Parameter Identification of Nonlinear Mechanical Systems Using Proper Orthogonal Decomposition

V. Lenaerts, G. Kerschen and J.C. Golinval

University of Liège, LTAS

Vibrations et Identification des Structures

1, Grande Traverse, B52

B-4000 Liège (Belgium)

e-mail : v.lenaerts@ulg.ac.be

ABSTRACT

Proper Orthogonal Decomposition (POD), also known as Karhunen-Loeve decomposition, or principal components analysis (PCA), is emerging as a useful experimental tool in dynamics and vibrations. The POD is a statistical pattern analysis technique for finding the dominant structures in an ensemble of spatially distributed data. These structures can be used as an orthogonal basis for efficient representation of the ensemble. The Proper Orthogonal Modes (POM) have been interpreted mainly as empirical system modes. They have been shown to represent the optimal distributions of kinetic energy or power, and the proper orthogonal values indicate the power associated with these principal distributions. We investigate the use of the proper orthogonal modes of displacements for the identification of parameters of nonlinear dynamical structures with an optimization procedure based on the difference between the experimental and simulated POM. A numerical example of a beam with a local nonlinear component will illustrate the method.

1 INTRODUCTION

For modal analysis, most mechanical structures are approximated by a linear model. However, when these structures are subject to large displacement amplitudes, nonlinear effects may become important and the linear model consequently fails. Even when amplitudes remain small, some nonlinear distortions may occur due to the physical behavior of the structure (e.g. dry friction contact, backslash phenomenon, ...). Both reasons demonstrate why interest in nonlinear identification is increasing. The aim of identification is to generate a mathematical model of a system. Once the model parameters are identified, the model may be used afterwards to predict the behaviour of the system.

When an analytical model of the structure is available, a common practice for the structural analyst, the identification problem consists in a parameter estimation of the structural model. The present paper aims to investigate the use of the proper orthogonal modes (POM) of displacements for the identification of parameters of nonlinear dynamical structures using an optimization procedure based on the difference between the experimental and simulated POM.

2 PROPER ORTHOGONAL DECOMPOSITION

Proper Orthogonal Decomposition (POD), also known as Karhunen-Loeve (K-L) decomposition, is based on a statistical formulation, although it facilitates modal projections of partial differential equations into reduced-order deterministic models[1]. The K-L method has been applied successfully in the fields of fluid dynamic[2], thermics[3] and signal processing[4].

The Proper Orthogonal Decomposition is a means of extracting spatial information from a set of time series data available on a domain. The use of Karhunen-Loeve (K-L) transform is of great help in nonlinear settings where traditional linear techniques such as modal testing and power spectrum analyses cannot be applied. The advantage of this method lies in the fact that the modes (POM) obtained from the K-L decomposition for a given set of parameters, can be used to reconstruct the response of a system whose parameters take different values from the original system. The additional advantage of the K-L analysis is that it can be applied, not only to conservative systems, but also to dissipative ones and that it provide information about the spatial structure of the system dynamics as well as the energy contained in the system.

The method was first applied to turbulence problems by Lumley[5]. The POD allows to quantify spatial coherence in turbulence[2] and structures[6][7]. A recent work[8] has shown that the application of POD to measured displacements of a

discrete structure with a known mass matrix leads to an estimation of the normal modes. In reference[9] the K-L method is applied to vibroimpacting beams and rotors to create low dimensional models, via a Galerkin projection.

2.1 Mathematical formulation of the POD

The POM are shown here to be the eigenfunctions of the space correlation tensor. The definitions and formulation presented here follow closely the ones used in Azeez[9].

