Topology Optimization with Different Stress Limits in Tension and Compression

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1. Abstract
This report presents new progresses in topology optimization of continuum structures with stress constraints. One principal contribution consists in the consideration of equivalent stress criteria which can generalization of von Mises criterion and which are able to take into account non equal stress limits in tension and compression. A literature review led us to consider Raghava and Ishai criteria, which include a contribution of hydrostatic pressure. With the help of these criteria topology optimization can predict more realistic designs in which structural members are able to withstand better tension loads than compression loads, or vice-versa, as it is sometimes encountered in civil engineering or in composite material design. The implementation and sensitivity analysis aspects of Raghava and Ishai criteria in the Finite Element context are presented. We also present recent advanced developments to the solution of topology problems with stress constraints like the stress constraint relaxation technique and the numerical optimization procedure based on convex approximations and dual optimizers. Finally numerical applications will show the original character of the stress based topology designs ad versus compliance designs when there are unequal stress limits or when there are more than one load case.

2. Keywords
Topology optimization, stress constraints, Raghava criterion, Ishai criterion.

3. Introduction and formulation of the problem
Since the foundation work of Bendsøe and Kikuchi [1], most of the topology optimization work has been based on compliance type arguments. Recently a couple of researches (e.g. [2], [3]) led to the successful solution of the stress constrained topology problem. This paper continues along this work and presents new progresses in topology optimization of continuum structures.

Up to now topology optimization with stress constraints was based on the quadratic von Mises criterion. This criterion is problem is very usual, because it predicts very precisely failure for metals which are usually used in mechanical engineering. However there are many other cases where von Mises criterion is unable to predict real-life designs. Structures made of materials with unequal stress limits are good examples of this. Engineers know a lot of materials that have different behaviors in tension and compression: e.g. concrete, rocks, chalk, composite materials, etc. One also can remember that thin structural members, like cables or thin sheets, are also not able to sustain high compressive loads because of buckling. An indirect procedure to take into account this buckling in the preliminary design phase consists in restricting the compressive loads by reducing the stress limit in compression. From a practical point of view these different behaviors in tension and compression have been mastered for a long time and resulted in quite specific designs. For example since Roman time the arch technique allows the use of the high compressive resistance of bricks to build high buildings and bridges. On other hand civil engineers use cables, which have nearly no load bearing capability in compression for example by choosing suspension bridges where they work in a pure tensile stress state. Despite the fact that the phenomenon of unequal stress limits was quite well integrated in classical building techniques, topology optimization was nearly unable to take into account this effect and thus to predict realistic designs in this case, which means designs that stick to existing results that proved their efficiency. In order to give an appropriate answer to this problem one has be able to consider stress-constrained designs and to consider stress criteria able to cope with unequal stress limits. This particular type of problem is considered here.

The general problem we consider is the following: to find the structural topology design that supports the applied loads using a minimum amount of material while avoiding the material failure everywhere in the structure.

$$\begin{array}{ll} \min\limits_{0 \le r(x) \le 1} & \int_\Omega r(x)\, dx \\ s.t. & \| s^{eq}(r(x), x) \| \le T \quad if \;\; r(x) > 0 \end{array} \tag{1}$$

$r(x)$ is the material density at point **x**. $\| s^{eq}(r(x), x) \|$ is an equivalent stress criterion which predicts the failure of the material at point **x** while T is the stress limit. Up to now Von Mises was the only criterion that was considered, but other criteria are going to be introduced in the next paragraph. In this study a pure elastic regime is assumed, so that the criterion predicts the end of the elastic behavior everywhere in the material.

In order to solve numerically the problem, the design domain is divided into finite elements and a density variable is attached to each element. The discrete valued 0/1 problem is avoided and the density is allowed to vary continuously between void and solid, so composite materials of intermediate densities are included in the design. The continuous formulation presents the advantage to allow the use of sensitivity analysis and mathematical programming algorithms to solve the problem in an efficient way. Unfortunately the discretized problem is ill-posed and its numerical solutions are mesh-dependent. To overcome the difficulty, we use here a restriction method of the design space based on a bound over the perimeter [4], a low-pass filtering scheme [5], or a combination of both.

