MATH0024 – Modeling with PDEs

Separation of variables and spectral problem with applications to Laplace/Poisson equation

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- Separation of variables.

- Review of series and function series.

- Convergence and separation of variables.

- Sturm-Liouville problem.

- Summary and conclusion.

- References.

Separation of variables

Motivating example: solution of linear problem with real symmetric system matrix

- **Linear problem**: Let us consider a linear problem of the following form:
$$[A]\boldsymbol{x} = \boldsymbol{y}.$$

- **Eigenproblem**: Let us assume that $[A]$ is a real, symmetric, square n-dimensional matrix. Then, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $[A]$ are real and there is an orthonormal basis consisting of eigenvectors $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n$. Hence, with $[D] = [\mathrm{Diag}(\lambda_1, \ldots, \lambda_n)]$ and $[V] = [\boldsymbol{\varphi}_1 | \ldots | \boldsymbol{\varphi}_n]$, we obtain:

$$
\begin{aligned}
[A]\boldsymbol{\varphi}_k &= \lambda_k \boldsymbol{\varphi}_k, & 1 \le k \le n, \\
\boldsymbol{\varphi}_k \cdot \boldsymbol{\varphi}_\ell &= \delta_{k\ell}, & 1 \le k, \ell \le n.
\end{aligned}
$$

$$
\begin{aligned}
[A][V] &= [V][D], \\
[V]^{\mathrm{T}}[V] &= [V][V]^{\mathrm{T}} = [I].
\end{aligned}
$$

$$[A] = [V][D][V]^{\mathrm{T}}.$$

- **Series**: Because the eigenvectors form an orthonormal basis, we can expand $\boldsymbol{x}$ as follows:

$$\boldsymbol{x} = \sum_{k=1}^{n} b_k \boldsymbol{\varphi}_k \text{ with } b_k = \boldsymbol{x} \cdot \boldsymbol{\varphi}_k, \ 1 \le k \le n.$$

$$\boldsymbol{x} = [V]\boldsymbol{b} \quad \text{with} \quad \boldsymbol{b} = [V]^{\mathrm{T}} \boldsymbol{x}.$$

Motivating example: solution of linear problem with real symmetric system matrix (continued)

- **System of uncoupled equations ("diagonalization"):**

$$[A] \sum_{k=1}^{n} b_k \boldsymbol{\varphi}_k = \boldsymbol{y}.$$

$$[A][V]\boldsymbol{b} = \boldsymbol{y}.$$

$$\boldsymbol{y} = \sum_{k=1}^{n} c_k \boldsymbol{\varphi}_k \text{ with } c_k = \boldsymbol{y} \cdot \boldsymbol{\varphi}_k, \ 1 \leq k \leq n.$$

$$\boldsymbol{y} = [V]\boldsymbol{c} \quad \text{with} \quad \boldsymbol{c} = [V]^{\mathrm{T}}\boldsymbol{y}.$$

$$\sum_{k=1}^{n} b_k \underbrace{[A]\boldsymbol{\varphi}_k}_{=\lambda_k \boldsymbol{\varphi}_k} = \sum_{k=1}^{n} \lambda_k b_k \boldsymbol{\varphi}_k = \sum_{k=1}^{n} c_k \boldsymbol{\varphi}_k.$$

$$\underbrace{[A][V]}_{=[V][D]} \boldsymbol{b} = [V][D]\boldsymbol{b} = [V]\boldsymbol{c}.$$

$$\boxed{\lambda_k b_k = c_k, \quad 1 \leq k \leq n.}$$

$$\boxed{[D]\boldsymbol{b} = \boldsymbol{c}.}$$

- We can observe that the projection of a linear problem with a real symmetric system matrix onto the eigenvectors of this matrix breaks down this linear problem (harder to solve) into a system of uncoupled scalar equations (easier to solve). Once a solution $b_1, \ldots, b_n$ is available, a solution to the linear problem is obtained in the form of the series $\sum_{k=1}^{n} b_k \boldsymbol{\varphi}_k$ in the eigenvectors $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n$.

