Enhanced Laplace transform and holomorphic Paley-Wiener theorems

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Motivations

Let K be a non-empty convex compact set of $\mathbb{C}$ and $h_K : w \mapsto \sup_{z\in K} \Re(zw)$ its support function. The classical Polya's representation theorem states that there is a (topological) isomorphism between

$$\mathcal{O}^0(\mathbb{C}\setminus K) = \{u \in \mathcal{O}(\mathbb{C}\setminus K) : \lim_{|z|\to+\infty} u(z) = 0\}$$

and

$$\mathrm{Exp}(K) = \{v \in \mathcal{O}(\mathbb{C}) : \forall\varepsilon > 0,\ \sup_{w\in\mathbb{C}} |v(w)|e^{-h_K(w)-\varepsilon|w|} < \infty\},$$

given by

$$\mathcal{O}^0(\mathbb{C}\setminus K) \ni u \mapsto \left(w \mapsto \frac{1}{2i\pi}\int_{C(0,r)^+} e^{zw} u(z)\, dz\right) \in \mathrm{Exp}(K),$$

where $C(0,r)^+$ is a circle which encloses K.

This kind of holomorphic Laplace transform is deeply linked with the Borel transform. Indeed, take $K = \overline{D}(0, R)$ and

$$f(z) = \sum_{k=1}^{+\infty} a_k z^{-k} \in \mathcal{O}^0(\mathbb{C} \setminus K).$$

Then

$$\begin{aligned} \frac{1}{2i\pi} \int_{C(0,2R)^+} e^{zw} u(z)\, dz &= \frac{1}{2i\pi} \sum_{k=1}^{+\infty} a_k \int_{C(0,2R)^+} e^{zw} z^{-k}\, dz \\ &= \sum_{k=1}^{+\infty} \frac{a_k w^{k-1}}{(k-1)!}. \end{aligned}$$

There is a kind of non-compact analogue of Polya's theorem, due to Méril.

Let $S \subset \mathbb{C}$ be a non-empty closed convex non-compact set which contains no lines. Let us set

$$S_\infty = \{z \in \mathbb{C} : z + S \subset S\}$$

the asymptotic cone of S and

$$S_\infty^* = \{w \in \mathbb{C} : \Re(zw) \leq 0\}$$

the polar cone of S_∞. It is a closed convex proper cone of $\mathbb{C}$ with non-empty interior. Let ξ_0 be a fixed complex number on the bisector of S_∞^*.

Theorem (Méril, 1983)

There is a (topological) isomorphism between

$$\varinjlim_{\varepsilon'\to 0} \frac{\{u \in \mathcal{O}(\mathbb{C}\setminus S) : \forall r > \varepsilon > 0, \sup_{z\in S_r\setminus S_\varepsilon} |u(z)e^{\varepsilon'\xi_0 z}| < \infty)\}}{\{u \in \mathcal{O}(\mathbb{C}) : \forall r > 0, \sup_{z\in S_r} |u(z)e^{\varepsilon'\xi_0 z}| < \infty)\}}$$

and

$$\begin{aligned} \mathrm{Exp}(S) = &\{v \in \mathcal{O}((S^*_\infty)^\circ) : \forall \varepsilon, \varepsilon' > 0, \\ &\sup_{w\in S^\star_\infty + \varepsilon'\xi_0} |v(w)e^{-h_S(w)-\varepsilon|w|}| < \infty\}, \end{aligned}$$

given by

$$([u_{\varepsilon'}])_{\varepsilon'} \mapsto \frac{1}{2i\pi}\int_{\partial S^+_\varepsilon} e^{zw} u_{\varepsilon'}(z)\, dz,$$

where ∂S^+_ε is the positively oriented boundary of any thickening S_ε.

Aim of the work

The aim is to understand the cohomological framework which allows to obtain such kind of holomorphic Paler-Wiener theorems and to see how the contour integrations naturally appear.

At the end, we would like to have a kind of algorithmic device which provides Laplace isomorphisms between interesting functional spaces.

Strategy

1) To develop an abstract notion of Laplace transform for enhanced subanalytic sheaves and applying it to some sheaf of functions/distributions with growth conditions. (Whitney, Gevrey, tempered, ...)

2) To apply the isomorphisms obtained in the previous step to concrete cases (e.g. the Legendre transform of a convex function) to reveal functionals spaces which are isomorphic through the Laplace transform. The contour integrations will naturally appear when computing explicitly the cohomology groups.

Let $\mathbb{V}$ be a complex vector space of dimension n and consider the correspondance

$$\mathbb{V} \xleftarrow{p} \mathbb{V} \times \mathbb{V}^* \xrightarrow{q} \mathbb{V}^*$$

and the complex duality bracket $\langle , \rangle : \mathbb{V} \times \mathbb{V}^* \to \mathbb{C}$. The (negative) Laplace transform of a tempered distributions on $\mathbb{V}$ is the composition of three operations :

1) the pullback by p,

2) the multiplication by $e^{-\langle z,w \rangle}$,

3) the direct image (integration over fibers) along q.

Problem : The second step does not preserve the temperate condition. We introduce an additional variable to take care of the exponential.

Tempered distributions

Definition

Let M be a real analytic manifold. We define a subanalytic sheaf $\mathcal{D}b^{\mathrm{t}}_M$ by setting

$$\Gamma(U, \mathcal{D}b^{\mathrm{t}}_M) = \{u \in \mathcal{D}b_M(U) : u \text{ can be extended to } M\}.$$

On a complex manifold X, we note $\mathcal{O}^{\mathrm{t}}_X \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_X)$ the associated Dolbeault complex.

