Unitary transformation from color-spin to isospin-spin coupling schemes for six-quark color singlet states

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Abstract : We derive a unitary transformation relating color singlet six-quark states constructed in the color-spin (CS) and isospin-spin (TS) schemes. The chosen orbital symmetries are [42] in the TS = (01) sector and [33] in the TS = (00) sector. They are important in the description of the NN system.

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I. INTRODUCTION

There are two classification schemes which have been used in the literature to construct totally antisymmetric states of six quarks incorporating orbital (O), color (C), isospin (T) and spin (S) degrees of freedom. In the one called the CS coupling scheme one first combines the SU_3-color singlet [222] and the SU_2 spin $[f]_S$ representations to a given SU_6 symmetry $[f]_{CS}$. Subsequently one can couple[1,2] this to an SU_2-isospin representation $[f]_T$ to obtain an SU_{12} representation $[f]_{CST}$ adjoint to the orbital symmetry $[f]_O$ chosen. In constructing a totally antisymmetric state as above one can also permute the orbital and the isospin symmetries i.e. couple first CS to O to get a function of dual symmetry, $[\tilde{f}]_{CSO} = [f]_T$ as for example in Ref.3.

One can also build totally antisymmetric states starting from an intermediate representation $[f]_{TS}$ of the SU_4 isospin-spin group. The list of all contributing SU_4 representations associated with the orbital symmetries $[6]_O$, $[51]_O$, $[42]_O$ and $[33]_O$ can be found in Table 1 of Ref. 4. We shall call this classification scheme the TS scheme.

Each of these two schemes has its advantages. In the CS scheme the expectation value of the color magnetic operator

$$\Gamma_{cm} = - \sum_{i<j} \hat{\lambda}_i . \hat{\lambda}_j \, \vec{\sigma}_i . \vec{\sigma}_j \tag{1.1}$$

with $\hat{\lambda}_i$ the SU_3-color generators, can be easily calculated.[2] In the TS scheme one has the advantage of being able to interchange the values of T and S at the group theory level of calculation of the matrix elements. For example results for TS = (01) and (10) are identical before implementing the spin-spin matrix elements. Moreover in calculating the matrix elements of the six-quark hamiltonian one can use[5,6] fractional parentage coefficients available in the literature of the nuclear shell model.[7]

To understand the relationship between results obtained in the CS and TS schemes it is necessary to know the unitary transformation between bases for various sectors.

This question has been raised, for example, in Ref. 8.

To our knowledge, the present work is the first to accomplish such a task. Our results concern specific orbital symmetries, the $[42]_O$ in the TS = (01) sector and $[33]_O$ in the TS = (00) sector. They play a very important role in describing the NN interaction at short separation distances (see, e.g. Refs. 5,8). The other symmetries contributing to the NN state (Sec. III), i.e. the $[6]_O$ for TS = (01) and $[51]_O$ for TS = (00) produce only one state so they are identical in either scheme.

In the next section we shall sketch the method used in obtaining the transformation and shall give results of intermediate steps consisting of tables of $\overline{K}$ matrices associated with both schemes. They can be used subsequently in the calculation of matrix elements of the six-quark hamiltonians chosen for NN studies. Sec. III contains the two unitary transformations obtained in this work and a discussion.

II. THE METHOD

The states of interest for TS = (01) and (00) in the TS schemes are listed in Table 1 of Ref. 4. The corresponding spin-color symmetries $[f]_{CS}$ can be obtained from the inner product $[222]_C \times [f]_S$. This gives :

$$[f]_{CS} = [42], [321], [222], [3111] \text{ and } [21111] \qquad \text{for } T = 0, S = 1 \qquad (2.1)$$

$$[f]_{CS} = [33], [411], [2211] \text{ and } [1^6] \qquad \text{for } T = 0, S = 0 \quad . \qquad (2.2)$$

All symmetries of (2.1) can be combined with the orbital symmetry $[42]_O$ to produce a totally antisymmetric TS = (01) state and the same holds for (2.2) in obtaining a TS = (00) state of orbital symmetry $[33]_O$. We note that $[6]_O$ selects only $[222]_{CS}$ for TS = (01) states and $[51]_O$ only $[2211]_{CS}$ for TS = (00) states. Accordingly, the unitary matrices we are looking for have to be 5 x 5 for $[42]_O$ and 4 x 4 for $[33]_O$, respectively.

