Dualities for Boolean Contact Algebras

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The well-known de Vries duality, established by H. de Vries in 1962, states that the category of compact Hausdorff spaces is dually equivalent to that of complete *compingent* Boolean algebras [1].

The notion of Boolean *contact* algebra (BCA) was developed independently in the context of region-based theory of space. A BCA is a Boolean algebra B endowed with a binary relation $\mathcal{C}$ satisfying the following axioms:

C1 $a \,\mathcal{C}\, b \Rightarrow a \neq 0$;

C2 $a \neq 0 \Rightarrow a \,\mathcal{C}\, a$;

C3 $a \,\mathcal{C}\, b \Rightarrow c \,\mathcal{C}\, a$;

C4 $a \,\mathcal{C}\, b, b \leq c \Rightarrow a \,\mathcal{C}\, c$;

C5 $a \,\mathcal{C}\, (b \vee c) \Rightarrow a \,\mathcal{C}\, b$ or $a \,\mathcal{C}\, c$.

A BCA is *extensional* if it satisfies

C6 $a \not\leq b \Rightarrow \exists c \in B$ such that $a \,\mathcal{C}\, c$ and $c \perp b$,

where $\perp$ denotes the complement of the relation $\mathcal{C}$.

Düntsch and Winter established in [3] a representation theorem for extensional BCAs, showing that every extensional BCA is isomorphic to a dense subalgebra of the regular closed sets of a T_1 weakly regular space. It appears that BCAs are a direct generalization of de Vries' compingent algebras, and that the representation theorem for complete BCAs generalizes de Vries duality for objects. We turn this representation theorem into a duality, including morphisms, thus answering a question asked informally by Vakarelov.

We also provide a duality for general BCAs (satisfying C1-C5) through clans.

References

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