamics of large structures. In addition to the nonlinear coefficients, they compute a FRF-based model of the underlying linear structure directly from the experimental data, which facilitates the identification process.

– The NLRD method proposes to classify the modes into different categories (i.e., influenced or not by nonlinear effects, coupled or uncoupled in damping and/or nonlinearity), which enables the treatment of modes individually or in small groups. This technique does not decrease the number of parameters to be estimated, but it simplifies the parameter estimation process by targeting a multi-stage identification.

– Structural model updating techniques exploit the knowledge of the geometric and mechanical properties to determine an initial model of the structure, many parameters of which are usually accurately computed and do not have to be identified from experimental data.

All these methods have their own drawbacks, but they show promise in the challenging area of nonlinear system identification.

Besides rendering parameter estimation tractable, other important issues must be addressed adequately to progress toward the development of accurate, robust, reliable and predictive models of large, three-dimensional structures with multiple components and strong nonlinearities. The following discussion presents some of the key aspects that, we believe, will drive the development of nonlinear system identification in the years to come.

(i) We cannot stress enough the importance of having an accurate characterization of the nonlinear elastic and dissipative behavior of the physical structure prior to parameter estimation. Without a precise understanding of the nonlinear mechanisms involved, the identification process is bound to failure. Characterization is a very challenging step because nonlinearity may be caused by many different mechanisms and may result in plethora of dynamic phenomena. Some 'real-life' nonlinear effects only begin to be adequately modeled (e.g., the dynamics of structures with bolted joints [77, 91, 92]); some are still far from being understood [e.g, experiments reported in [93] showed that quasiperiodic responses in a frictionally excited beam may involve very low frequencies at subharmonic orders of 20 to 130]. The lack of knowledge about nonlinearity is sometimes circumvented by nonlinear black-box models such as those proposed in [94−96], but, in our opinion, a priori information and physics-based models should not be superseded by any 'blind' methodology. Careful and systematic studies of nonlinear dynamical effects such as those carried out in [4, 97, 98] are strongly encouraged and are a necessary step toward the development of accurate nonlinear structural models. Improving our knowledge and our modeling capabilities of the range of possible nonlinear behaviors [this also reduces the level of uncertainty and increases our confidence in the model; see (iii)] is therefore a crucial need, especially because structural dynamics is becoming increasingly nonlinear, addressing multi-physics phenomena [80, 99].

(ii) Most of the analytical techniques currently available are limited to the steady-state response of weakly nonlinear oscillators. On the other hand, because strong nonlinearity is more and more encountered in practical applications, new dynamical phenomena are observed that have to be accounted for. For example, it is only recently that resonance capture phenomena which are mainly of a transient nature have been reported in the structural dynamics literature [100−102]. As a result, there is the need for new analytical developments enabling the study of the transient dynamics of strongly nonlinear oscillators. Such developments will provide better insight into the dynamics of interest, thereby facilitating the characterization of the nonlinear behavior discussed in (i).

(iii) The concept of NNM offers a solid theoretical and mathematical framework for analyzing and interpreting a wide class (but not the entirety!) of nonlinear dynamical phenomena, and yet it has a clear and simple conceptual relation to the classical linear normal mode, with which practicing vibration engineers are familiar. Viewed in this context, the concept of NNM can provide the appropriate framework for closer collaboration and mutual understanding between Academia and Industry. To formulate practical NNM-based nonlinear system identification techniques, advances in a number of critical research areas need to be accomplished including

- The development of efficient computational algorithms for studying the NNMs of practical (multi-DOF, flexible or large-scale) mechanical systems and their bifurcations;
- The study of possible exact or approximate (for example, asymptotic) NNM-based superposition principles for expressing nonlinear responses as nonlinear superpositions of component responses;
- The study of possible exact or approximate (energy dependent) orthogonality relations satisfied by NNMs that would permit their use as bases for order reduction of the nonlinear dynamics; we mention at this point the computational studies of S. Shaw, C. Pierre and co-workers [103−107] that show that (ad hoc) NNM-based Galerkin expansions lead to more accurate numerical computations of the responses of flexible systems, compared to linear eigenfunction-based expansions;
- The examination of the relation of NNMs to computational bases extracted by techniques such as wavelet analysis and linear or nonlinear POD [some preliminary results on relation between NNMs and POMs, and between NNMs and nonlinear POMs are reported in [108−111]];
- The examination of the relation between NNMs and Volterra series expansions / HOFRFs; also, of the relation of NNMs to already studied nonlinear superposition techniques for special classes of dynamical systems.