The modelling of the intermediate density properties is based on the power-law approach (also called SIMP model). If the script * denotes the effective properties of the porous material and the index 0 is relative to the solid material properties, the effective Young's modulus E^* is given in term of the density r by:

$$E^* = r^p E^0 \tag{2}$$

The factor p>1 is introduced to penalize the intermediate densities in order to end up with 'black and white' designs. Moreover to consider stress constraints in continuous topology optimization, one also needs the definition of a relevant stress measure in the porous composites. Following the approach developed in [3] we consider an overall stress measure that controls the stress state in the microstructure. For the SIMP model of stiffness, a careful study [3] showed that a power-law model with the same power p is a consistent model for the micro-stresses s_{ij}:

$$s_{ij} = s^*_{ij} / r^p \tag{3}$$

Therefore the overall failure criteria in the porous composites $\| s^{eq} \|$ that predicts the first failure in the microstructure is given by:

$$\| s^{eq}(r) \| = {}^*s^{eq} / r^p \leq T \tag{4}$$

4. Failure criteria for unequal stress limits in tension and compression

Treating different behaviors in tension and compression requires particular failure criteria. For brittle materials one can consider a principal stress criterion. s_I , s_{II} and s_{III} are defined as the principal stresses. T and C are the stress limits (in absolute value) respectively in tension and compression. One defines also 's' the ratio between the stress limits in compression and in tension: $s = C/T$. The material fails when the maximum principal stress does not satisfy:

$$-C \leq \max_{K=I,II,II} s_K \leq T \tag{5}$$

The principal stress criterion is quite realistic for brittle materials (like glass), but this criterion does not fit so well to experiments for ductile materials. Furthermore one would prefer to use smooth criteria which render better the physics of the failure for a lot of materials. We also prefer to find a quadratic criterion that can generalize the well-known von Mises theory. Quadratic character is quite interesting, because it insures that the criterion is smooth and convex. Smoothness is very good for differentiability in sensitivity analysis. A review work (see for example [6] and [7]) was necessary to scan different criteria that are able to cater with unequal compressive and tensile stress limits. From the comparison of the different criteria, we selected the Raghava criterion and the Ishai criterion for this study in topology optimization.

As the two criteria are functions of the first stress invariant J_1 and the second deviator stress invariant J_{2D} we remind the reader with the definition of these two stress invariant numbers:

$$J_1 = s_I + s_{II} + s_{III} \quad J_{2D} = \frac{(s_I - s_{II})^2 + (s_{II} - s_{III})^2 + (s_{III} - s_I)^2}{6} \tag{6}$$

The second invariant is directly related to the distortional shear stress t_0 and to the equivalent von Mises stress s_{VM}: $3\,J_{2D} = t_0^2 = s_{VM}^2$. This presence of the second invariant is obvious because the criteria have to render the von Mises criterion when the stress limits are equal. The first invariant is related to the hydrostatic pressure $J_1 = 3\,s_h$ and its presence in the criterion is essential since it introduces the dependence upon the sign of the stress state and so the different behaviors in tension and compression.

<u>The Raghava criterion</u> [8] is a quadratic failure criterion that is generally used with adhesive materials. In terms of the stress invariants, the Raghava stress criterion and the related equivalent stress can be written as:

$$s^{eq}_{RAG} = \frac{J_1(s-1) + \sqrt{J_1^2(s-1)^2 + 12 J_{2D} s}}{2s} \leq T \tag{7}$$

As it is demonstrated in [6], the Raghava criterion can also be viewed as a particular case of Tsai-Wu criterion of orthotropic materials when one assumes that the stress limits along the 3 orthotropy directions are the same.

An interesting alternative to Raghava is <u>the Ishai criterion</u> [9]. With the notations that has been defined, one writes the Ishai equivalent stress and its failure criterion:

$$s^{eq}_{ISH} = \frac{(s+1)\sqrt{3J_{2D}} + (s-1)J_1}{2s} \leq T \tag{8}$$

As shown in [6] this criterion belongs to the Drucker-Prager plasticity criterion family that is of a general use for chalk, rocs and soils materials.