Our procedure is straightforward. We write each basis vector in the TS scheme as a linear combination of basis vectors in the CS scheme. Then we explicit the content of each basis state in terms of Young tableaux associated with orbital, color, isospin and spin degrees of freedom and by identification of these tableaux we obtain equations for the coefficients entering the linear combinations. In all cases the system of equation of overdetermined.

Each antisymmetric state can be expanded as :

$$\psi_{[1^6]} = \frac{1}{\sqrt{\eta_f}} \sum_Y (-)^{P_Y} | fY > | \tilde{f}\tilde{Y} > \qquad (2.3)$$

where Y is the Young tableau associated with the representation [f] of the symmetric group S_6 and $[\tilde{f}]$ is the adjoint representation of [f] i.e. $\tilde{Y}$ is obtained from Y by interchanging rows and columns. The phase $(-)^{P_Y}$ depends on the number P of transpositions to restore the normal order in the Young tableau Y. Together with the factor $\eta_f^{-1/2}$ where η_f is the dimension of [f] it represents the Clebsch-Gordan coefficient of the inner product of $[f] \times [\tilde{f}]$ leading to $[1^6]$. The equation (2.3) is subsequently rewritten in terms of the diagonalized Young-Yamanouchi-Rutherford representation[7] where the last pair of particles has definite symmetry. The next step is to explicit the content of [f] and $[\tilde{f}]$ as inner products of various S_6 representations. In the TS scheme $[f] = [f]_O \times [f]_C$ and $[\tilde{f}] = [f]_T \times [f]_S$. In the CS scheme $[f] = [f]_O$ and $[\tilde{f}] = \{[f]_C \times [f]_S\} \times [f]_T$. An inner product of two S_6 representations [f'] and [f"] can be expanded by using the corresponding S_6 Clebsch-Gordan coefficients. We use the factorization property[9] of these coefficients into a K-matrix and a S_{n-1} coefficient and write them as in Ref. 4 as a product of a $\overline{K}$ matrix resulting from two sequent factorizations and an S_4 Clebsch-Gordan coefficient. Following the notations given in the Appendix A of Ref. 4 we have :

$$|fY> = \sum \overline{K} \, ([f']p'q' \, [f'']p''q'' \, |[f]pq) \; S \, (f_{p'q'} y' \, f''_{p''q''} y'' \, | f_{pq} y)$$

$$\times \; |[f']p'q'y'> |[f'']p''q''y''> \qquad (2.4)$$

where the summation runs over p'q', p"q", y' and y". Here p labels the row of the 6^{th} particle, q the row of the 5^{th} particle and y is the remaining of the Young tableau Y after the removal of the 6^{th} and 5^{th} particle, i.e. a tableau of the S_4 representation f_{pq}. In the diagonalized Young Yamanouchi-Rutherford representation the indices pq appear either in a symmetric $\overline{pq}$ or in an antisymmetric $\widetilde{pq}$ combination only. The factor S in (2.4) is the Clebsch-Gordan coefficient needed in the inner product $[f_{pq}] = [f'_{p'q'}] \times [f''_{p''q''}]$ of S_4.

In this work a consistent phase convention of the 6 particle states is absolutely necessary. We have constructed the S_4 Clebsch-Gordan coefficients and the $\overline{K}$ matrices following conventions and using symmetry properties of the Clebsch-Gordan coefficients as introduced in Ref. 9. In calculating matrix elements of the six-quark hamiltonian the problem reduces to the calculation of two-body matrix elements and only the knowledge of the $\overline{K}$ matrices is necessary because the S_4 Clebsch-Gordan coefficients sum up in orthogonality relations.[4] In view of this practical aspect we give in Tables 1-6 the $\overline{K}$ matrices resulting from this study. They complement[10] the Table 3 of Ref. 6. Our Tables 5 and 6 recover Table 5 of Ref. 4. One can see that in few cases we disagree on the phase. Up to a phase convention we also obtain agreement with Tables 1 and 2 of Ref. 1 and parts of Table 3 of Ref. 4 where they appear under the name of two-body fractional parentage coefficients. The correspondence with Ref. 1 is obvious and that with Table 3 of Ref. 4 can also be established easily by using the notations of the head columns of Ref. 1.