(iv) All systems referenced in this paper are assumed to be deterministic. Because there will always be some degree of uncertainty in the numerical models due to unknown physics, environmental variability, economics of modeling for parameter estimation, uncertain inputs, manufacturing tolerances, assembly procedures, idealization errors, etc., the issues of uncertainty quantification and propagation, and of numerical predictability are central questions when it comes to assessing whether a simulation is capable of reproducing with acceptable accuracy the experiment it is supposed to replace. To this end, fundamental questions such as the following need to be addressed [112]:

1. Are the experiments and simulations consistent statistically speaking ?
2. What is the degree of confidence associated with the first answer ?
3. If additional data sets are available, by how much does the confidence increase ?

Such questions are progressively being addressed in the structural dynamics community by considering nonlinear system identification as an integral part of the V&V process.

(v) Research should focus more on testing of practical structures in their own operating environment, rather than on laboratory tests of representative structures. Algorithms for optimally deploying sensors and exciters along the structure are not yet fully developed. The ability to use vibrations induced by ambient environmental or operating loads is an area that merits further investigation; this will demand to reduce the dependence upon measurable excitation forces, as attempted in [73, 113]. On-line identification is also important for applications such as structural health monitoring [114, 115].

To conclude this paper, it is fair to say that, even if one cannot foresee the arrival of a paradigm shift, it can be safely predicted that during the next ten years a ‘universal’ technique capable of addressing nonlinear dynamical phenomena of every possible type in every possible structural configuration will not be developed. It is therefore likely that nonlinear system identification will have to retain its current ‘toolbox’ philosophy, with (hopefully) more powerful methodologies, techniques and algorithms of increased sophistication being added. In the future, the stage will be (hopefully) reached, where attempts to unify and combine the most powerful and reliable methods will be initiated.

ACKNOWLEDGMENTS

One of the authors (GK) is supported by a grant from the Belgian National Fund for Scientific Research (FNRS) which is gratefully acknowledged.

REFERENCES

[1] D.J. Ewins, *Modal Testing: Theory, Practice and Application*, Research Studies Press LTD (2nd edition), Hertfordshire, 2000.

[2] N.M.M. Maia, J.M.M. Silva, *Theoretical and Experimental Modal Analysis*, Research Studies Press LTD, Taunton, 1997.

[3] R. Singh, P. Davies, A.K. Bajaj, Identification of nonlinear and viscoelastic properties of flexible polyurethane foam, *Nonlinear Dynamics* 34 (2003), 319-346.

[4] C.M. Richards, R. Singh, Characterization of rubber isolator nonlinearities in the context of single- and multi-degree-of-freedom experimental systems, *Journal of Sound and Vibration* 247 (2001), 807-834.

[5] T.K. Caughey, A. Vijayaraghavan, Free and forced oscillations of a dynamic system with linear hysteretic damping, *International Journal of Non-Linear Mechanics* 5 (1970), 533-555.

[6] V.I. Babitsky, V.L. Krupenin, *Vibrations of Strongly Nonlinear Discontinuous Systems*, Springer Verlag, Berlin, 2001.

[7] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*. Wiley-Interscience, New York, 1979.

[8] S.H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering*, Addison-Wesley, Reading, 1994.

[9] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*. Springer-Verlag, Berlin, 2nd edition, 1999.

[10] R. Rand, *Lecture Notes on Nonlinear Vibrations*, Cornell, 2003. Notes freely available at http://tam.cornell.edu/Rand.html.

[11] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields*. Springer Verlag, New York, (1983).

[12] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer Verlag, New York, 1990.