Define a random field $u(x,t)$ on some domain Ω. This field is first decomposed into mean $(U(x) =< u >)$ and time varying $(v(x,t))$ parts. This is represented as :

$$u(x,t) = U(x) + v(x,t) \tag{1}$$

This fields are sampled at finite number of points in time. Hence at a fixed time t_n, the system displays an instantaneous snapshot $v_n(x)$, that is a continuous function of x, with $x \in \Omega$. Now we are looking for a representative structure $\phi(x)$ of the ensemble of N snapshots. This coherent structure is computed by minimizing the objective function λ:

$$Minimize \quad \{\lambda = \sum_{n=1}^{N} (\phi(x) - v_n(x))^2\} \quad \forall x \in \Omega \tag{2}$$

Equation (2) can be compactly expressed as the maximization problem :

$$Maximize \quad \{\lambda = \frac{< (\phi, v_n)^2 >}{(\phi,\phi)}\} \quad \forall x \in \Omega \tag{3}$$

with the following notations :

$$(f,g) \equiv \int_\Omega f(x)\, g(x)\, d\Omega \qquad \text{inner product of f and g}$$
$$< v_n > \equiv \frac{1}{N} \sum_{n=1}^{N} v_n(x) \qquad \text{average of snapshots}$$

This equation is equivalent to the following integral eigenvalue problem[9][2]:

$$\int_\Omega K(x,x')\, \phi(x')\, dx' = \lambda\, \phi(x') \tag{4}$$

where the two point correlation function (K) is defined as

$$K(x,x') = \frac{1}{N} \sum_{n=1}^{N} v_n(x)\, v_n(x') \tag{5}$$

Equation (4) has a finite number of orthogonal solutions $\phi^i(x)$with corresponding real and positive eigenvalues λ^i. The eigenvalue with the largest magnitude is the maximum which is achieved in the maximization problem (3). The second largest eigenvalue is the maximum of the same problem restricted to the space orthogonal to the first eigenfunction, and so on. In order to make the computation unique, the eigenfunctions are normalized. Therefore we can use it as a basis for the decomposition of the field :

$$v(x,t) = \sum_{i=1}^{N} a^i(t)\, \phi^i(x) \tag{6}$$

It should be noted that time-dependent part in equation (6) forms orthogonal modes[2]. Thus the POD can be viewed as a bi-orthogonal decomposition because of the space-time symmetry of the decomposition.

For an accurate approximation of the tensor (5) it is necessary to perform a long and expensive simulation. The computation of the eigenfunctions is even more expensive. Two methods exist to solve the problem : the direct[10] and the snapshot method[9].

2.2 Discrete formulation

Suppose S linear snapshots v_i of size M obtained for instance by measurements of the acceleration on a beam at M locations. The $M \times M$ covariance matrix C[4] is defined as:

$$C = \frac{1}{S} \sum_{i=0}^{S-1} v_i v_i^T \tag{7}$$

Its eigenvectors ϕ_k satisfying

$$C\phi_k = \lambda_k \phi_k, \qquad k = 0, \ldots, S-1 \tag{8}$$

with

$$\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{S-1} > 0 \tag{9}$$

form the proper orthogonal decomposition we are looking for. Each λ_k corresponds to the vector ϕ_k and represents the relative importance of that vector in the data. This decomposition is the best basis in term of de-correlation. Due to the space-time symmetry property, one can use an alternative method for the resolution of (8). This method is called the "method of snapshots". Providing that the eigenvectors ϕ_k are unique linear combinations of linearly independent snapshots v_i :

$$\phi_k = \sum_{j=0}^{S-1} a_{k,j} v_j \qquad k = 0, \ldots, S-1. \tag{10}$$

The eigenvalue problem (8) becomes :

$$\overbrace{\frac{1}{S} \sum_{i=0}^{S-1} v_i v_i^T}^{C} \overbrace{\sum_{j=0}^{S-1} a_{k,j} v_j}^{\phi_k} = \lambda_k \overbrace{\sum_{j=0}^{S-1} a_{k,j} v_j}^{\phi_k} \tag{11}$$

$$\frac{1}{S} \sum_{i=0}^{S-1} v_i \left(\sum_{j=0}^{S-1} v_i^T v_j a_{k,j} \right) = \sum_{i=0}^{S-1} \lambda_k a_{k,i} v_i \tag{12}$$