The two criteria can be compared in Fig. 1 and 2 where their failure envelopes are plotted in the space (σ_h , τ_0) of the hydrostatic pressure and of the distortional shear stress. At first the failure surface of the Raghava approach is a parabolic curve whereas the Ishai criterion is a hyperbolic one. But from the figures, one also notices that the two theories predict the failure for different maximum hydrostatic pressures and for different pure distortional shear stresses. The pure hydrostatic stress limit predicted by Ishai's theory is twice Raghava's one. We remind the reader that this hydrostatic failure stress is infinite in von Mises' theory. Failure under a pure shear stress state is also different in the two theories. The pure distortional shear stress limit for Raghava's theory is always greater than Ishai's one because $\sqrt{s} \geq 2s/(s+1)$

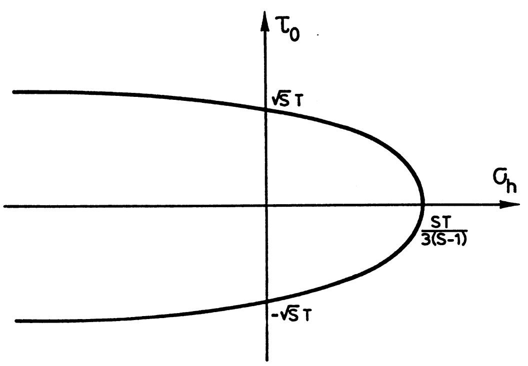

Figure 1: Raghava failure criterion

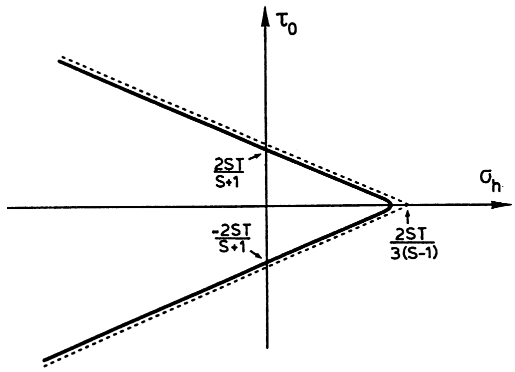

Figure 2: Ishai failure criterion

5. Numerical implementation and sensitivity analysis of Ishai and Raghava criteria

If the implementation of von Mises criterion in an optimization code coupled with a finite element analysis is well known, it may be quite interesting here to describe the guidelines of the implementation of Ishai's and Raghava's criteria. This is the opportunity to underline the specific character of the two criteria.

The equilibrium equation after discretization by finite element writes $\boldsymbol{K}\,\boldsymbol{q} = \boldsymbol{f}$ where $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{q}$ is the generalized displacement vector and $\boldsymbol{f}$ is the load vector. When writing the overal stress in the finite element in a vector form, it is given by $s = \boldsymbol{T}\,\boldsymbol{q}$ where $\boldsymbol{T}$ is the stress matrix of the element. The two first invariants of the stress in the finite element that are required to calculate the equivalent stress criteria can be written in matrix form:

$$J_1^* = 3s_h^* = w^T\, s = W\, q \quad and \quad J_{2D}^* = 1/3\,\left(s_{VM}^*\right)^2 = 1/3\, q^T\, V\, q \qquad (9)$$

As the stress matrix depends linearly upon the stiffness properties, one can also write $T = r^p\, T^0$ where $\boldsymbol{T^0}$ is the stress matrix of an element of relative density of 1. In the same way the influence of density can be put in evidence in matrices $\boldsymbol{W}$ and $\boldsymbol{V}$: $V = r^{2p} V^0$ and $W = r^p W^0$ where matrices the $\boldsymbol{V^0}$ and $\boldsymbol{W^0}$ are the von Mises stress matrix and the hydrostatic stress matrices of the solid element.

Therefore one can evaluate easily the effective failure criteria based on Ishai and Raghava theories with the following expressions:

$$\|\, s_{ISH}^{eq} \,\| = {}^*s_{ISH}^{eq} \,/\, r^p = \frac{s-1}{2s} W^0\, q + \frac{s+1}{2s}\sqrt{q^T\, V^0\, q} \qquad (10)$$

$$\|\, s_{RAG}^{eq} \,\| = {}^*s_{RAG}^{eq} \,/\, r^p = \frac{(s-1)\, W^0\, q + \sqrt{(s-1)^2\,(W^0\, q)^2 + 4s\, q^T\, V^0\, q}}{2s} \qquad (11)$$

Sensitivity analysis can also be implemented efficiently for Ishai and Raghava criteria as it can be done for Von Mises theory. As only the guidelines of the implementation are presented we give only the main results of the derivation calculus.