III. THE UNITARY TRANSFORMATION

In Tables 7 and 8 we exhibit the two unitary matrices obtained through the procedure described in the previous section. The rows contain the coefficients c_i of the linear combinations of ψ_i^{CS} states defining a TS scheme state as :

$$[f]_O \{f\}_{TS} = \sum c_i \, \psi_i^{CS} \qquad (3.1)$$

where the l.h.s. denotes a TS scheme state in the compressed notation[4] used in the literature. In the CS scheme the notation for the TS = (01) basis vectors ψ_i^{CS}(i = 1,...5) is :

$$
\begin{aligned}
\psi_1^{CS} &= [42]_O\{42\}_{CS} = \{[42]_O \times \{[42]_{CS} \times [33]_T\}_{[2211]}\}_{[1^6]} \\
\psi_2^{CS} &= [42]_O\{321\}_{CS} = \{[42]_O \times \{[321]_{CS} \times [33]_T\}_{[2211]}\}_{[1^6]} \\
\psi_3^{CS} &= [42]_O\{3111\}_{CS} = \{[42]_O \times \{[3111]_{CS} \times [33]_T\}_{[2211]}\}_{[1^6]} \\
\psi_4^{CS} &= [42]_O\{222\}_{CS} = \{[42]_O \times \{[222]_{CS} \times [33]_T\}_{[2211]}\}_{[1^6]} \\
\psi_5^{CS} &= [42]_O\{21111\}_{CS} = \{[42]_O \times \{[21111]_{CS} \times [33]_T\}_{[2211]}\}_{[1^6]}
\end{aligned}
\tag{3.2}
$$

and for the TS = (00) sector the ψ_i^{CS} (i = 1,...,4) are :

$$
\begin{aligned}
\psi_1^{CS} &= [33]_O\{33\}_{CS} = \{[33]_O \times \{[33]_{CS} \times [33]_T\}_{[222]}\}_{[1^6]} \\
\psi_2^{CS} &= [33]_O\{411\}_{CS} = \{[33]_O \times \{[411]_{CS} \times [33]_T\}_{[222]}\}_{[1^6]} \\
\psi_3^{CS} &= [33]_O\{2211\}_{CS} = \{[33]_O \times \{[2211]_{CS} \times [33]_T\}_{[222]}\}_{[1^6]} \\
\psi_4^{CS} &= [33]_O\{1^6\}_{CS} = \{[33]_O \times \{[1^6]_{CS} \times [33]_T\}_{[222]}\}_{[1^6]} \ .
\end{aligned}
\tag{3.3}
$$

In view of the application of this transformation it is useful to recall the separation given by Harvey[4] into "asterisked" and "non-asterisked" SU_4 symmetries. The first can lead to di-baryon states NN, NΔ and ΔΔ, the latter do not but, as hidden color states CC, can contribute to the three-quark clusters energy at short separations. For TS = (01) the asterisked symmetries are $\{33\}_{TS}$ and $\{51\}_{TS}$. They define the NN, ΔΔ and CC states by the linear combinations[4]

$$NN = \sqrt{\tfrac{1}{9}}\,([6]_O\{33\}_{TS}) + \sqrt{\tfrac{4}{9}}\,([42]_O\{33\}_{TS}) - \sqrt{\tfrac{4}{9}}\,([42]_O\{51\}_{TS})$$

$$\Delta\Delta = \sqrt{\tfrac{4}{45}}\,([6]_O\{33\}_{TS}) + \sqrt{\tfrac{16}{45}}\,([42]_O\{33\}_{TS}) + \sqrt{\tfrac{25}{45}}\,([42]_O\{51\}_{TS})$$

$$CC = \sqrt{\tfrac{4}{5}}\,([6]_O\{33\}_{TS}) - \sqrt{\tfrac{1}{5}}\,([42]_O\{33\}_{TS}) \quad . \qquad (3.4)$$

Using Table 7 we rewrite these linear combinations in terms of the CS basis vectors. The result is exhibited in Table 9 where we have used the identity

$$[6]_O\{33\}_{TS} = [6]_O\{222\}_{CS} \quad . \qquad (3.5)$$

One can see that among the $[42]_O$ states the largest contributions to NN come from $\{2111\}_{CS}$ (54.9 %) and $\{222\}_{CS}$ (33 %) symmetries, the rest of the other $[42]_O$ states contributing together with 1 %.