Tempered distributions form a subanalytic flabby sheaf. For example, one has

$$\Gamma(\mathbb{R}^n, \mathcal{D}b^{\mathrm{t}}_{S^n}) = \mathcal{S}'(\mathbb{R}^n).$$

Let us note $\overline{\mathbb{V}}$ the projective compactification of $\mathbb{V}$.

Enhanced distributions

Let us note $\mathsf{P} = \mathbb{R} \cup \{\infty\}$ and $\mathbb{V}_\infty \times \mathbb{R}_\infty = (\mathbb{V} \times \mathbb{R}, \overline{\mathbb{V}} \times \mathsf{P})$.

Definition

We set

$$\Gamma(U, \mathcal{D}b^{\mathsf{T}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty}) = \{u \in \Gamma(U, \mathcal{D}b^{\mathsf{t}}_{\overline{\mathbb{V}} \times \mathsf{P}}) : \partial_t u = u\},$$

for all subanalytic open set $U \subset \mathbb{V} \times \mathbb{R}$.

We call it the subanalytic sheaf (in the bordered sense) of *enhanced distributions*.

We note $\mathcal{O}^{\mathsf{T}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty} \in \mathsf{D}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty})$ the associated Dolbeault complex and $\mathcal{O}^{\mathsf{E}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty}$ the associated object in $\mathsf{E}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty})$.

Pullback of distributions

Let us note $p_{\mathbb{R}} = p \times \mathrm{id}_{\mathbb{R}}$. For all $k, l \in \mathbb{Z}$, one has a morphism

$$p_{\mathbb{R}}^{*} : p_{\mathbb{R}}^{-1} \mathcal{D}b^{\mathsf{T},(k,l)}_{\mathbb{V}_\infty \times \mathbb{R}_\infty} \to \mathcal{D}b^{\mathsf{T},(k,l)}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty \times \mathbb{R}_\infty},$$

given by the pullback of distributions by the submersion $p_{\mathbb{R}}$.

In 1984, M. Kashiwara proved that the pullback of a tempered distribution is still tempered. Moreover,

$$\begin{aligned}\langle \partial_t p_{\mathbb{R}}^* u, \omega \rangle = -\langle p_{\mathbb{R}}^* u, \partial_t \omega \rangle &= -\left\langle u, \int_{\mathbb{V}^*} \partial_t \omega \right\rangle \\ &= -\left\langle u, \partial_t \int_{\mathbb{V}^*} \omega \right\rangle = \left\langle \partial_t u, \int_{\mathbb{V}^*} \omega \right\rangle \\ &= \left\langle u, \int_{\mathbb{V}^*} \omega \right\rangle = \langle p_{\mathbb{R}}^* u, \omega \rangle.\end{aligned}$$

Multiplication by the exponential kernel

Let us define

$$\mu_{\Re\langle,\rangle} : (z, w, t) \in \mathbb{V} \times \mathbb{V}^* \times \mathbb{R} \mapsto (z, w, t + \Re\langle z, w\rangle) \in \mathbb{V} \times \mathbb{V}^* \times \mathbb{R}\,.$$

One has a morphism of subanalytic sheaves

$$\mu_{\Re\langle,\rangle *}\,\mathcal{D}b^{\mathsf{T},(k,l)}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty \times \mathbb{R}_\infty} \to \mathcal{D}b^{\mathsf{T},(k,l)}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty \times \mathbb{R}_\infty}$$

given by $u(z, w, t) \mapsto u(z, w, t - \Re\langle z, w\rangle)$. Since u is enhanced, there is a unique v such that $u(z, w, t) = e^t v(z, w)$. Hence

$$u(z, w, t - \Re\langle z, w\rangle) = e^{t-\Re\langle z,w\rangle} v(z, w) = e^{-\Re\langle z,w\rangle} u(z, w, t).$$

Then one composes this morphism with the multiplication by $e^{-i\Im\langle z,w\rangle}$.

Direct image (integration) of distributions

Let us note $q_{\mathbb{R}} = q \times \mathrm{id}_{\mathbb{R}}$. For all $k, l \in \mathbb{Z}$ one has a morphism

$$\int_{q_{\mathbb{R}}} : q_{\mathbb{R}!!}\, \mathcal{D}b^{\mathsf{T},(k,l)}_{\mathbb{V}_\infty \times \mathbb{V}^*_\infty \times \mathbb{R}_\infty} \to \mathcal{D}b^{\mathsf{T},(k-n,l-n)}_{\mathbb{V}^*_\infty \times \mathbb{R}_\infty},$$

given by the direct image of distributions by $q_{\mathbb{R}}$.