For Ishai equivalent stress, we get:

$$\frac{\partial \| s_{ISH}^{eq} \|}{\partial r_i} = \left\{ \frac{s-1}{2s} W^0 \quad + \quad \frac{s+1}{2s} \frac{1}{\sqrt{q^T V^0 q}} V^0 \; q \right\}^T \quad \frac{\partial q}{\partial r_i} \tag{12}$$

Raghava's criterion is a bit more difficult. After some calculus the derivative of the averaged stress criterion can be written as:

$$\frac{\partial \| s_{RAG}^{eq} \|}{\partial r_i} = \left\{ \frac{s-1}{2s} \frac{\| s_{RAG}^{eq} \|}{\sqrt{(s-1)^2 (W^0 q)^2 + 4s\, q^T V^0 q}} W^0 \quad + \quad \frac{2}{\sqrt{(s-1)^2 (W^0 q)^2 + 4s\, q^T V^0 q}} V^0 \; q \right\}^T \quad \frac{\partial q}{\partial r_i} \tag{13}$$

In a direct approach of the sensitivity analysis one would evaluate all the derivatives of the generalized displacements $\partial q / \partial r_i$, which could be a major job in topology optimization. However the number of active stress constraints is generally smaller than the number of design variables and the adjoined method is generally preferred because only one additional load case is required per active constraint. In this case the adjoined load vector is the vector that is put between curly braces.

6. Numerical solution of stress-constrained problems

As it has been shown in [10] and [3], the topology optimization with stress constraints is subject to the 'singularity' phenomenon. At short the paradox comes from the fact that the optimization procedure is often unable to remove or to add some vanishing members without violating the stress constraints although one would end up with a perfectly feasible design if they were removed or added. From a mathematical point of view the classical algorithms are unable to reach some optimum configurations because of the degeneracy of the design space. In order to turn around the difficulty one has to use a perturbation technique of the stress constraints, generally known as the ε-relaxation technique [10] that results in a relaxation of the stress limits in the low-density regions. We use here the following ε-relaxed formulation of the overall stress criterion that is slightly modified for density variables:

$$\frac{1}{T} \frac{s^{eq}}{r^p} - \frac{e}{r} + e \le 1 \tag{14}$$

The advantage of this relaxation formulation is that the influence of the perturbation always cancels for $r = 1$ whatever is the value of relaxation parameter ε. Therefore one can stop the optimization procedure with any value of ε with a feasible design for the stress constraints in hands. In addition to this new relaxation formulation, an automatic strategy to reduce the ε parameter is also provided. The program without any interaction of the user makes the reduction of the perturbation parameter. The parameter is reduced as soon as the optimization problem is solved with a sufficiently small tolerance.

A second interesting aspect of the procedure is the use of a mathematical programming approach suited to the solution of large-scale systems. A robust approximation strategy has been elaborated on the basis on the Method of Moving Asymptotes (MMA) [11] while the core of the solution algorithm relies on a dual algorithm [12]. In fact the separability of the approximations is one of the key ideas to manage efficiently the solution of large-scale problems. The MMA algorithm has been customized in order to give rise to small trust regions and avoid constraint violations. The primal-dual optimizer from Fleury [12] showed itself as extremely robust for large-scale problem solution. Optimal solutions are generally produced within a computation time that of the same order of magnitude than the finite element analysis for all our numerical applications.

7. Applications

<u>Two-bar truss benchmark</u>

The first application is the classic two-bar truss problem. In fact the two-bar truss problem is not a trivial example in the framework of topology optimization with stress constraints, moreover with unequal stress limits. Indeed even if the lay-out of the solution is the obvious two-bar truss for equal stress constraints and the von Mises stress criterion (see Fig. 3 right), one have to be able to overcome the problem of the singular stress paradox. The part of the design domain that is enclosed inside the two bars remains highly strained and thus highly stressed while all its material vanishes. Without the ε-relaxation technique it is impossible to remove totally this part. In the framework of unequal stress limits the second point of interest is to be able to check the analytical solution for this problem. The analytical solution has been exhibited by Rozvany [13] while discussing shortcoming of Michell's truss theory. When adopting a tensile stress limit is 3 times higher than the compressive stress limit, the solution is still a two bar truss, but the two bars make an angle of 30° and 60° degrees respectively with the foundation wall. Ishai equivalent stress is adopted to take into account the different stress limits effect. The optimized material distribution is presented in Fig.3 (left). The numerical result totally matches the theoretical prediction, which validates the method.