[13] G. Kerschen, K. Worden, A.F. Vakakis, J.C. Golinval, Past, present and future of nonlinear system identification in structural dynamics, *Mechanical Systems and Signal Processing* 20 (2006), 505-592.

[14] T.K. Caughey, Equivalent linearisation techniques, *Journal of the Acoustical Society of America* 35 (1963), 1706-1711.

[15] W.D. Iwan, A generalization of the concept of equivalent linearization, *International Journal of Non-Linear Mechanics* 8 (1973), 279-287.

[16] R.M. Rosenberg, The normal modes of nonlinear n-degree-of-freedom systems, *Journal of Applied Mechanics* 29 (1962), 7-14.

[17] R.M. Rosenberg, On nonlinear vibrations of systems with many degrees of freedom, *Advances in Applied Mechanics* 9 (1966), 155-242.

[18] R. Rand, A direct method for nonlinear normal modes, *International Journal of Non-Linear Mechanics* 9 (1974), 363-368.

[19] S.W. Shaw, C. Pierre, Normal modes for non-linear vibratory systems, *Journal of Sound and Vibration* 164 (1993), 85-124.

[20] A.F. Vakakis, L.I. Manevitch, Y.V. Mikhlin, V.N. Pilipchuk, A.A. Zevin, *Normal Modes and Localization in Nonlinear Systems*, John Wiley & Sons, New York, 1996.

[21] P.J. Roache, *Verification and Validation in Computational Science and Engineering*, Hermosa Publications, Albuquerque, 1998.

[22] M. Link, M.I. Friswell, *Working group 1: Generation of validated structural dynamic models — Results of a benchmark study utilising the GARTEUR SM-AG19 test bed*, Mechanical Systems and Signal Processing 17 (2003), 9-20.

[23] I. Babuska, J.T. Oden, Verification and validation in computational engineering and science: basic concepts, *Computer Methods in Applied Mechanics and Engineering* 193 (2004), 4057-4066.

[24] P. Ibanez, Identification of dynamic parameters of linear and non-linear structural models from experimental data, *Nuclear Engineering and Design* 25 (1973), 30-41.

[25] S.F. Masri, T.K. Caughey, A nonparametric identification technique for nonlinear dynamic problems, *Journal of Applied Mechanics* 46 (1979), 433-447.

[26] K. Worden, G.R. Tomlinson, *Nonlinearity in Structural Dynamics: Detection, Identification and Modelling*. Institute of Physics Publishing, Bristol and Philadelphia, 2001.

[27] K. Worden, G. Kerschen, A.F. Vakakis, J.C. Golinval, Nonlinear system identification in structural dynamics: a short history. Proceedings of IMAC XXV, Orlanddo, 2007.

[28] H.J. Rice, J.A. Fitzpatrick, A generalised technique for spectral analysis of non-linear systems, *Mechanical Systems and Signal Processing* 2 (1988), 195-207.

[29] H. Esmonde, J.A. Fitzpatrick, H.J. Rice, F. Axisa, Analysis of non-linear squeeze film dynamics: part I — physical theory and modelling. *Proceedings of ASME PVP Conference*, Nashville, 1990.

[30] H. Esmonde, F. Axisa, J.A. Fitzpatrick, H.J. Rice, Analysis of non-linear squeeze film dynamics: part II — experimental measurement and model verification. *Proceedings of ASME PVP Conference*, Nashville, 1990.

[31] J.S. Bendat, *Nonlinear System Analysis and Identification from Random Data*, John Wiley & Sons, New York, 1990.

[32] H.J. Rice, J.A. Fitzpatrick, The measurement of nonlinear damping in single-degree-of-freedom systems, *Journal of Vibration and Acoustics* 113 (1991), 132-140.

[33] J.S. Bendat, R.N. Coppolino, P.A. Palo, Identification of physical parameters with memory in non-linear systems, *International Journal of Non-Linear Mechanics* 30 (1995), 841-860.

[34] B.A. Zeldin, P.D. Spanos, Spectral identification of nonlinear structures, *Journal of Engineering Mechanics* 124 (1998), 728-733.