One easily see that the tempered condition is preserved thanks to the compactness of $\overline{\mathbb{V}} \times \overline{\mathbb{V}}^* \times \mathsf{P}$. Moreover,

$$\begin{aligned} \left\langle \partial_t \int_{q_{\mathbb{R}}} u, \omega \right\rangle &= -\left\langle \int_{q_{\mathbb{R}}} u, \partial_t \omega \right\rangle = -\langle u, q_{\mathbb{R}}^* \partial_t \omega \rangle \\ &= -\langle u, \partial_t q_{\mathbb{R}}^* \omega \rangle = \langle \partial_t u, q_{\mathbb{R}}^* \omega \rangle \\ &= \langle u, q_{\mathbb{R}}^* \omega \rangle\rangle = \left\langle \int_{q_{\mathbb{R}}} u, \omega \right\rangle . \end{aligned}$$

Laplace transform of enhanced distributions

The negative Laplace transform (with parameter t) is encoded by

$$\begin{aligned} q_{\mathbb{R}!!}(\mu_{\Re\langle,\rangle_*} p_{\mathbb{R}}^{-1}\,\mathcal{D}b^{\mathsf{T},(n,l+n)}_{\mathbb{V}_\infty\times\mathbb{R}_\infty}) &\to q_{\mathbb{R}!!}(\mu_{\Re\langle,\rangle_*}\,\mathcal{D}b^{\mathsf{T},(n,l+n)}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty\times\mathbb{R}_\infty}) \\ &\to q_{\mathbb{R}!!}(\mathcal{D}b^{\mathsf{T},(n,l+n)}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty\times\mathbb{R}_\infty}) \\ &\to \mathcal{D}b^{\mathsf{T},(0,l)}_{\mathbb{V}^*_\infty\times\mathbb{R}_\infty}. \end{aligned}$$

This gives derived morphisms

$$\mathrm{R}q_{\mathbb{R}!!}(\mathrm{R}\mu_{\Re\langle,\rangle_*} p_{\mathbb{R}}^{-1}\Omega^{\mathsf{T}}_{\mathbb{V}_\infty\times\mathbb{R}_\infty})[n] \to \mathcal{O}^{\mathsf{T}}_{\mathbb{V}^*_\infty\times\mathbb{R}_\infty},$$

$$\mathrm{E}q_{!!}(\mathbb{C}_{\{t=\Re\langle z,w\rangle\}}\overset{+}{\otimes}\mathrm{E}p^{-1}\Omega^{\mathsf{E}}_{\mathbb{V}_\infty\times\mathbb{R}_\infty})[n] \to \mathcal{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty\times\mathbb{R}_\infty}.$$

Fourier-Sato functor

Definition

The enhanced Fourier-Sato functor

$$ {}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}} : \mathsf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty}) \to \mathsf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}^*_\infty \times \mathbb{R}_\infty}) $$

is defined by

$$ {}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}(F) = \mathsf{E}q_{!!}(\mathbb{C}_{\{t=\Re\langle z,w\rangle\}} \overset{+}{\otimes} \mathsf{E}p^{-1}F) $$

Theorem (Kashiwara, Schapira, 2016)

The enhanced Fourier-Sato functor ${}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}$ *is an equivalence of categories. In particular, one has an isomorphism*

$$ \mathrm{RHom}^{\mathsf{E}}(F_1, F_2) \simeq \mathrm{RHom}^{\mathsf{E}}({}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}(F_1), {}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}(F_2)), $$

functorial in $F_1, F_2 \in \mathsf{E}^{\mathrm{b}}(\mathbb{C}^{\mathrm{sub}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty})$.

The Laplace isomorphism

Theorem

The morphism

$$ {}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}(\Omega^{\mathsf{E}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty})[n] \to \mathcal{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty \times \mathbb{R}_\infty} $$

is an isomorphism in $\mathsf{E}^{\mathsf{b}}(\mathbb{C}^{\mathsf{sub}}_{\mathbb{V}^*_\infty \times \mathbb{R}_\infty})$.

Idea : Use the enhanced Riemann-Hilbert correspondence to go back to the famous result

$$ \mathsf{D}p_*(\mathcal{L} \overset{\mathsf{D}}{\otimes} \mathsf{D}q^*(\mathscr{D}_{\overline{\mathbb{V}}^*}(*\mathbb{H}^*)) \simeq \mathscr{D}_{\overline{\mathbb{V}}}(*\mathbb{H}) \otimes \det(\mathbb{V}) $$

of Katz and Laumon (1985).

Legendre transform

(i) We say that $f : \mathbb{V} \to \mathbb{R} \cup \{+\infty\}$ is a closed proper convex function on $\mathbb{V}$ if its epigraph $\{(z, t) \in \mathbb{V} \times \mathbb{R} : t \geq f(z)\}$ is closed, convex and non-empty.

(ii) For any $f \in \mathrm{Conv}(\mathbb{V})$, we set $\mathrm{dom}(f) = f^{-1}(\mathbb{R})$ and call it the domain of f. This set is convex and non-empty.

(iii) For any $f \in \mathrm{Conv}(\mathbb{V})$, we define a function $f^* : \mathbb{V}^* \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f^*(w) = \sup_{z \in \mathrm{dom}(f)} (\Re\langle z, w\rangle - f(z)).$$

We call it the Legendre transform of f. It is an element of $\mathrm{Conv}(\mathbb{V}^*)$.