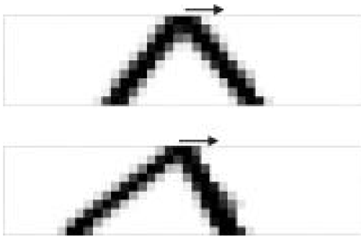

Figure 3: Two-bar truss solutions for equal stress limits (T=C)
and different stress limits (T=3C)

Four-bar truss problem

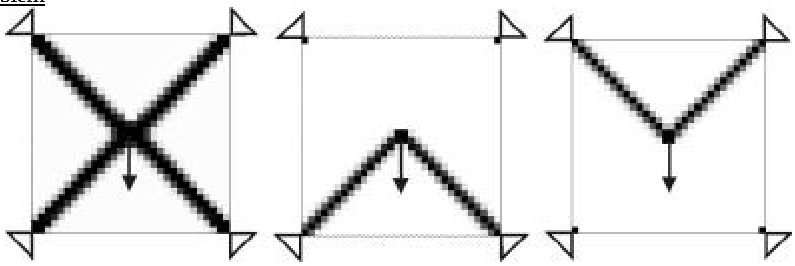

Figure 4: Four-bar truss solutions for equal stress limits (C=T) (left), high compressive strength (C=4T) (center) and high tension strength (T=4C) (right)

The second example stems from an application that was initially suggested by Swan and Kosaka [12] to demonstrate that ultimate strength optimization can lead to substantially different results from a minimum elastic compliance design. In the present study the same example is revisited in the framework of first failure stress constraints and elastic behavior. The design domain is a square (L= 1m) clamped in the four corners. A unit load is applied downwards in the center of the square domain. Using a normalized material (E=100N/m² and ν=0.3) one bounds at first the von Mises equivalent stress to T=C=10 N/m² (see Fig 4-left). The same optimal topologies could be obtained for minimum compliance as for minimum volume subject to stress constraints. It is cross-like structure that withstands the load in tension (upper members) and compression (lower members). Now if one uses a material that has a better strength limit in compression than in tension (T=6 N/m², C=24 N/m²), the optimal structure (Fig 4-center) is an arch that works only in compression. Conversely if the material works better in tension than in compression (T=24 N/m², C=6 N/m²) one gets a structure (Fig. 4-right) working exclusively in tension like cables in suspension bridges. We have also to remark that the results for unequal stress limits have been made with the Ishai criterion, but the same results can be obtained with the Raghava criterion since they give exactly the same strength for members under a one dimensional stress state as here.

Three-bar truss problem

The last example is the three-bar truss problem [3], which lends itself to demonstrate the specific character of the stress-constrained design when there is more than one load cases. The sizes and material data of the problem are normalized (L=1 m, W =2.5 m, E=100 N/m², ν=0.3). The minimum compliance design leads to a two-bar truss design (see Fig. 5 top, left). The von Mises stress constraints (with T=C=150 N/m²) give rise to a different layout with a three-bar truss design (Fig. 5 top, right). A posteriori computation of the stress level in the two-bar truss solution shows that this configuration is over-stressed by nearly a factor 1.5 compared to the three-bar truss, which proves that the 3-bar truss topology is not a local optimum. One may also remark that introducing unequal stress limits with Ishai criterion (T=150 N/m² and C=450 N/m² in Fig 5 bottom left and T=450 N/m² and C=150 N/m² in Fig.5 bottom right) leads again to a two-bar truss configuration. This demonstrates the different nature of minimum compliance design and strength maximization when there are several load cases.

8. Conclusion

Although most of structural problems can be successfully studied with compliance minimization or local von Mises constraints, there are many cases where the effect of unequal stress limits is crucial upon structural layout. When the structure is made of materials with different stress limits, topology predictions matching practical engineering experience can be made with Raghava and Ishai equivalent stresses. These quadratic failure criteria generalize von Mises theory in the case of unequal stress limits, by introducing a dependence upon the first invariant of stresses. Numerical applications validated the quality of the results. These criteria are not difficult to implement in topology optimization codes.