[35] J.S. Bendat, *Nonlinear System Techniques and Applications*, John Wiley & Sons, New York, 1998.

[36] J.S. Bendat, Spectral techniques for nonlinear system analysis and identification, *Shock and Vibration* 1 (1993), 21-31.

[37] H.J. Rice, J.A. Fitzpatrick, A procedure for the identification of linear and non-linear multi-degree-of-freedom systems, *Journal of Sound and Vibration* 149 (1991), 397-411.

[38] C.M. Richards, R. Singh, Identification of multi-degree-of-freedom non-linear systems under random excitations by the reverse-path spectral method, *Journal of Sound and Vibration* 213 (1998), 673-708.

[39] G. Kerschen, J.C. Golinval, Generation of accurate finite element models of nonlinear systems - Application to an aeroplane-like structure, *Nonlinear Dynamics* 39 (2005), 129-142.

[40] J.A. Fitzpatrick, H.J. Rice, Comments on ”Identification of multi-degree-of-freedom non-linear systems under random excitations by the ’reverse path’ spectral method”, *Journal of Sound and Vibration* 237 (2000), 357-358.

[41] C.M. Richards, R. Singh, Comments on ”Identification of multi-degree-of-freedom non-linear systems under random excitations by the ’reverse path’ spectral method” — Authors’ reply, *Journal of Sound and Vibration* 237 (2000), 358-360.

[42] C.M. Richards, R. Singh, Comparison of two non-linear system identification approaches derived from ”reverse path” spectral analysis, *Journal of Sound and Vibration* 237 (2000), 361-376.

[43] C.M. Richards, R. Singh, Feasibility of identifying non-linear vibratory systems consisting of unknown polynomial forms, *Journal of Sound and Vibration* 220 (1999), 413-450.

[44] G. Kerschen, V. Lenaerts, S. Marchesiello, A. Fasana, A frequency domain versus a time domain identification technique for nonlinear parameters applied to wire rope isolators, *Journal of Dynamic Systems, Measurement, and Control* 123 (2001), 645-650.

[45] G. Kerschen, V. Lenaerts, J.C. Golinval, Identification of a continuous structure with a geometrical non-linearity, part I: conditioned reverse path method, *Journal of Sound and Vibration* 262 (2003), 889-906.

[46] L. Garibaldi, Application of the conditioned reverse path method, *Mechanical Systems and Signal Processing* 17 (2003), 227-236.

[47] S. Marchesiello, Application of the conditioned reverse path method, *Mechanical Systems and Signal Processing* 17 (2003), 183-188.

[48] D.E. Adams, R.J. Allemang, A frequency domain method for estimating the parameters of a non-linear structural dynamic model through feedback, *Mechanical Systems and Signal Processing* 14 (2000), 637-656.

[49] J.S. Bendat, A.G. Piersol, *Random Data: Analysis and Measurement Procedures*, Wiley Interscience, New York, 3rd edition, 2000.

[50] J.C. Golinval, G. Kerschen, V. Lenaerts, F. Thouverez, P. Argoul, European COST action F3 on structural dynamics. Working group 3: identification of non-linear systems; Introduction and conclusions, *Mechanical Systems and Signal Processing* 17 (2003), 177-178, 251-254.

[51] J.R.F. Arruda, J.M.C. Santos, Mechanical joint parameter estimation using frequency response functions and component mode synthesis, *Mechanical Systems and Signal Processing* 7 (1993), 493-508.

[52] E. Balmès, A finite element updating procedure using frequence response functions; application to the MIT/SERC interferometer testbed, *Proceedings of the 11th Internation Modal Analysis Conference*, Kissimmee, 176-182, 1993.

[53] R. Lin, D.J. Ewins, Analytical model improvement using frequency response functions, *Mechanical Systems and Signal Processing* 8 (1994), 437-458.

[54] R. Williams, J. Crowley, H. Vold, The multivariate mode indicator function in modal analysis, *Proceedings of the 4th Internation Modal Analysis Conference*, Los Angeles, 1986.