The summations can be dropped as the v_i are linearly independent and we obtain :

$$\sum_{j=0}^{S-1} v_i^T v_j \, a_{k,j} = \lambda_k^* \, a_{k,i} \qquad \left\{ \begin{array}{rcl} i & = & 0, \ldots, S-1, \\ k & = & 0, \ldots, S-1, \\ \lambda_k^* & = & S\lambda_k \end{array} \right. \tag{13}$$

Defining the matrix C^* and the vector A_k as

$$C^* = \left[c_{i,j}^*\right] = \left[v_i^T v_j\right] \quad (i,j \quad = 0, \ldots, S-1) \tag{14}$$

$$A_k = [a_{k,i}] \quad (i \quad = 0, \ldots, S-1) \tag{15}$$

an eigenvalue problem equivalent to (8) is obtained:

$$C^* A_k = \lambda_k^* A_k, \qquad with \quad \lambda_k^* = S\lambda_k. \tag{16}$$

Depending on whether the number S of snapshots is greater or smaller than the number M of elements in one snapshot, the resolution,respectively, of (8) or (16) will be more suited.

If $S > M$, the snapshots v_i are not linearly independent and the matrix C^* has a rank $r \leq M$. The method of snapshots still works because for $k = r, \ldots, S-1$:

- the eigenvalues λ_k will be zero due to the rank deficiency of C^*
- the corresponding eigenvectors A_k will be unit vectors,
- the additional eigenmodes ϕ_k will be zero.

2.3 computation of the POD using Singular Value Decomposition(SVD)

The complete bi-orthogonal decomposition of the data may be obtained by use of the SVD[11]. For instance the SVD which is related to Principal Component Analysis, is used in reference[12][13] to compute modal metrics to solve model updating problems in an optimization procedure. Let $x(t)$ denote a response time-history, where x is a vector containing the displacement, velocity or acceleration at M discrete locations. The discrete matrix X is formed:

$$X = \left[\begin{array}{ccc} x_1(t_1) & \cdots & x_1(t_N) \\ \vdots & \ddots & \vdots \\ x_M(t_1) & \cdots & x_M(t_N) \end{array} \right] \tag{17}$$

So, each row corresponds to a time history at one location and each column corresponds to a snapshot of the system at a specific time. Now the singular value decomposition of matrix X can be written as:

$$X = U\Sigma V^T \tag{18}$$

with U an orthonormal matrix (size $M \times M$) of eigenvectors of XX^T and V an orthonormal matrix (size $N \times N$) of eigenvectors of $X^T X$. The size of the matrix Σ is $M \times N$ but only the main diagonal has non-zero elements that are the singular values of X, sorted in descending order. If the matrix X is rank deficient, i.e. some rows (or columns) can be generated by a linear superposition of the others, some of the singular values will be zero. The SVD can be used to estimate the rank of a matrix and filter out the measurement noise by discarding the modes associated with singular values smaller than a threshold value related to the presence of noise. In this paper we use the SVD to compute the POM's and the normalized basic shapes including the response time-histories[13]. This bi-orthonormal decomposition of space and time data will be used for the identification of parameters of nonlinear dynamical structures with an optimisation procedure based on the difference between the experimental and simulated decomposition.

3 APPLICATION TO PARAMETER IDENTIFICATION

The identification of the nonlinear parameters of a structure is based on the solution of an optimization problem which consists in minimizing the difference between the bi-orthogonal decompositions of the measured and simulated data respectively. Defining the objective function F as

$$F = \sum_i \sum_j (\Delta U_{ij})^2 + \sum_j (\Delta \Sigma_{jj})^2 + \sum_j \sum_k (\Delta V_{jk})^2 \tag{19}$$

where ΔU_{ij}, $\Delta \Sigma_{jj}$ and ΔV_{jk} are the differences between the matrices containing the bi-orthogonal decompositions. It must be stated that the full decomposition is not retained in the objective function. Only the terms corresponding to the higher singular values are considered, which means that we take the proper orthogonal modes that contain the greatest amount of energy in the signal. Then the objective function F may be minimized by standard optimization algorithms.