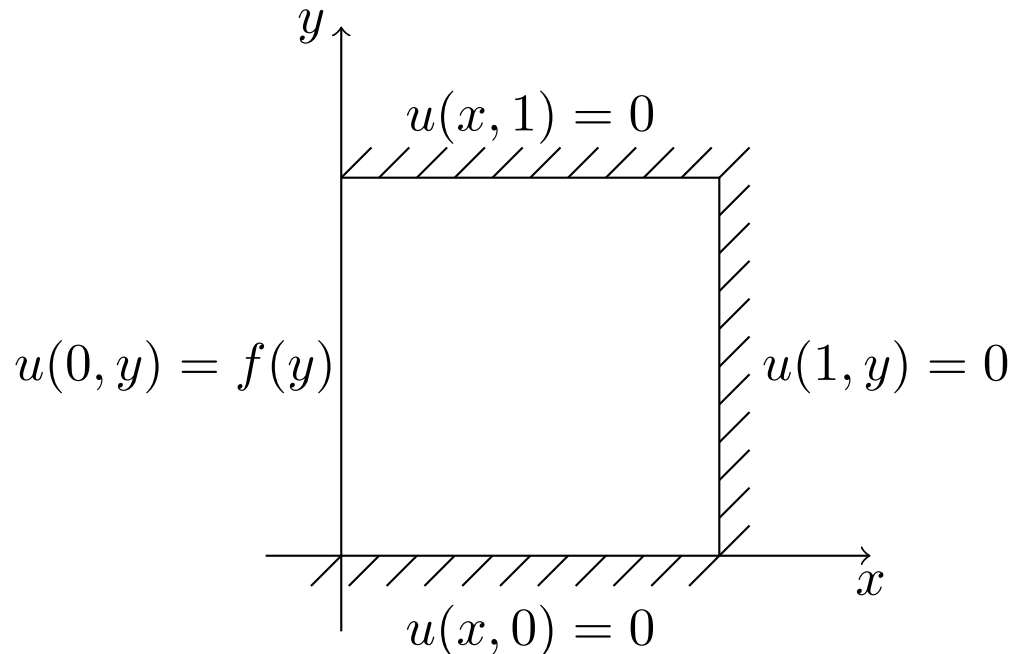## Notion of separation of variables

■ **Separation of variables** refers to a family of solution methods that share the property that a solution is sought in the form of a (series of) product(s) of functions of fewer independent variables.

■ This (series of) product(s) of functions of fewer independent variables is often constructed by using eigenfunctions obtained by solving an **eigenproblem**. A particular advantage of eigenfunctions is that their use often transforms the problem under consideration into a system of subsidiary problems that involve fewer independent variables ("**diagonalization**").

Example: Laplace's equation in a square

■ **Dirichlet problem**: Let us consider the following Dirichlet problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in }]0,1[\times]0,1[,$$

$$u(x,0) = 0 \qquad \text{on }]0,1[\times\{y=0\},$$

$$u(1,y) = 0 \qquad \text{on } \{x=1\}\times]0,1[,$$

$$u(x,1) = 0 \qquad \text{on }]0,1[\times\{y=1\},$$

$$u(0,y) = f(y) \qquad \text{on } \{x=0\}\times]0,1[.$$

The figure shows the unit square with boundary conditions: $u(x,1)=0$ on the top, $u(0,y)=f(y)$ on the left, $u(1,y)=0$ on the right, and $u(x,0)=0$ on the bottom.

Example: Laplace's equation in a square (continued)

■ **Eigenproblem:** With reference to the boundary conditions $u(x, 0) = 0$ at $y = 0$ and $u(x, 1) = 0$ at $y = 1$, we begin by solving the following eigenproblem, also called **spectral problem**:

$$\begin{cases} -\dfrac{d^2\varphi}{dy^2}(y) = \lambda\varphi(y) & \text{in }]0, 1[, \\[2ex] \varphi(0) = \varphi(1) = 0 & \text{at } y = 0 \text{ and } y = 1. \end{cases}$$

◆ Given a value of λ, the homogeneous ODE is solved by any linear combination of two linearly independent elementary solutions given by

$$\exp(-\sqrt{-\lambda}y), \qquad \exp(\sqrt{-\lambda}y), \qquad \text{if } \lambda < 0,$$
$$1, \qquad y, \qquad \text{if } \lambda = 0,$$
$$\cos(\sqrt{\lambda}y), \qquad \sin(\sqrt{\lambda}y), \qquad \text{if } \lambda > 0.$$

◆ For $\lambda < 0$ and $\lambda = 0$, the only linear combination of the elementary solutions that satisfies the boundary conditions is the trivial solution equal to zero everywhere.