[55] J.R. Wright, J.E. Cooper, M. Desforges, Normal mode force appropriation — Theory and application, *Mechanical Systems and Signal Processing* 13 (1999), 217-240.

[56] S. Naylor, M.F. Platten, J.R. Wright, J.E. Cooper, Identification of multi-degree-of-freedom systems with nonproportional damping using the resonant decay method, *Journal of Vibration and Acoustics* 126 (2004), 298-306.

[57] J.R. Wright, M.F. Platten, J.E. Cooper, M. Sarmast, Identification of multi-degree-of-freedom weakly non-linear systems using a model based in modal space, *Proceedings of the International Conference on Structural System Identification*, Kassel, 49-68, 2001.

[58] Y. Ben-Haim, Identification of certain polynomial nonlinear structures by adaptive selectively-sensitive excitation, *Journal of Vibration and Acoustics* 115 (1993), 246-255.

[59] P. Atkins, J.R. Wright, K. Worden, An extension of force appropriation to the identification of non-linear multi-degree-of-freedom systems, *Journal of Sound and Vibration* 237 (2000), 23-43.

[60] O. Zienkiewicz, *The Finite Element Method*. McGraw-Hill, London, 1977.

[61] A. Berman, E.J. Nagy, Theory of incomplete models of dynamic structures, *AIAA Journal* 9 (1971), 1481-1487.

[62] M. Baruch, Optimization procedure to correct stiffness and flexibility matrices using vibration test, *AIAA Journal* 16 (1978), 1208-1210.

[63] H.G. Natke, *Einfuhrung in Theorie und Praxis der Zeitreihen und Modalanalyse*, Vieweg Verlag, Braunschweig/Wiesbaden, 1992.

[64] M.I. Friswell, J.E. Mottershead, *Finite Element Model Updating in Structural Dynamics*, Kluwer Academic Publishers, London, 1995.

[65] M. Link, *Updating of Analytical Models - Basic Procedures and Extensions* in *Modal Analysis & Testing*, J.M.M. Silva and N.M.M. Maia, editors, Kluwer Academic Publishers, Dordrecht, 1999.

[66] R. Schmidt, Updating non-linear components, *Mechanical Systems and Signal Processing* 8 (1994), 679-690.

[67] R.K. Kapania, S. Park, Parametric identification of nonlinear structural dynamic systems using time finite element method, *AIAA Journal* 35 (1997), 719-726.

[68] K.D. Dippery, S.W. Smith, An optimal control approach to nonlinear system identification, *Proceedings of the 16th International Modal Analysis Conference*, Santa Barbara, 637-643, 1998.

[69] A. Kyprianou, *Non-linear Parameter Estimation of Dynamic Models using Differential Evolution: Application to Hysteretic Systems and Hydraulic Engine Mounts*, Ph.D. Thesis, University of Sheffield, 1999.

[70] A. Kyprianou, K. Worden, M. Panet, Identification of hysteretic systems using the differential evolution algorithm, *Journal of Sound and Vibration* 248 (2001), 289-314.

[71] S. Meyer, M. Link, Modelling local non-linear behaviour — Simultaneous updating of linear and non-linear parameters using frequency response residuals, *Proceedings of International Conference on Structural Dynamics Modelling*, Funchal, 2002.

[72] S. Meyer, M. Weiland, M. Link, Modelling and updating of local non-linearities using frequency response residuals, *Mechanical Systems and Signal Processing* 17 (2003), 219-226.

[73] K.V. Yuen, J.L. Beck, Updating properties of nonlinear dynamical systems with uncertain input, *Journal of Engineering Mechanics* 129 (2003), 9-20.

[74] D.R. Mulville, *Pyroshock Test Criteria*, NASA Technical Standard, Report NASA-STD-7003, 1999.

[75] S. Doebling, F.M. Hemez, W. Rhee, Statistical model updating and validation applied to nonlinear transient structural dynamics, *Proceedings of the European COST F3 Conference on System Identification & Structural Health Monitoring*, Madrid, 409-418, 2000.

[76] J.F. Schultze, F.M. Hemez, S.W. Doebling, H. Sohn Application of non-linear system model updating using feature extraction and parameter effect analysis, *Shock and Vibration* 8 (2001), 325-337.