4 NUMERICAL EXAMPLE

To illustrate the method, let us consider a clamped steel beam with a local nonlinearity at the free end (see figure 1).

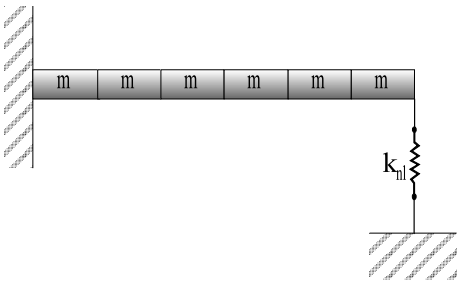

Figure 1: Clamped beam with a local nonlinearity.

The steel beam is 0.593 meter length and has a square section of $0.196 \times 10^{-3} m^2$. The beam is modeled with six finite elements. Each node posses two degrees of freedom: one for the vertical deflection and one for the rotation. The whole structure has 12 degrees of freedom. The nonlinearity is a spring that exhibits a cubic stiffness.

4.1 Numerical simulation : the modified Newmark method

Although nonlinear forces are locally distributed in the system, the dynamic behavior of the whole structure is nonlinear due to interactions. The numerical simulation of such systems with the whole set of degrees of freedom can be very time consuming. Therefore a modified Newmark[14] method that is unconditionally stable is used to integrate the responses of the system. In this formulation, the nonlinear iterations only need to be performed on the localized nonlinear part of the system equations.

The equations of motion can be written as:

$$M\ddot{x} + C\dot{x} + Kx = g + f^{nl}(x, \dot{x}) \tag{20}$$

where $M,\ C,\ K,\ \in R^{n \times n}$ are respectively the mass matrix, damping matrix and stiffness matrix; $x(t)$ is the displacement vector, and $g(t)$ is the external force vector. $f_{nl}(x, \dot{x})$ is the nonlinear spatially localized part so that (20) takes the form :

$$\begin{bmatrix} M_a & M_{ab} \\ M_{ba} & M_b \end{bmatrix} \begin{Bmatrix} \ddot{x}_a \\ \ddot{x}_b \end{Bmatrix} + \begin{bmatrix} C_a & C_{ab} \\ C_{ba} & C_b \end{bmatrix} \begin{Bmatrix} \dot{x}_a \\ \dot{x}_b \end{Bmatrix}$$

$$+ \begin{bmatrix} K_a & K_{ab} \\ K_{ba} & K_b \end{bmatrix} \begin{Bmatrix} x_a \\ x_b \end{Bmatrix} = \begin{Bmatrix} g_a \\ g_b \end{Bmatrix} + \begin{Bmatrix} f_a^{nl}(x_a, \dot{x}_a) \\ 0 \end{Bmatrix} \tag{21}$$

where the partition vector x_a is directly concerned by the nonlinear internal forces while x_b is indirectly nonlinear, due to the coupling of the equations.

When applying an implicit integration scheme[15], the displacements, velocities and accelerations involved in the equilibrium equations (21) are linked by the integration operator. Let us rewrite the equilibrium equations as a relationship in terms of displacements $x(t)$.

$$r(x) = M\ddot{x}(t) + f(x, \dot{x}) - g(x, t) = 0 \tag{22}$$

where r is the residual vector. By substituting the Newmark's time integration relationships into (22), the residual equation at time step $n+1$ is expressed in terms of x_{n+1} only:

$$r(x_{n+1}) = 0 \tag{23}$$

To solve this set of nonlinear equations, linearization techniques are used. Let us denote x_{n+1}^k an approximate value of x_{n+1} resulting from iteration k. In the neighborhood of this value the residual equation can be replaced with enough accuracy by the linear expression

$$r_L(x_{n+1}^{k+1}) = r(x_{n+1}^k) + S(x_{n+1}^k)(x_{n+1}^{k+1} - x_{n+1}^k) \tag{24}$$

in terms of the jacobian matrix

$$S(x_{n+1}^k) = \left[\frac{\partial r}{\partial x}\right]_{x_{n+1}^k} \tag{25}$$