(iv) For any $f \in \mathrm{Conv}(\mathbb{V})$, we denote by $H(f)$ the real affine space generated by $\mathrm{dom}(f)$ and we set $E(f) = H(f^*)^\perp$. We also set

$$d(f) = \dim E(f) = \mathrm{codim}\, H(f^*).$$

Lemma (Kashiwara, Schapira, 2016)

Let $f \in \mathrm{Conv}(\mathbb{V})$. One has an isomorphism

$$ {}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}(\mathbb{C}_{\{t \geq f(z)\}}) \simeq \mathbb{C}_{\{t \geq -f^*(-w), -w \in \mathrm{dom}^\circ(f^*)\}} \otimes or_{E(f)}[d(f)]. $$

Assume $H(f) = \mathbb{V}^*$. As a consequence one gets an isomorphism

$$ \begin{aligned} & \mathrm{RHom}^{\mathsf{E}}(\mathbb{C}_{\{t \geq f(z)\}}, \Omega^{\mathsf{E}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty})[n] \\ & \simeq \mathrm{RHom}^{\mathsf{E}}(\mathbb{C}_{\{t \geq -f^*(-w), -w \in \mathrm{dom}^\circ(f^*)\}}, {}^{\mathsf{E}}\mathcal{F}_{\mathbb{V}}\Omega^{\mathsf{E}}_{\mathbb{V}_\infty \times \mathbb{R}_\infty})[n] \\ & \xrightarrow{\sim} \mathrm{RHom}^{\mathsf{E}}(\mathbb{C}_{\{t \geq -f^*(-w), -w \in \mathrm{dom}^\circ(f^*)\}}, \mathcal{O}^{\mathsf{E}}_{\mathbb{V}^*_\infty \times \mathbb{R}_\infty}) \end{aligned} $$

given by the enhanced Laplace transform. One can show that these complexes are concentrated in degree 0. How can we compute explicitly the degree 0 morphism using Dolbeault complexes ?

Almost subanalytic functions

Definition

Let M be a real analytic manifold and $U \subset M$ a subanalytic open subset of M. A function $f : U \to \mathbb{R}$ is subanalytic if its graph $\Gamma_f \subset U \times \mathbb{R}$ is subanalytic in $M \times \overline{\mathbb{R}}$. A continuous function $f : U \to \mathbb{R}$ is *almost $\mathcal{C}^\infty$-subanalytic* if there is a subanalytic $\mathcal{C}^\infty$-function $g : U \to \mathbb{R}$ such that

$$\exists\, C > 0, \forall x \in U : |f(x) - g(x)| < C.$$

In this case, we say that g is in the (ASA)-class of f.

M. Kashiwara and P. Schapira conjecture that any continuous subanalytic function $f : U \to \mathbb{R}$ is almost subanalytic.

Exponential growth condition

Definition

Let $f : U \to \mathbb{R}$ be a continuous almost $\mathcal{C}^\infty$-subanalytic function. For any open subanalytic set $V \subset M$ and any $p \in \mathbb{Z}$, we set

$$e^{-f}\mathcal{D}b^{\mathrm{t},p}_M(V) = \{u \in \mathcal{D}b^p_M(U \cap V) : e^g u \in \mathcal{D}b^{\mathrm{t},p}_M(U \cap V)\},$$

where g is in the (ASA)-class of f. The correspondence $V \mapsto e^{-f}\mathcal{D}b^{\mathrm{t},p}_M(V)$ clearly defines a subanalytic flabby sheaf on M.

Let U be an open set of complex manifold X and $f : U \to \mathbb{R}$ a continuous almost $\mathcal{C}^\infty$-subanalytic function. For each $p \in \mathbb{Z}$, we define the subanalytic sheaf $e^f\Omega^{\mathrm{t},p}_X$ by the Dolbeault complex

$$0 \to e^{-f}\mathcal{D}b^{\mathrm{t},(p,0)}_X \xrightarrow{\bar\partial} e^{-f}\mathcal{D}b^{\mathrm{t},(p,1)}_X \to \cdots \to e^{-f}\mathcal{D}b^{\mathrm{t},(p,d_X)}_X \to 0.$$

Proposition

Let $f : M \to \mathbb{R} \cup \{+\infty\}$ be a function such that $f^{-1}(\mathbb{R})$ is a closed subset of M. Assume moreover that there is a subanalytic function $g \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $g|_{f^{-1}(\mathbb{R})} = f|_{f^{-1}(\mathbb{R})}$. There is an isomorphism

$$\Gamma_{f^{-1}(\mathbb{R})}(e^{-g}\,\mathcal{D}b_M^{\mathsf{t},p}) \simeq \mathrm{R}\mathscr{I}hom^{\mathsf{E}}(\mathbb{C}_{\{t \geq f(x)\}}, \mathcal{D}b^{\mathsf{E},p}_{M\times\mathbb{R}_\infty})$$

for each $p \in \mathbb{Z}$. In particular, the right hand side is concentrated in degree 0.

We prove it on sections. Set $S = f^{-1}(\mathbb{R})$. Let U be an open subanalytic set of M. We can show that u is in

$$\mathscr{A} := \{u \in \mathcal{D}b_M^p(U) : \mathrm{supp}(u) \subset S, e^g u \in \mathcal{D}b_M^{\mathsf{t},p}(U)\}$$

if and only if $e^t u$ is in

$$\mathscr{B} := \{u \in \mathcal{D}b^{\mathsf{t},p}_{M\times\mathrm{P}}((U\times\mathbb{R})\cap\{t < g(x)\}) : \mathrm{supp}(u) \subset S,\ \partial_t u = u\}.$$

Proposition

Let M be a real analytic manifold and $U \subset M$ a subanalytic open subset of M. Let $h : U \to \mathbb{R}$ be a continuous almost $\mathcal{C}^\infty$-subanalytic function. There is an isomorphism

$$e^{-h}\mathcal{D}b_M^{\mathsf{t},p} \simeq \mathrm{R}\mathscr{I}hom^{\mathsf{E}}(\mathbb{C}_{\{t\geq h(x), x\in U\}}, \mathcal{D}b_{M\times\mathbb{R}_\infty}^{\mathsf{E},p})$$

for each $p \in \mathbb{Z}$. In particular, the right hand side is concentrated in degree 0.