[77] Y. Song, C.J. Hartwigsen, D.M. McFarland, A.F. Vakakis, L.A. Bergman, Simulation of dynamics of beam structures with bolted joints using adjusted Iwan beam elements, *Journal of Sound and Vibration* 273 (2004), 249-276.

[78] X. Ma, A.F. Vakakis, Karhunen-Loève decomposition of the transient dynamics of a multibay truss, *AIAA Journal* 37 (1999), 939-946.

[79] T.K. Hasselman, M.C. Anderson, W.G. Gan, Principal component analysis for nonlinear model correlation, updating and uncertainty evaluation, *Proceedings of the 16th International Modal Analysis Conference*, Santa Barbara, 644-651, 1998.

[80] F.M. Hemez, S.W. Doebling, Review and assessment of model updating for non-linear, transient dynamics, *Mechanical Systems and Signal Processing* 15 (2001), 45-74.

[81] G. Kerschen, J.C. Golinval, A.F. Vakakis, L.A. Bergman, The method of proper orthogonal decomposition for dynamical characterization and order reduction of mechanical systems: an overview, *Nonlinear Dynamics* 41 (2005), 147-170.

[82] V. Lenaerts, G. Kerschen, J.C. Golinval, Proper orthogonal decomposition for model updating of non-linear mechanical systems, *Mechanical Systems and Signal Processing* 15 (2001), 31-43.

[83] V. Lenaerts, G. Kerschen, J.C. Golinval, Identification of a continuous structure with a geometrical non-linearity, part II: proper orthogonal decomposition, *Journal of Sound and Vibration* 262 (2003), 907-919.

[84] G. Kerschen, *On the Model Validation in Non-linear Structural Dynamics*, Ph.D. Thesis, University of Liège, 2003.

[85] G. Kerschen, J.C. Golinval, A model updating strategy of non-linear vibrating structures, *International Journal for Numerical Methods in Engineering* 60 (2004), 2147-2164.

[86] M.A. Kramer, Nonlinear principal component analysis using autoassociative neural networks, *A.I.Ch.E. Journal* 37 (1991), 233-243.

[87] D.C. Zimmerman, T. Hasselman, M. Anderson, Approximation and calibration of nonlinear structural dynamics, *Nonlinear Dynamics* 39 (2005).

[88] C. Mares, J.E. Mottershead, M.I. Friswell, Results obtained by minimising natural frequency errors and using physical reasoning, *Mechanical Systems and Signal Processing* 17 (2003), 39-46.

[89] P. Holmes, J.L. Lumley, G. Berkooz, *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge, New York, 1996.

[90] F.M. Hemez, Y.W. Ben-Haim, Info-gap robustness for the correlation of tests and simulations of a non-linear transient, *Mechanical Systems and Signal Processing* 18 (2004), 1443-1467.

[91] L. Gaul, J. Lenz, Nonlinear dynamics of structures assembled by bolted joints, *Acta Mechanica* 125 (1997), 169-181.

[92] C.J. Hartwigsen, Y. Song, D. M. McFarland, L.A. Bergman, A.F. Vakakis, Experimental study of non-linear effects in a typical shear lap joint configuration. *Journal of Sound and Vibration* 277 (2004), 327-351.

[93] J.L. Quinby, B.F. Feeny, Low-frequency phenomena in a frictionally excited beam, *ASME International Mechanical Engineering Congress*, Anaheim, 2004.

[94] E.B. Kosmatopoulos, A.W. Smyth, S.F. Masri, A.G. Chassiakos, Robust adaptive neural estimation of restoring forces in nonlinear structures, *Journal of Applied Mechanics* 68 (2001), 880-893.

[95] Y. Fan, C.J. Li, Non-linear system identification using lumped parameter models with embedded feedforward neural networks, *Mechanical Systems and Signal Processing* 16 (2002), 357-372.

[96] M. Peifer, J. Timmer, H.U. Voss, Nonparametric identification of nonlinear oscillating systems, *Journal of Sound and Vibration* 267 (2003), 1157-1167.