In the case of the partitioned set of equations (21), the residual equation takes the form:

$$\begin{bmatrix} S_a(x_{a,n+1}^k) & S_{ab} \\ S_{ba} & S_b \end{bmatrix} \begin{bmatrix} \triangle x_a^k \\ \triangle x_b^k \end{bmatrix} = - \begin{bmatrix} r_a(x_{n+1}^k) \\ r_b(x_{n+1}^k) \end{bmatrix} \tag{26}$$

where the second equation is completely linear. In fact, only the submatrix S_a depends on the displacements. The other matrices are constant because they correspond to the linear part of the system. For sake of simplicity, subscripts corresponding to time step $(n+1)$ and iteration step (k) are eliminated and (26) can be written:

$$\begin{bmatrix} S_a(x) & S_{ab} \\ S_{ba} & S_b \end{bmatrix} \begin{bmatrix} \Delta x_a \\ \Delta x_b \end{bmatrix} = - \begin{bmatrix} r_a \\ r_b \end{bmatrix} \tag{27}$$

If we substitute the second equation of (27)

$$\Delta x_b = -S_b^{-1} r_b - S_b^{-1} S_{ba} \Delta x_a \tag{28}$$

into the first one, we obtain:

$$\left[S_a(x) - S_{ab} S_b^{-1} S_{ba}\right] \Delta x_a = -r_a + S_{ab} S_b^{-1} r_b \tag{29}$$

Taking a constant time step, the projection $S_{ab} S_b^{-1} S_{ba}$ is computed only one time so that the iterative solution of (29) is done only on a few degrees of freedom.

4.2 Identification of the nonlinear parameter

In this section, the linear part is supposed to be known so that the only parameter to identify is the nonlinear cubic stiffness. The free vibration of the beam is simulated with an initial displacement given by a static force applied at the end of the beam. The simulation is performed over a time period of 0.1 seconds, with a time step of 2×10^{-4} seconds. The robustness of the technique has been tested adding a gaussian noise with an amplitude of 1% of the initial displacement.

The objective function is written in terms of the first POM only. The normalized value of the nonlinear parameter in the reference case is 50. This value is perturbed to start the optimization process. Different starting points are tested which lead to different minima. To obtain the correct optimum, the optimization process needs to start with an initial value for the nonlinear coefficient of about 80% of the correct value. The comparison between the original and the reconstructed signals at the starting point is given in figure 2a, while in figure 2b, the comparison is shown after the optimization. The reconstructed signal (shown in dashed line) matches very closely the original one (in solid line).

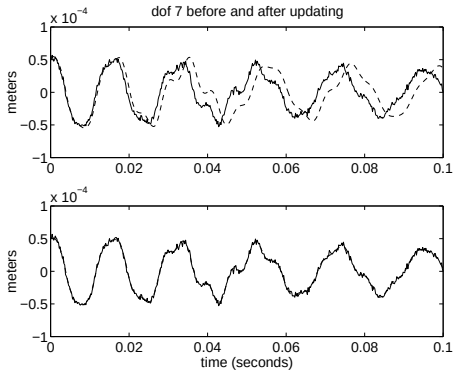

Figure 2: Comparison of displacements before and after optimization.

4.3 Improvement of the objective function (OF)

To improve the optimization process, the influence of different parameters were tested, such as the total time of the simulation, the number of POD or the type of objective function are modified. For each test the objective function has been calculated for a large range of variation of the nonlinear parameter.

4.3.1 DURATION OF THE SIMULATION

The simulation period has a great influence on the picture of the objective function because the time decomposition of the data is included in the objective function (19).When the simulation duration is long, the time decomposition contains a lot of oscillations involving more oscillations in the plot of the objective function. In figure 3 two objective functions are plotted with a simulation time of $0.02\ second$ and $0.2\ second$ respectively. The figure clearly shows that a lot of minima appear in the case of a long simulation time which can be bad for the research of the global minimum during the optimization process. Of course a small simulation time (e.g. one oscillating period of the proper orthogonal mode) gives too little information on the system so that the proper orthogonal modes loose physical significance. One solution consists to drop the time decomposition in the objective function (19). The plot of the objective function in figure 4 shows a more suited curve for the optimization process. The drawback is again the lost of temporal information over the system.