Let us go back to $\mathbb{V}$ and $f \in \mathrm{Conv}(\mathbb{V})$. Let us assume that

(i) The convex sets $\mathrm{dom}(f)$ and $\mathrm{dom}^\circ(f^*)$ are subanalytic.

(ii) There is a subanalytic $\mathcal{C}^\infty$-function g on $\mathbb{V}$ such that $g|_{\mathrm{dom}(f)} = f|_{\mathrm{dom}(f)}$.

(iii) The function $f^* : \mathrm{dom}^\circ(f^*) \to \mathbb{R}$ is almost $\mathcal{C}^\infty$-subanalytic.

Theorem

There is an isomorphism

$$H^n_{\mathrm{dom}(f)}(\mathbb{V}, e^{-g}\Omega^{\mathrm{t}}_{\overline{\mathbb{V}}}) \simeq H^0(\mathbb{V}^*, e^{f^*}\mathcal{O}^{\mathrm{t}}_{\overline{\mathbb{V}}^*})$$

explicitly given by

$$[u] \in \left(\frac{\Gamma_{\mathrm{dom}(f)} e^{-g}\, \mathcal{D}b^{\mathrm{t},(n,n)}_{\overline{\mathbb{V}}}}{\bar{\partial}\Gamma_{\mathrm{dom}(f)} e^{-g}\, \mathcal{D}b^{\mathrm{t},(n,n-1)}_{\overline{\mathbb{V}}}} \right)$$

$$\mapsto \mathcal{L}^+ u \in \{v \in \mathcal{O}_{\mathbb{V}^*}(\mathrm{dom}^\circ(f^*)) : e^{-f^*} v \text{ is tempered on } \overline{\mathbb{V}}^*\}.$$

Let $\mathbb{V} = \mathbb{C}$. We identify $\mathbb{V}^*$ to $\mathbb{C}$ in such a way that $\langle z, w \rangle = zw$. Let K be a non-empty convex compact set of $\mathbb{C}$. For all $\varepsilon > 0$, we define a function $f_\varepsilon : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f_\varepsilon(z) = \begin{cases} 0 & \text{if } z \in K_\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, this function is convex of domain K_ε and is the restriction to K_ε of a subanalytic $\mathcal{C}^\infty$-function defined on $\mathbb{C}$. Moreover, its Legendre transform is given by

$$f_\varepsilon^*(w) = \sup_{z \in K_\varepsilon} \Re(zw) = h_{K_\varepsilon}(w) = h_K(w) + h_{\overline{D}(0,\varepsilon)}(w) = h_K(w) + \varepsilon|w|$$

for all $w \in \mathbb{C}$. In particular, $\mathrm{dom}^\circ(f_\varepsilon^*) = \mathbb{C}$.

Identifying 0-forms and 1-forms and under suitable hypothesis, one gets an isomorphism

$$H^1_{K_\varepsilon}(\mathbb{C}, \mathcal{O}^{\mathrm{t}}_{\mathbb{P}}) \xrightarrow{\sim} \{v \in \mathcal{O}(\mathbb{C}) : v \in e^{h_{K_\varepsilon}} \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C})\}$$

for all $\varepsilon > 0$.

Proposition

Let $\varepsilon > 0$. One has a canonical isomorphism

$$H^1_{K_\varepsilon}(\mathbb{C}, \mathcal{O}^{\mathrm{t}}_{\mathbb{P}}) \simeq \frac{\mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C} \setminus K_\varepsilon) \cap \mathcal{O}(\mathbb{C} \setminus K_\varepsilon)}{\mathbb{C}[z]}.$$

Corollary

One has an isomorphism

$$\varinjlim_{\varepsilon \to 0} H^1_{K_\varepsilon}(\mathbb{C}, \mathcal{O}^{\mathrm{t}}_{\mathbb{P}}) \simeq \frac{\{u \in \mathcal{O}(\mathbb{C} \setminus K) : u \textit{ is tempered at } \infty\}}{\mathbb{C}[z]}.$$

Proposition

Let $\varepsilon > 0$ and let $u \in \mathcal{D}b^{\mathsf{t}}_{\mathbb{P}}(\mathbb{C} \setminus K_\varepsilon) \cap \mathcal{O}(\mathbb{C} \setminus K_\varepsilon)$. Let us note $\underline{u}$ the extension of u to $\mathbb{C}$. Then

$$[\bar{\partial}_z \underline{u}] \in \frac{\Gamma_{K_\varepsilon}(\mathbb{C}, \mathcal{D}b^{\mathsf{t}}_{\mathbb{P}})}{\bar{\partial}_z \Gamma_{K_\varepsilon}(\mathbb{C}, \mathcal{D}b^{\mathsf{t}}_{\mathbb{P}})} \simeq H^1_{K_\varepsilon}(\mathbb{C}, \mathcal{O}^{\mathsf{t}}_{\mathbb{P}})$$

is the image of

$$[u] \in \frac{\mathcal{D}b^{\mathsf{t}}_{\mathbb{P}}(\mathbb{C} \setminus K_\varepsilon) \cap \mathcal{O}(\mathbb{C} \setminus K_\varepsilon)}{\mathbb{C}[z]}.$$