The important role of the $[42]_O$ state in the NN problem comes from the fact that the hyperfine interaction compensates for the extra kinetic energy of the two p-shell quarks and brings it nearly degenerate to the $[6]_O$ state.[5,8] The hyperfine interaction which is given approximately by the operator (1.1) has its lowest (negative) expectation value for the symmetry $\{42\}_{CS}$. This can justify the choice of Ref. 1 to study the contribution of this state only. But the CS composition of the NN state given above shows that the symmetries $\{222\}_{CS}$ and $\{21111\}_{CS}$ cannot a priori be neglected.

For the TS = (00) sector the NN, ΔΔ and CC states are :[4,11]

$$NN = -\sqrt{\tfrac{5}{45}}\,([51]_O\{42\}_{TS}) - \sqrt{\tfrac{4}{45}}\,([33]_O\{42\}_{TS}) - \sqrt{\tfrac{36}{45}}\,([33]_O\{6\}_{TS})$$

$$\Delta\Delta = -\sqrt{\tfrac{20}{45}}\,([51]_O\{42\}_{TS}) - \sqrt{\tfrac{16}{45}}\,([33]_O\{42\}_{TS}) + \sqrt{\tfrac{9}{45}}\,([33]_O\{6\}_{TS})$$

$$CC = -\sqrt{\tfrac{4}{9}}\,([51]_O\{42\}_{TS}) + \sqrt{\tfrac{5}{9}}\,([33]_O\{42\}_{TS}) \quad . \qquad (3.6)$$

Their content in CS states obtained using Table 8 is presented in Table 10 where :

$$[51]_O\{2211\}_{CS} = [51]_O\{42\}_{TS} \quad .$$

This shows that the $\{33\}_{CS}$, $\{411\}_{CS}$ and $\{2211\}_{CS}$ symmetries contribute with 25 %, 50 % and 14 %, respectively, while the amplitude of the $\{1^6\}_{CS}$ symmetry cancels identically for NN.

In both cases when calculating expectation values with respect to NN for a model hamiltonian one should include both the contributions of the diagonal and non-diagonal matrix elements between CS states. On the other hand the NN state has a strong coupling to other physical ΔΔ, NΔ or hidden color CC states at short separation distances.[5,6] The final conclusion about the role of various CS symmetries has then to be given by a full dynamical analysis.

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TABLE CAPTIONS

Table 1 : The $\bar{K}([42]pq\ [33]p'q'\ |\ [f'']p''q'')$ matrices used in the S x T coupling for S = 1, T = 0.

Table 2 : The $\bar{K}([33]pq\ [33]p'q'\ |\ [f'']p''q'')$ matrices used in the S x T coupling for S = 0, T = 0.

Table 3 : The $\bar{K}([f]pq\ [33]p'q'\ |\ [2211]p''q'')$ matrices used in the CS x T coupling for S = 1, T = 0.

Table 4 : The $\bar{K}([f]pq\ [33]p'q'\ |\ [222]p''q'')$ matrices used in the CS x T coupling for S = 0, T = 0.

Table 5 : The $\bar{K}([42]pq\ [222]p'q'\ |\ [f'']p''q'')$ matrices used in the S x C coupling for S = 1, O x C coupling for $[42]_O$ or CS x T coupling for $[222]_{CS}$, T = 0.

Table 6 : The $\bar{K}([f]pq\ [222]p'q'\ |\ [f'']p''q'')$ matrices used in the S x C coupling for S = 0 or O x C coupling for $[51]_O$ or $[33]_O$.

Table 7 : The unitary transformation between the CS and TS basis vectors of orbital symmetry $[42]_O$, isospin T = 0 and spin S = 1.

Table 8 : The unitary transformation between the CS and TS basis vectors of orbital symmetry $[33]_O$, isospin T = 0 and spin S = 0.