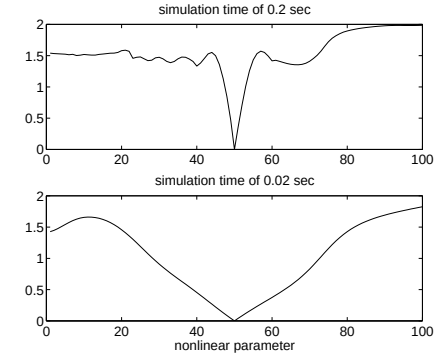

Figure 3: Objective Function for two simulation time.

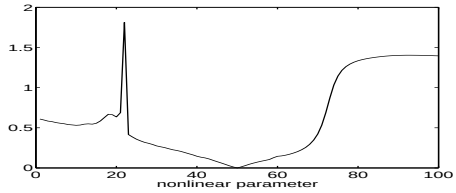

Figure 4: Objective Function without the time decomposition.

4.3.2 WAVELET TRANSFORM OF THE TIME DECOMPOSITION

In order to retain the time information but without the drawback of its oscillatory nature, the wavelet transform of the decomposition may be performed. The "instantaneous frequencies" of the signals are extracted and included into the objective function (4)in place of the right-singular vectors. The comparison between the two formulations is shown in figure 5. The time simulation is one second. The objective function without the wavelet transform shows a quite "horizontal" line connected to a narrow valley that contains the global minimum. When the wavelet transform is applied, the objective function decreases from the start to the optimum, showing a large valley.

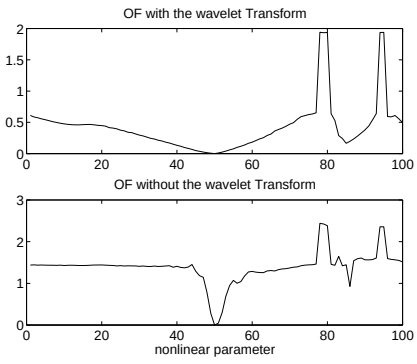

Figure 5: OF with and without wavelet transform.

The first part of each curve is zoomed in figure 6. The decrease of function in the case of the wavelet transform is very smooth, which allows to reach easily the optimum of the function. On the other hand, the horizontal line is oscillating before the narrow valley, which may compromise the efficiency of the optimization procedure.

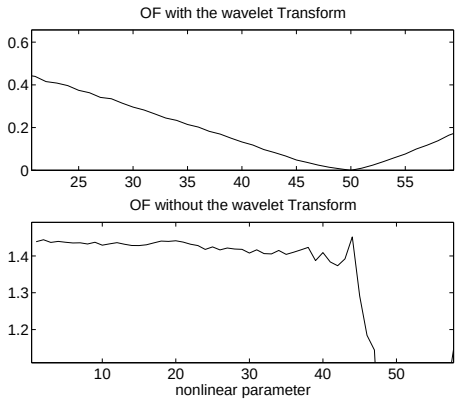

Figure 6: Zoom of the Objective Function with and without wavelet transform.

4.3.3 NUMBER OF POM

In the previous sections only the first proper orthogonal mode was included in the objective function. Since no external excitation is applied, this mode contains more than 60% of the total amount of energy, which is sufficient for the identification of the nonlinear parameter. Some tests have been performed with more than one proper orthogonal mode and the same conclusions than before can be formulated.

5 CONCLUSIONS

In this paper, the proper orthogonal decomposition has been used to identify parameters of nonlinear dynamical structures. An optimization procedure based on the difference between the experimental and simulated POM has been used. A numerical example of a beam with a local nonlinear component illustrates the method. The wavelet transform is then used to improve the optimization procedure.