Key of the proof : Use the distinguished triangle

$$\mathrm{R}\Gamma_{K_\varepsilon}(\mathbb{C}, \mathcal{O}^{\mathsf{t}}_{\mathbb{P}}) \to \mathrm{R}\Gamma(\mathbb{C}, \mathcal{O}^{\mathsf{t}}_{\mathbb{P}}) \to \mathrm{R}\Gamma(\mathbb{C} \setminus K_\varepsilon, \mathcal{O}^{\mathsf{t}}_{\mathbb{P}}) \xrightarrow{+1}$$

and compute the mapping cone of the restriction morphism

$$\rho_{K_\varepsilon} : \mathcal{D}b^{\mathsf{t}}_{\mathbb{P}}(\mathbb{C}) \to \mathcal{D}b^{\mathsf{t}}_{\mathbb{P}}(\mathbb{C} \setminus K_\varepsilon).$$

Corollary

Let $\varepsilon > 0$ *and let* ψ_ε *be a* $\mathcal{C}^\infty$*-regularizing function which is equal to* 1 *on* $\mathbb{C} \setminus K_\varepsilon$ *and to* 0 *on* $K_{\varepsilon/2}$*. Let* $u \in \mathcal{O}(\mathbb{C} \setminus K)$ *be a tempered function at* ∞*. Then the image of* $[u]$ *through the canonical map*

$$\frac{\{u \in \mathcal{O}(\mathbb{C} \setminus K) : u \text{ is tempered at } \infty\}}{\mathbb{C}[z]} \to \frac{\mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C} \setminus K_\varepsilon) \cap \mathcal{O}(\mathbb{C} \setminus K_\varepsilon)}{\mathbb{C}[z]}$$

$$\xrightarrow{\sim} H^1_{K_\varepsilon}(\mathbb{C}, \mathcal{O}^{\mathrm{t}}_{\mathbb{P}})$$

is given by $[\bar{\partial}_z(\psi_\varepsilon u)]$.

Definition

We set

$$\mathrm{Exp}^{\mathrm{t}}(K) = \varprojlim_{\varepsilon \to 0} \{v \in \mathcal{O}(\mathbb{C}) : v \in e^{h_{K_\varepsilon}} \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathbb{C})\}.$$

Theorem

There is an isomorphism of $\mathbb{C}$-vector spaces

$$\frac{\{u \in \mathcal{O}(\mathbb{C} \setminus K) : u \text{ is tempered at } \infty\}}{\mathbb{C}[z]} \xrightarrow{\sim} \mathrm{Exp}^{\mathrm{t}}(K)$$

given by

$$[u] \mapsto \left(w \in \mathbb{C} \mapsto - \int_{C(0,r)^+} e^{zw} u(z)\, dz \right) \in \mathrm{Exp}^{\mathrm{t}}(K),$$

where $C(0,r)^+$ is a circle of center 0 and radius r, positively oriented, which encloses K.

Idea of the proof :

$$\begin{aligned}
\mathcal{L}_w(\bar{\partial}_z(\psi_\varepsilon u)) &= \int_{\mathbb{C}} e^{zw}\bar{\partial}_z(\psi_\varepsilon u)(z)\, dz \wedge d\bar{z} \\
&= \int_{\mathbb{C}} \bar{\partial}_z(e^{zw}\psi_\varepsilon u)(z)\, dz \wedge d\bar{z} \\
&= \int_{\overline{D(0,r)}} \bar{\partial}_z(e^{zw}\psi_\varepsilon u)(z)\, dz \wedge d\bar{z} \\
&= -\int_{C(0,r)^+} e^{zw}\psi_\varepsilon(z)u(z)\, dz \\
&= -\int_{C(0,r)^+} e^{zw}u(z)\, dz.
\end{aligned}$$

Tempered Méril's theorem

Let us fix $S \subset \mathbb{C}$ a non-empty closed convex non-compact set which contains no lines and ξ_0 a point on the bisector of S^*_∞. For all $\varepsilon, \varepsilon' > 0$, we define a function $f_{\varepsilon,\varepsilon'} : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f_{\varepsilon,\varepsilon'}(z) = \begin{cases} \varepsilon' \Re(\xi_0 z) & \text{if } z \in S_\varepsilon \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly, this function is convex of domain S_ε and is the restriction to S_ε of a subanalytic $\mathcal{C}^\infty$-function defined on $\mathbb{C}$. Moreover, its Legendre transform is given by

$$f^*_{\varepsilon,\varepsilon'}(w) = \sup_{z \in S_\varepsilon} \Re(z(w - \varepsilon'\xi_0)) = h_{S_\varepsilon}(w - \varepsilon'\xi_0)$$

for all $w \in \mathbb{C}$. Since it is well-known that the domain of h_S is $S^\star_\infty$, one immediately gets that $\mathrm{dom}^\circ(f^*_{\varepsilon,\varepsilon'}) = (S^\star_\infty)^\circ + \varepsilon'\xi_0$. In particular, since this open cone is non-empty, the generated affine space is $\mathbb{C}$.