Table 9 : The content of NN, ΔΔ and CC states in the CS scheme for TS = (01).

Table 10 : Same as Table 9, but for TS = (00).

TABLE 1

[42] [33]

		$\overline{11}\,\overline{22}$	$\overline{22}\,\overline{22}$	$\overline{12}\,\overline{22}$	$\widetilde{12}\,\widetilde{12}$
[33]	$\overline{22}$	$\sqrt{\frac{10}{108}}$	$\sqrt{\frac{8}{108}}$	$\sqrt{\frac{60}{108}}$	$\sqrt{\frac{30}{108}}$
[51]	$\overline{11}$	$-\sqrt{\frac{4}{27}}$	$\sqrt{\frac{5}{27}}$	$\sqrt{\frac{6}{27}}$	$-\sqrt{\frac{12}{27}}$
	$\overline{12}$	0	0	1	0
[411]	$\overline{11}$	$\sqrt{\frac{1}{6}}$	0	$\sqrt{\frac{2}{6}}$	$-\sqrt{\frac{3}{6}}$
	$\overline{13}$	$-\sqrt{\frac{25}{54}}$	$-\sqrt{\frac{20}{54}}$	$\sqrt{\frac{6}{54}}$	$\sqrt{\frac{3}{54}}$
[2211]	$\overline{24}$	$-\sqrt{\frac{3}{10}}$	0	$\sqrt{\frac{6}{10}}$	$\sqrt{\frac{1}{10}}$
[321]	$\overline{12}$	$\sqrt{\frac{8}{15}}$	0	$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{6}{15}}$
	$\overline{13}$	0	0	-1	0
	$\overline{23}$	$-\sqrt{\frac{8}{27}}$	$\sqrt{\frac{10}{27}}$	$-\sqrt{\frac{3}{27}}$	$\sqrt{\frac{6}{27}}$

[42] [33]

		$\overline{11}\,\widetilde{12}$	$\overline{22}\,\widetilde{12}$	$\overline{12}\,\widetilde{12}$	$\widetilde{12}\,\overline{22}$
[33]	$\widetilde{12}$	$\sqrt{\frac{5}{12}}$	$-\sqrt{\frac{2}{12}}$	0	$\sqrt{\frac{5}{12}}$
[51]	$\widetilde{12}$	$-\sqrt{\frac{2}{3}}$	0	0	$-\sqrt{\frac{1}{3}}$
[411]	$\widetilde{23}$	$-\sqrt{\frac{1}{3}}$	0	0	$\sqrt{\frac{2}{3}}$
	$\widetilde{13}$	0	0	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$
[2211]	$\widetilde{12}$	1	0	0	0
	$\widetilde{24}$	0	0	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
	$\widetilde{34}$	$-\sqrt{\frac{1}{20}}$	$\sqrt{\frac{10}{20}}$	0	$\sqrt{\frac{9}{20}}$
[321]	$\widetilde{12}$	0	0	$\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$
	$\widetilde{13}$	$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{5}{15}}$	0	$-\sqrt{\frac{2}{15}}$
	$\widetilde{23}$	0	0	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{2}{3}}$

TABLE 2

		[33] [33] $\overline{22}\,\overline{22}$	[33] [33] $\widetilde{12}\,\widetilde{12}$
[6]	$\overline{11}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$
[42]	$\overline{11}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
	$\overline{12}$	-1	0
	$\overline{22}$	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
[222]	$\overline{33}$	$-\sqrt{\frac{3}{4}}$	$\sqrt{\frac{1}{4}}$
[3111]	$\overline{11}$	0	-1
	$\overline{14}$	-1	0

		[33] [33] $\overline{22}\,\widetilde{12}$	[33] [33] $\widetilde{12}\,\overline{22}$
[42]	$\widetilde{12}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$
[222]	$\widetilde{12}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
[3111]	$\widetilde{14}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$
	$\widetilde{34}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{2}}$

TABLE 3

[321] [33]