◆ For $\lambda > 0$, the solution need not be equal to zero everywhere if and only if

$$\sin(\sqrt{\lambda}) = 0.$$

◆ Thus, we find a sequence of **eigenvalues** λ_k, with corresponding **eigenfunctions** φ_k,

$$\lambda_k = k^2\pi^2 \quad \text{with} \quad \varphi_k = \sqrt{2}\sin(k\pi y), \qquad \text{where } k = 1, 2, 3, \ldots.$$

Example: Laplace's equation in a square (continued)

◆ Using trigonometric identities, it can be easily verified that the eigenfunctions are orthogonal; in fact, thanks to the factor $\sqrt{2}$ in their expression, the eigenfunctions are orthonormal:

$$\int_0^1 \varphi_k(y)\varphi_\ell(y)dy = \delta_{k\ell}.$$

where $\delta_{k\ell}$ is the Kronecker delta equal to 1 if $k = \ell$ and 0 otherwise.

◆ From the theory of Fourier series, we know that the functions $\{\sqrt{2}\sin(k\pi y)\}_{k=1}^{+\infty}$, hence, the eigenfunctions of the aforementioned eigenproblem, constitute an **orthonormal basis** for $L^2(]0,1[)$, that is, any square-integrable function g from $]0,1[$ into $\mathbb{R}$ can be represented as

$$g(y) = \sum_{k=1}^{+\infty} g_k \varphi_k(y), \quad \text{where} \quad g_k = \int_0^1 g(y)\varphi_k(y)dy,$$

where the convergence is such that

$$\lim_{\ell \to +\infty} \int_0^1 \left| g(y) - \sum_{k=1}^{\ell} g_k \varphi_k(y) \right|^2 dy = 0.$$

◆ Here, $\{\sqrt{2}\sin(k\pi y)\}_{k=1}^{+\infty}$ denotes the sequence $\sqrt{2}\sin(\pi y), \sqrt{2}\sin(2\pi y), \sqrt{2}\sin(3\pi y),\ldots$

Example: Laplace's equation in a square (continued)

■ **Function series**: Given the eigenfunctions $\{\varphi_k\}_{k=1}^{+\infty}$, we seek a solution of the following form:

$$u(x,y) = \sum_{k=1}^{+\infty} b_k(x)\varphi_k(y);$$

thus, we seek a solution of the form of a series of products of functions of fewer independent variables, namely, a solution of the form of a series of products of functions b_k of only x (yet to be determined) and the eigenfunctions φ_k of only y (already known).

■ **System of uncoupled equations ("diagonalization")**: Inserting this function series into the BVP and assuming that the second-order partial derivatives of this function series can be obtained by term-by-term differentiation, that is, by interchanging differentiation and summation, we obtain

$$\begin{cases} \displaystyle\sum_{k=1}^{+\infty}\left(\frac{d^2 b_k}{dx^2}(x)\varphi_k(y) + b_k(x)\frac{d^2\varphi_k}{dy^2}(y)\right) = 0 & \text{in }]0,1[\times]0,1[, \\[2em] \displaystyle\sum_{k=1}^{+\infty} b_k(0)\varphi_k(y) = f(y) & \text{on } \{x=0\}\times]0,1[, \\[2em] \displaystyle\sum_{k=1}^{+\infty} b_k(1)\varphi_k(y) = 0 & \text{on } \{x=1\}\times]0,1[. \end{cases}$$

Example: Laplace's equation in a square (continued)

◆ Because φ_k is an eigenfunction of the aforementioned eigenproblem, we have

$$-\frac{d^2\varphi_k}{dy^2}(y) = \lambda_k\varphi_k(y) \quad \text{in }]0,1[;$$

further, assuming that f is a square-integrable function from $]0,1[$ into $\mathbb{R}$, we have

$$f(y) = \sum_{k=1}^{+\infty} f_k\varphi_k(y), \quad \text{where} \quad f_k = \int_0^1 f(y)\varphi_k(y)dy;$$

thus, we obtain

$$\begin{cases} \displaystyle\sum_{k=1}^{+\infty}\left(\frac{d^2 b_k}{dx^2}(x) - b_k(x)\lambda_k\right)\varphi_k(y) = 0 & \text{in }]0,1[\times[0,1], \\[2em] \displaystyle\sum_{k=1}^{+\infty} b_k(0)\varphi_k(y) = \sum_{k=1}^{+\infty} f_k\varphi_k(y) & \text{on } \{x=0\}\times]0,1[, \\[2em] \displaystyle\sum_{k=1}^{+\infty} b_k(1)\varphi_k(y) = 0 & \text{on } \{x=1\}\times]0,1[. \end{cases}$$