Definition

We set

$$\begin{aligned}\mathrm{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S) =&\{v \in \mathcal{O}((S^\star_\infty)^\circ + \varepsilon'\xi_0) : \\ &v \in e^{h_{S_\varepsilon}(w-\varepsilon'\xi_0)}\mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}((S^\star_\infty)^\circ + \varepsilon'\xi_0)\}\end{aligned}$$

as well as

$$\mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S) = \varprojlim_{\varepsilon\to 0} \mathrm{Exp}^{\mathrm{t}}_{\varepsilon,\varepsilon'}(S), \quad \mathrm{Exp}^{\mathrm{t}}(S) = \varprojlim_{\varepsilon'\to 0} \mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S).$$

Note that

$$\begin{aligned}\mathrm{Exp}^{\mathrm{t}}(S) =&\{v \in \mathcal{O}((S^\star_\infty)^\circ) : \forall \varepsilon, \varepsilon' > 0, \\ &v \in e^{h_{S_\varepsilon}}\mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}((S^\star_\infty)^\circ + \varepsilon'\xi_0)\}.\end{aligned}$$

Thanks to the Mittag-Leffler theorem, one gets

Lemma

Let $\varepsilon, \varepsilon' > 0$. One has a canonical isomorphism

$$H^1_{S_\varepsilon}(\mathbb{C}, e^{-\varepsilon'\xi_0 z}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}}) \simeq \varprojlim_{r \to +\infty} H^1_{S_\varepsilon}(\mathring{S}_r, e^{-\varepsilon'\xi_0 z}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}}).$$

Proposition

Let $\varepsilon' > 0$. One has an isomorphism

$$\varprojlim_{\varepsilon \to 0} H^1_{S_\varepsilon}(\mathbb{C}, e^{-\varepsilon'\xi_0 z}\,\mathcal{O}^{\mathrm{t}}_{\mathbb{P}})$$

$$\simeq \frac{\{u \in \mathcal{O}(\mathbb{C} \setminus S) : \forall r > \varepsilon > 0,\ e^{\varepsilon'\xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r \setminus S_\varepsilon)\}}{\{u \in \mathcal{O}(\mathbb{C}) : \forall r > 0,\ e^{\varepsilon'\xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r)\}}.$$

Let $\varepsilon_1' > \varepsilon' > 0$. Then there is a well defined map from

$$\frac{\{u \in \mathcal{O}(\mathbb{C} \setminus S) : \forall r > \varepsilon > 0,\ e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r \setminus S_\varepsilon)\}}{\{u \in \mathcal{O}(\mathbb{C}) : \forall r > 0,\ e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r)\}}$$

to

$$\frac{\{u \in \mathcal{O}(\mathbb{C} \setminus S) : \forall r > \varepsilon > 0,\ e^{\varepsilon_1' \xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r \setminus S_\varepsilon)\}}{\{u \in \mathcal{O}(\mathbb{C}) : \forall r > 0,\ e^{\varepsilon_1' \xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r)\}},$$

namely $[u] \mapsto [u]$. Indeed, if $e^{\varepsilon' \xi_0 z} u$ is tempered on $\mathring{S}_r \setminus S_\varepsilon$ (resp. on $\mathring{S}_r$), then

$$e^{\varepsilon_1' \xi_0 z} u = e^{(\varepsilon_1' - \varepsilon') \xi_0 z} e^{\varepsilon' \xi_0 z} u$$

is also tempered on $\mathring{S}_r \setminus S_\varepsilon$ (resp. on $\mathring{S}_r$), since $\Re((\varepsilon_1' - \varepsilon') \xi_0 z) < 0$ for all $z \in \mathring{S}_r$ with large enough module. Hence, this gives rise to a projective system indexed by ε'.

Proposition

Let $\varepsilon > 0$ *and let* ψ_ε *be a* $\mathcal{C}^\infty$*-regularizing function which is equal to* 1 *on* $\mathbb{C} \setminus S_\varepsilon$ *and to* 0 *on* $S_{\varepsilon/2}$*. Let* $u \in \mathcal{O}(\mathbb{C} \setminus S)$ *such that for all* $r > \varepsilon > 0$, $e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathsf{t}}_{\mathbb{P}}(\mathring{S}_r \setminus S_\varepsilon)$*. Then the image of* $[u]$ *through the canonical map*

$$\frac{\{u \in \mathcal{O}(\mathbb{C} \setminus S) : \forall r > \varepsilon > 0,\ e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathsf{t}}_{\mathbb{P}}(\mathring{S}_r \setminus S_\varepsilon)\}}{\{u \in \mathcal{O}(\mathbb{C}) : \forall r > 0,\ e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathsf{t}}_{\mathbb{P}}(\mathring{S}_r)\}}$$
$$\to H^1_{S_\varepsilon}(\mathbb{C}, e^{-\varepsilon' \xi_0 z}\, \mathcal{O}^{\mathsf{t}}_{\mathbb{P}})$$

is given by $[\bar{\partial}_z(\psi_\varepsilon u)]$.

Theorem

Let $\varepsilon' > 0$. There is an isomorphism of $\mathbb{C}$-vector spaces

$$\frac{\{u \in \mathcal{O}(\mathbb{C} \setminus S) : \forall r > \varepsilon > 0,\ e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r \setminus S_\varepsilon)\}}{\{u \in \mathcal{O}(\mathbb{C}) : \forall r > 0,\ e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r)\}} \xrightarrow{\sim} \mathrm{Exp}^{\mathrm{t}}_{\varepsilon'}(S)$$