		$\overline{12}\,\overline{22}$	$\overline{13}\,\overline{22}$	$\overline{23}\,\overline{22}$	$\widetilde{12}\,\widetilde{12}$	$\widetilde{13}\,\widetilde{12}$	$\widetilde{23}\,\widetilde{12}$
[2211]	$\overline{24}$	$\sqrt{\frac{15}{40}}$	$-\sqrt{\frac{2}{40}}$	$-\sqrt{\frac{9}{40}}$	$-\sqrt{\frac{5}{40}}$	0	$\sqrt{\frac{9}{40}}$

[321] [33]

		$\overline{12}\,\widetilde{12}$	$\overline{13}\,\widetilde{12}$	$\overline{23}\,\widetilde{12}$	$\widetilde{12}\,\overline{22}$	$\widetilde{13}\,\overline{22}$	$\widetilde{23}\,\overline{22}$
	$\widetilde{12}$	0	$-\sqrt{\frac{3}{8}}$	0	$-\sqrt{\frac{5}{8}}$	0	0
[2211]	$\widetilde{24}$	$-\sqrt{\frac{15}{80}}$	0	$\sqrt{\frac{27}{80}}$	$-\sqrt{\frac{5}{80}}$	$-\sqrt{\frac{30}{80}}$	$\sqrt{\frac{3}{80}}$
	$\widetilde{34}$	0	$-\sqrt{\frac{6}{20}}$	0	$\sqrt{\frac{5}{20}}$	0	$-\sqrt{\frac{9}{20}}$

[3111] [33]

		$\overline{11}\,\overline{22}$	$\overline{14}\,\overline{22}$	$\widetilde{14}\,\widetilde{12}$	$\widetilde{34}\,\widetilde{12}$
[2211]	$\overline{24}$	$\sqrt{\frac{4}{20}}$	$-\sqrt{\frac{6}{20}}$	$-\sqrt{\frac{1}{20}}$	$-\sqrt{\frac{9}{20}}$

[3111] [33]

		$\overline{11}\,\widetilde{12}$	$\overline{14}\,\widetilde{12}$	$\widetilde{14}\,\overline{22}$	$\widetilde{34}\,\overline{22}$
	$\widetilde{12}$	0	0	1	0
[2211]	$\widetilde{24}$	0	$-\sqrt{\frac{6}{10}}$	$\sqrt{\frac{1}{10}}$	$\sqrt{\frac{3}{10}}$
	$\widetilde{34}$	$\sqrt{\frac{6}{40}}$	0	$\sqrt{\frac{25}{40}}$	$-\sqrt{\frac{9}{40}}$

[21111] [33]

		$\overline{15}\,\overline{22}$	$\widetilde{15}\,\widetilde{12}$	$\widetilde{45}\,\widetilde{12}$
[2211]	$\overline{24}$	$\sqrt{\frac{1}{5}}$	0	$-\sqrt{\frac{4}{5}}$

[21111] [33]

		$\overline{15}\,\widetilde{12}$	$\widetilde{15}\,\overline{22}$	$\widetilde{45}\,\overline{22}$
	$\widetilde{12}$	0	0	1
[2211]	$\widetilde{24}$	0	$-\sqrt{\frac{3}{5}}$	$-\sqrt{\frac{2}{5}}$
	$\widetilde{34}$	$-\sqrt{\frac{3}{5}}$	0	$-\sqrt{\frac{2}{5}}$

[222] [33]

		$\overline{33}\,\overline{22}$	$\widetilde{23}\,\widetilde{12}$
[2211]	$\overline{24}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

[222] [33]

		$\widetilde{23}\,\overline{22}$	$\overline{33}\,\widetilde{12}$
	$\widetilde{12}$	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
[2211]	$\widetilde{24}$	1	0
	$\widetilde{34}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$

TABLE 4

[33] [33]

		$\overline{22}\,\overline{22}$	$\widetilde{12}\,\widetilde{12}$
[222]	$\overline{33}$	$\sqrt{\frac{3}{4}}$	$-\sqrt{\frac{1}{4}}$

		$\overline{22}\,\widetilde{12}$	$\widetilde{12}\,\overline{22}$
[222]	$\widetilde{23}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

[411] [33]

		$\overline{11}\,\overline{22}$	$\overline{13}\,\overline{22}$	$\widetilde{13}\,\widetilde{12}$	$\widetilde{23}\,\widetilde{12}$
[222]	$\overline{33}$	$\sqrt{\frac{3}{8}}$	$\sqrt{\frac{3}{8}}$	0	$-\sqrt{\frac{2}{8}}$