The extension of the technique to the optimization of several parameters is straightforward. The method has been tested only on simulated examples and will be verified on experimental data in the next future.

ACKNOWLEDGMENTS

This work is supported by a grant from the Walloon government as a part of the research convention n 9613419 ”Analyse intégrée de résultats numériques et expérimentaux en dynamique des structures”.

Part of this text also presents research results of the Belgian programme on Inter-University Poles of Attraction initiated by the Belgian state, Prime Minister’s Office, Science Policy Programming. The scientific responsibility is assumed by its authors.

REFERENCES

[1] **Azeez, M. F. A.** and **Vakakis, A. F.**, *Numerical and experimental analysis of the nonlinear dynamics due to impacts of a continuous overhung rotor*, *Proceedings of DETC’97, ASME Design Engineering Technical Conferences, Sacramento, California*, 1997.

[2] **Cazemier, W.**, Proper orthogonal decomposition and Low Dimensional Models for Turbulent Flows, Ph.D. thesis, Rijksuniversiteit Groningen, 1997.

[3] **Newman, A.** and **Krishnaprasad, P. S.**, *Nonlinear Model Reduction for RTCVD*, *Proceedings of the 32nd Conference on Information Sciences and Systems, Princeton, NJ*, 1998.

[4] **Uytterhoeven, G.**, Wavelets: Software and Applications, Ph.D. thesis, Katholieke Universiteit Leuven, 1999.

[5] **Lumley, J. L.**, *The structure of inhomogeneous turbulent flows*, Atmospheric Turbulence and Radio Wave Propagation, 1967.

[6] **Cusumano, J. P.** and **Bai, B.-Y.**, *Period-infinity periodic motions, chaos, and spatial coherence in a 10 degree of freedom impact oscillator*, Chaos solitons fractals, Vol. 3, pp. 515–536, 1993.

[7] **Cusumano, J. P., Sharkady, M. T.** and **Kimble, B. W.**, *Spatial coherence measurements of a chaotic flexible-beam impact oscillator*, Aerospace Structures: Nonlinear Dynamics and System Response, ASME, Vol. 33, pp. 13–22, 1993.

[8] **Feeny, B. F.**, *Interpreting Proper orthogonal Modes in Vibrations*, *Proceedings of DET’97 ASME Design Engineering Technical Conferences*, 1997.

[9] **Azeez, M. F. A.** and **Vakakis, A. F.**, *Proper orthogonal decomposition (POD) of a class of vibroimpact oscillations*, internal report, Department of Mechanical and Industrial Engineering, University of Illinois at Urbana Champaign, U.S.A., 1998.

[10] **Xianghong M., Azeez, M. F. A.** and **Vakakis, A. F.**, *Nonlinear Normal Modes and Nonparametric System Identification of Nonlinear Oscillators*, Modal Analysis and Testing, NATO-Advanced Study Institute at Sesimbra, Portugal, 1998.

[11] **Press, W. H., Teukolsky, S. A., Vetterling, W. T.** and **Flannery, B. P.**, Numerical Recipes in C, The Art of Scientific Computing, Cambridge University Press, 2nd edn., 1997.

[12] **Hemez, F. M.** and **Doebling, S. W.**, *Test Analysis Correlation and finite Element Model Updating for Nonlinear Transient Dynamics*, *Proceedings of the 17th International Modal Analysis Conference*, 1999.

[13] **Hasselman, T., Anderson, M.** and **Wenshui, G.**, *Principal Components Analysis for Nonlinear Model Correlation, Updating and Uncertainty Evaluation*, *Proceedings of the 16th International Modal Analysis Conference*, pp. 644–651, Santa Barbara, California, 1998.

[14] **Zheng, T.** and **Hasebe, N.**, *An Efficient Analysis of High-Order Dynamical System with Local Nonlinearity*, Journal of Vibration and Acoustics, Vol. 121, 1999.

[15] **Géradin, M.** and **Rixen, D.**, Mechanical Vibrations, John Wiley & Sons Ltd, 2nd edn., 1997.