[97] P. Malatkar, A.H. Nayfeh, A plethora of nonlinear dynamics phenomena observed in a simple cantilever plate, *Proceedings of the 2003 ASME Design Engineering Technical Conferences*, Chicago, 2003, DETC2003/VIB-48619.

[98] J.P. Caffrey, S.F. Masri, F. Tasbihgoo, A.W. Smyth, A.G. Chassiakos, A Re-Configurable Test Apparatus for Complex Nonlinear Dynamic Systems, *Nonlinear Dynamics* 36 (2004), 181-201.

[99] J.K. Hammond, *Non-Structural Dynamics* in *Structural Dynamics @ 2000: Current Status and Future Directions*, D.J. Ewins and D. Inman, editors, Research Studies Press, Baldock, 2001.

[100] D. Quinn, R. Rand, J. Bridge, The dynamics of resonance capture, *Nonlinear Dynamics* 8 (1995), 1-20.

[101] D. Quinn, Resonance capture in a three degree of freedom mechanical system, *Nonlinear Dynamics* 14 (1997), 309-333.

[102] A.F. Vakakis, O. Gendelman, Energy pumping in nonlinear mechanical oscillators: Part II — resonance capture, *Journal of Applied Mechanics* 68 (2001), 42-48.

[103] E. Pesheck, N. Boivin, C. Pierre, S.W. Shaw, Nonlinear modal analysis of structural systems using multi-mode invariant manifolds, *Nonlinear Dynamics* 25 (2001a), 183-205.

[104] E. Pesheck, C. Pierre, S.W. Shaw, Accurate reduced-order models for a simple rotor blade model using nonlinear normal modes, *Mathematical and Computer Modelling* 33 (2001), 1085-1097. (

[105] E. Pesheck, C. Pierre, S.W. Shaw, Modal reduction of a nonlinear rotating beam through nonlinear normal modes, *Journal of Vibration and Acoustics* 124 (2002), 229-236.

[106] E. Pesheck, C. Pierre, S.W. Shaw, A new Galerkin-based approach for accurate non-linear normal modes through invariant manifolds, *Journal of Sound and Vibration* 249 (2002), 971-993.

[107] P. Apiwattanalunggarn, S.W. Shaw, C. Pierre, Finite-element-based nonlinear modal reduction of a rotating beam with large-amplitude motion, *Journal of Vibration and Control* 9 (2003), 235-263.

[108] B.F. Feeny, R. Kappagantu, On the physical interpretation of proper orthogonal modes in vibrations, *Journal of Sound and Vibration* 211 (1998), 607-616.

[109] B.F. Feeny, On the proper orthogonal modes and normal modes of continuous vibration systems , *Journal of Vibration and Acoustics* 124 (2002), 157-160.

[110] G. Kerschen, J.C. Golinval, Physical interpretation of the proper orthogonal modes using the singular value decomposition, *Journal of Sound and Vibration* 249 (2002), 849-865.

[111] G. Kerschen, J.C. Golinval, Feature extraction using auto-associative neural networks, *Smart Materials and Structures* 13 (2004), 211-219.

[112] F.M. Hemez, S.W. Doebling, Inversion of structural dynamics simulations: state-of-the-art and orientations of the research, *Proceedings of the International Seminar on Modal Analysis (ISMA)*, Leuven, 2000.

[113] M. Haroon, D.E. Adams, Y.W. Luk, A.A. Ferri, A time and frequency domain approach for identifying nonlinear mechanical system models in the absence of an input measurement, *Journal of Sound and Vibration* 283 (2005), 1137-1155.

[114] A.W. Smyth, S.F. Masri, E.B. Kosmatopoulos, A.G. Chassiakos, T.K. Caughey, Development of adaptive modeling techniques for non-linear hysteretic systems, *International Journal of Non-Linear Mechanics* 37 (2002), 1437-1451.

[115] J.N. Yang, S. Lin, On-line identification of non-linear hysteretic structures using an adaptive tracking technique, *International Journal of Non-Linear Mechanics* 39 (2004), 1481-1491.