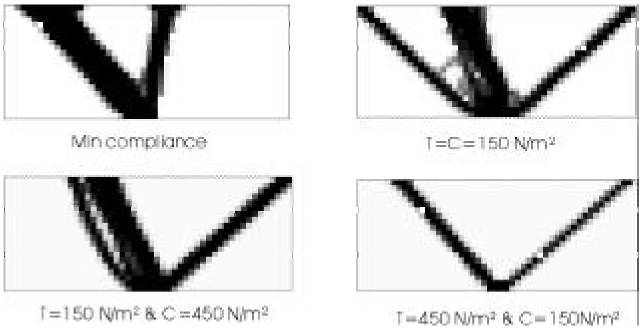

Figure 5. Three-bar truss designs for minimum compliance (top, left) and for stress constraints (top, right), and for unequal stress limits: high compressive strength (bottom left) and high tensile strength (bottom right).

The numerical applications presented herein lead also to demonstrate the original character of stress-constrained designs compared to compliance distributions. Even if engineers know that minimum compliance designs and stress-constrained designs can differ, most of the examples from literature are cases in which minimum compliance and maximum strength designs are the same. Indeed the examples are generally based on one load case, the same stress limits in tension and compression and one material (and void). Here our examples have demonstrated two important cases for which structural topologies are different for maximum stiffness designs and maximum strength design:

- When one considers unequal stress limits (even if one load case);
- When there are several load cases (even if equal stress limits).

New researches will be devoted to topology problems in which the structure is made of more than one material (and the void) in order to illustrate the third category.

9. Acknowledgements

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10. References

[1] Bendsøe, M.P. and Kikuchi N. (1988). Generating Optimal Topologies in Structural Design Using a Homogenization Method. *Computer Methods in Applied Mechanics and Engineering* vol. 71, p. 197-224.

[2] Shim P.Y. and Manoocheri S. (1997). Generating Optimal Configurations in Structural Design Using Simulated Annealing. *International Journal for Numerical Methods in Engineering*, vol. 40, p. 1053-1069.

[3] Duysinx P. and M.P. Bendsøe. (1997). Topology Optimization of Continuum Structures With Local Stress Constraints. *International Journal for Numerical Methods in Engineering*. To appear.

[4] Haber R.B., C.S. Jog and M.P. Bendsøe. (1996). A New Approach to Variable-Topology Shape Design Using a Constraint on Perimeter. *Structural Optimization*, vol. 11, p. 1-12.

[5] Sigmund O. (1997). On the Design of Compliant Mechanisms Using Topology Optimization", *Mechanics of Machines and Structures*, vol. 25, no 4, p. 493-526.

[6] Laschet G. (1994). Prédiction par Eléments Finis du Comportement non Linéaire et de la Résistance D'assemblage Multimatériaux Collés. Ph.D. thesis. University of Liège (in French).

[7] Boresi A.P., Schmidt R.T. and O.M. Sidebottom. (1993). Advanced Mechanics of Materials. 5th Edition. John Wiley and Sons, New York.

[8] Raghava R., R.M. Caddel and G.S. Yeh. (1973). The Macroscopic Yield Behaviour of Polymers. *Journal of . Material Sciences,* vol. 8, 225-232.

[9] Geli S., G. Dolex and O. Ishai. (1981). An Effective Stress/Strain Concept in the Mechanical Characterization of the Structural Adhesive Bonding. *Int. J. of Adhesion & Adhesives*, 1981, p 135-140.

[10] Cheng G.D. and X. Guo. (1997). ε-Relaxed Approach in Structural Topology Optimization. *Structural Optimization*, vol. 13 , p 258-266.

[11] Svanberg K. (1987). The Method of Moving Asymptotes – A New Method for Structural Optimization. *International Journal for Numerical Methods in Engineering*. vol. 24, p. 359-373.

[12] Fleury C. (1993). Mathematical Programming Methods for Constrained Optimization: Dual Methods. In volume 150 of *Progresses in Astronautics and Aeronautics* (M.P. Kamat ed.) chap 9, p 183-208, AIAA.

[13] Rozvany G.I.N (1996). Some Shortcomings in Michell's Truss Theory. *Structural Optimization*, vol.12, p.244-250.

[14] Swan C.C. and I. Kosaka. (1997). Voigt-Reuss Topology Optimization for Structures with Nonlinear Material Behaviors. *International Journal for Numerical Methods in Engineering*, vol. 40, p. 3785-3814.