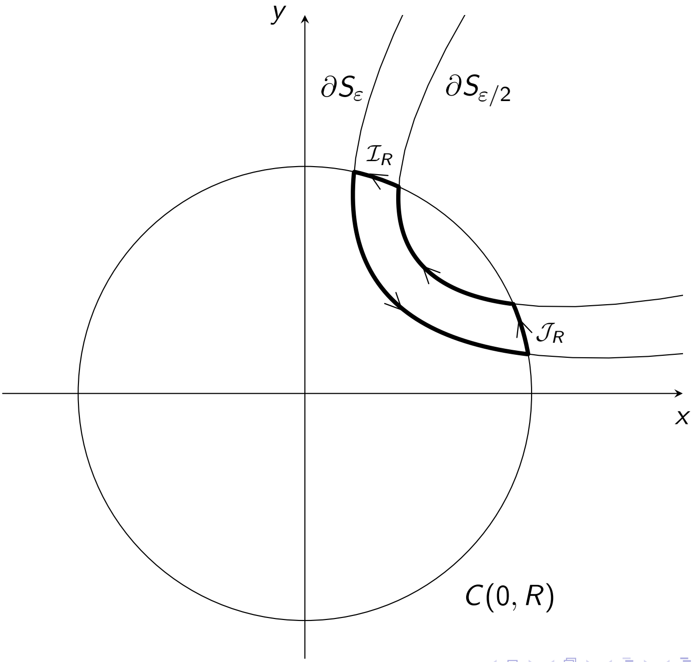
given by

$$[u] \mapsto \left(w \in (S^*_\infty)^\circ + \varepsilon' \xi_0 \mapsto - \int_{\partial S^+_\varepsilon} e^{zw} u(z)\, dz \right),$$

where ∂S^+_ε is the positively oriented boundary of any thickening S_ε.

Idea of the proof : Let us fix $w \in (S^\star_\infty)^\circ + \varepsilon'\xi_0$. One has

$$
\begin{aligned}
&\int_{\mathbb{C}} e^{zw}\bar{\partial}_z(\psi_\varepsilon(z)u(z))dz \wedge d\bar{z}\\
&= \int_{\mathbb{C}} \bar{\partial}_z(e^{zw}\psi_\varepsilon(z)u(z))\, dz \wedge d\bar{z}\\
&= \int_{S_\varepsilon \setminus \mathring{S}_{\varepsilon/2}} \bar{\partial}_z(e^{zw}\psi_\varepsilon(z)u(z))\, dz \wedge d\bar{z}\\
&= \lim_{R\to+\infty} \int_{(S_\varepsilon \setminus \mathring{S}_{\varepsilon/2})\cap \overline{D(0,R)}} \bar{\partial}_z(e^{zw}\psi_\varepsilon(z)u(z))\, dz \wedge d\bar{z}\\
&= -\lim_{R\to+\infty} \int_{\partial((S_\varepsilon \setminus \mathring{S}_{\varepsilon/2})\cap \overline{D(0,R)})^+} e^{zw}\psi_\varepsilon(z)u(z)\, dz.
\end{aligned}
$$


y
∂S_ε
$\partial S_{\varepsilon/2}$
$\mathcal{I}_R$
$\mathcal{J}_R$
x
$C(0, R)$

$$\left|\int_{\mathcal{I}_R} e^{zw}\psi_\varepsilon(z)u(z)\,dz\right| < 2\pi R \sup_{z\in\mathcal{I}_R} |e^{zw}u(z)|$$
$$= 2\pi R \sup_{z\in\mathcal{I}_R} |e^{(w-\varepsilon'\xi_0)z}| \sup_{z\in\mathcal{I}_R} |e^{\varepsilon'\xi_0 z}u(z)|.$$

Thanks to the tempered condition on u, one has

$$\sup_{z\in\mathcal{I}_R} |e^{\varepsilon'\xi_0 z}u(z)| \leq cR^N, \quad R \gg 0.$$

Moreover, one can write

$$\sup_{z\in\mathcal{I}_R} |e^{(w-\varepsilon'\xi_0)z}| = e^{\Re((w-\varepsilon'\xi_0)z_R)} = e^{|w-\varepsilon'\xi_0|R\cos(\theta_R)}.$$

Since

$$(w-\varepsilon'\xi_0 \in (S^*_\infty)^\circ, z_R \in S_\varepsilon) \Longrightarrow (\exists\delta > 0, \cos(\theta_R) < -\delta, \forall R \gg 0),$$

one gets

$$\left|\int_{\mathcal{I}_R} e^{zw}\psi_\varepsilon(z)u(z)\,dz\right| < 2\pi c R^{N+1} e^{-|w-\varepsilon'\xi_0|\delta R} \underset{R\to+\infty}{\longrightarrow} 0.$$

Corollary

There is an isomorphism of $\mathbb{C}$*-vector spaces*

$$\varprojlim_{\varepsilon' \to 0} \frac{\{u \in \mathcal{O}(\mathbb{C} \setminus S) : \forall r > \varepsilon > 0,\ e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r \setminus S_\varepsilon)\}}{\{u \in \mathcal{O}(\mathbb{C}) : \forall r > 0,\ e^{\varepsilon' \xi_0 z} u \in \mathcal{D}b^{\mathrm{t}}_{\mathbb{P}}(\mathring{S}_r)\}} \xrightarrow{\sim} \mathrm{Exp}^{\mathrm{t}}(S)$$

given by

$$([u_{\varepsilon'}])_{\varepsilon'} \mapsto \left(w \in (S^*_\infty)^\circ \mapsto - \int_{\partial S^+_\varepsilon} e^{zw} u_{\varepsilon'}(z)\, dz \right),$$

where ∂S^+_ε *is the positively oriented boundary of any thickening* S_ε*.*

Limitations of the approach

1) This method does not provide the explicit inverse map which is also interesting in practice.

2) The isomorphism is only algebraic. The topological part has to be checked by hand.

3) There are some subanalytic hypothesis that are perhaps not necessary if one proves the theorems by hand.