		$\overline{11}\,\widetilde{12}$	$\overline{13}\,\widetilde{12}$	$\widetilde{13}\,\overline{22}$	$\widetilde{23}\,\overline{22}$
[222]	$\widetilde{23}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	0

[2211] [33]

		$\overline{24}\,\overline{22}$	$\widetilde{12}\,\widetilde{12}$	$\widetilde{34}\,\widetilde{12}$
[222]	$\overline{33}$	$-\sqrt{\frac{5}{12}}$	$-\sqrt{\frac{2}{12}}$	$\sqrt{\frac{5}{12}}$

		$\widetilde{24}\,\overline{22}$	$\widetilde{12}\,\overline{22}$	$\widetilde{34}\,\overline{22}$	$\overline{24}\,\widetilde{12}$
[222]	$\widetilde{23}$	$\sqrt{\frac{30}{54}}$	$-\sqrt{\frac{4}{54}}$	$-\sqrt{\frac{5}{54}}$	$-\sqrt{\frac{15}{54}}$

$[1^6]$ [33]

		$\widetilde{56}\,\widetilde{12}$
[222]	$\overline{33}$	-1

		$\widetilde{56}\,\overline{22}$
[222]	$\widetilde{23}$	1

TABLE 5

[42] [222]		$\overline{11}\,\overline{33}$	$\overline{12}\,\overline{33}$	$\overline{22}\,\overline{32}$	$\widetilde{12}\,\widetilde{23}$
[222]	$\overline{33}$	$-\sqrt{\frac{5}{12}}$	0	$-\sqrt{\frac{2}{12}}$	$\sqrt{\frac{5}{12}}$
[21111]	$\overline{15}$	$\sqrt{\frac{2}{3}}$	0	0	$-\sqrt{\frac{1}{3}}$

[42] [222]		$\overline{11}\,\widetilde{23}$	$\overline{12}\,\widetilde{23}$	$\overline{22}\,\widetilde{23}$	$\widetilde{12}\,\overline{33}$
[222]	$\widetilde{23}$	$\sqrt{\frac{5}{54}}$	$\sqrt{\frac{30}{54}}$	$\sqrt{\frac{4}{54}}$	$\sqrt{\frac{15}{54}}$
[21111]	$\widetilde{15}$	0	1	0	0
	$\widetilde{45}$	$-\sqrt{\frac{4}{27}}$	$\sqrt{\frac{6}{27}}$	$\sqrt{\frac{5}{27}}$	$-\sqrt{\frac{12}{27}}$

TABLE 6

[51] [222]

		$\overline{11}\,\overline{33}$	$\overline{12}\,\overline{33}$	$\widetilde{12}\,\widetilde{23}$
[2211]	$\overline{24}$	$\sqrt{\frac{4}{5}}$	0	$\sqrt{\frac{1}{5}}$

[51] [222]

		$\overline{11}\,\widetilde{23}$	$\overline{12}\,\widetilde{23}$	$\widetilde{12}\,\overline{33}$
	$\widetilde{12}$	-1	0	0
[2211]	$\widetilde{24}$	$\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$	0
	$\widetilde{34}$	$\sqrt{\frac{2}{5}}$	0	$\sqrt{\frac{3}{5}}$

[33] [222]

		$\overline{22}\,\overline{33}$	$\widetilde{12}\,\widetilde{23}$
[2211]	$\overline{24}$	$-\sqrt{\frac{1}{2}}$	$-\sqrt{\frac{1}{2}}$

[33] [222]

		$\overline{22}\,\widetilde{23}$	$\widetilde{12}\,\overline{33}$
	$\widetilde{12}$	$-\sqrt{\frac{2}{5}}$	$\sqrt{\frac{3}{5}}$
[2211]	$\widetilde{24}$	1	0
	$\widetilde{34}$	$-\sqrt{\frac{1}{4}}$	$\sqrt{\frac{3}{4}}$
$[1^6]$	$\widetilde{56}$	$\sqrt{\frac{3}{5}}$	$\sqrt{\frac{2}{5}}$

TABLE 7

	ψ_1^{CS}	ψ_2^{CS}	ψ_3^{CS}	ψ_4^{CS}	ψ_5^{CS}
$[42]_O\{33\}_{TS}$	$\frac{9\sqrt{5}}{36}$	$-\frac{8\sqrt{5}}{36}$	$-\frac{5\sqrt{2}}{36}$	$\frac{11}{36}$	$-\frac{20}{36}$
$[42]_O\{51\}_{TS}$	$\frac{9\sqrt{5}}{45}$	$-\frac{8\sqrt{5}}{45}$	$-\frac{5\sqrt{2}}{45}$	$-\frac{25}{45}$	$\frac{25}{45}$
$[42]_O\{411\}_{TS}$	$\frac{9\sqrt{10}}{180}$	$-\frac{8\sqrt{10}}{180}$	$\frac{170}{180}$	$-\frac{25\sqrt{2}}{180}$	$-\frac{20\sqrt{2}}{180}$
$[42]_O\{2211\}_{TS}$	$\frac{11}{20}$	$\frac{8}{20}$	$\frac{\sqrt{10}}{20}$	$\frac{5\sqrt{5}}{20}$	$\frac{4\sqrt{5}}{20}$
$[42]_O\{321\}_{TS}$	$-\frac{18}{45}$	$-\frac{29}{45}$	$\frac{2\sqrt{10}}{45}$	$\frac{10\sqrt{5}}{45}$	$\frac{8\sqrt{5}}{45}$

TABLE 8

	ψ_1^{CS}	ψ_2^{CS}	ψ_3^{CS}	ψ_4^{CS}
$[33]_O \{6\}_{TS}$	$-\frac{\sqrt{5}}{5}$	$\frac{\sqrt{10}}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
$[33]_O \{42\}_{TS}$	$-\frac{3\sqrt{5}}{20}$	$\frac{3\sqrt{10}}{20}$	$-\frac{11}{20}$	$-\frac{12}{20}$
$[33]_O \{222\}_{TS}$	$-\frac{15}{20}$	$-\frac{5\sqrt{2}}{20}$	$-\frac{3\sqrt{5}}{20}$	$\frac{4\sqrt{5}}{20}$
$[33]_O \{3111\}_{TS}$	$-\frac{5\sqrt{2}}{20}$	$-\frac{10}{20}$	$\frac{3\sqrt{10}}{20}$	$-\frac{4\sqrt{10}}{20}$

TABLE 9

	$[6]_O\{222\}_{CS}$	$[42]_O\{42\}_{CS}$	$[42]_O\{321\}_{CS}$	$[42]_O\{3111\}_{CS}$	$[42]_O\{222\}_{CS}$	$[42]_O\{21111\}_{CS}$
NN	$\frac{1}{3}$	$\frac{1}{6\sqrt{5}}$	$-\frac{4}{27\sqrt{5}}$	$-\frac{1}{27\sqrt{2}}$	$\frac{31}{54}$	$-\frac{20}{27}$
ΔΔ	$\frac{2}{3\sqrt{5}}$	$\frac{2}{3}$	$-\frac{16}{27}$	$-\frac{2\sqrt{10}}{27}$	$-\frac{14}{27\sqrt{5}}$	$\frac{\sqrt{5}}{27}$
CC	$\frac{2}{\sqrt{5}}$	$-\frac{1}{4}$	$\frac{2}{9}$	$\frac{\sqrt{10}}{36}$	$-\frac{11}{36\sqrt{5}}$	$\frac{\sqrt{5}}{9}$

TABLE 10

	$[51]_O\{2211\}_{CS}$	$[33]_O\{33\}_{CS}$	$[33]_O\{411\}_{CS}$	$[33]_O\{2211\}_{CS}$	$[33]_O\{1^6\}_{CS}$
NN	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5}}{6}$	0
ΔΔ	$-\frac{2}{3}$	0	0	$\frac{4}{3\sqrt{5}}$	$\frac{1}{\sqrt{5}}$
CC	$-\frac{2}{3}$	$-\frac{1}{4}$	$\frac{\sqrt{2}}{4}$	$-\frac{11}{12\sqrt{5}}$	$-\frac{1}{\sqrt{5}}$