Example: Laplace's equation in a square (continued)

◆ Clearly, the previous equations hold if the following system of uncoupled equations holds:

$$\begin{cases} -\dfrac{d^2 b_k}{dx^2}(x) = -\lambda_k b_k(x) & \text{in }]0,1[\\[2mm] b_k(0) = f_k & \text{at } x = 0 \\[2mm] b_k(1) = 0 & \text{at } x = 1 \end{cases}, \quad \text{where } k = 1,2,3,\ldots.$$

It can be easily verified that the solution to this system of uncoupled equations is given by

$$b_k(x) = \frac{f_k}{\sinh(k\pi)} \sinh\big(k\pi(1-x)\big), \quad \text{where } k = 1,2,3,\ldots.$$

■ As a conclusion, we obtain the following representation of the solution to the aforementioned Dirichlet problem as a series of products of functions in fewer independent variables:

$$u(x,y) = \sum_{k=1}^{+\infty} \frac{f_k}{\sinh(k\pi)} \sinh\big(k\pi(1-x)\big)\sqrt{2}\sin(k\pi y) \quad \text{with} \quad f_k = \int_0^1 f(y)\sqrt{2}\sin(k\pi y)\,dy.$$

Example: Laplace's equation in a square (continued)

- In the previous example, the method of separation of variables worked because we were able to "separate" the original BVP involving two independent variables (x and y) in a useful way into subsidiary BVPs involving only one independent variable (either x or y).

  It is not in general possible to "separate" PDEs or BVPs into PDEs or BVPs with fewer independent variables. BVPs that can be "separated" into BVPs with fewer independent variables are typically BVPs defined on simple geometries, such as squares, disks, or cylinders.

  Thus, as a method for solving BVPs, the range of applicability of separation of variables is typically limited to BVPs defined on simple geometries, such as squares, disks, or cylinders.

- To show mathematically fully rigorously that the previous function series indeed solves the aforementioned Dirichlet problem, we still need to show that it converges in an appropriate sense, as well as that its second-order partial derivatives can be obtained by term-by-term differentiation.

  In the following, we will highlight how notions of convergence of series and function series from earlier courses in the engineering curriculum can be used to study these matters further.

Review of series and function series

This is not a lecture but rather a summary of key elements relevant to series and function series. For a more complete treatment of series and function series, please refer to MATH0002 "Analyse Mathématique" (E. Delhez) and MATH0007 "Analyse Mathématique II" (F. Bastin).

Series and convergence of series

■ With a sequence $\{a_k\}_{k=1}^{+\infty}$ of real scalars a_k, one can associate a **series** $\sum_{k=1}^{+\infty} a_k$.

If the sequence $\{s_k\}_{k=1}^{+\infty}$ of **partial sums** $s_\ell = \sum_{k=1}^{\ell} a_k$ converges to a limit s, then the series $\sum_{k=1}^{+\infty} a_k$ is said to converge to s, also called the **sum of the series**.

Please note that the sum of a series must be understood as the limit of a sequence of partial sums and is not obtained simply by addition.

■ The **root test** for the convergence of series asserts that

◆ if $\limsup_{k \to +\infty} \sqrt[k]{a_k} < 1$, then the series $\sum_{k=1}^{+\infty} a_k$ converges;

◆ if $\limsup_{k \to +\infty} \sqrt[k]{a_k} > 1$, then the series $\sum_{k=1}^{+\infty} a_k$ diverges;

◆ if $\limsup_{k \to +\infty} \sqrt[k]{a_k} = 1$, then the root test gives no information.

■ The **ratio test** for the convergence of series asserts that

◆ if $\limsup_{k \to +\infty} |a_{k+1}/a_k| < 1$, then the series $\sum_{k=1}^{+\infty} a_k$ converges;

◆ if $|a_{k+1}/a_k| \geq 1$ for all $k \geq \tilde{k}$, where $\tilde{k}$ is a fixed integer, then the series $\sum_{k=1}^{+\infty} a_k$ diverges.

■ Let f be a function from $[0, +\infty[$ into $\mathbb{R}$ that is positive and monotonically decreasing. Then, the **integral test** asserts that $\sum_{k=1}^{+\infty} f(k)$ converges if and only if the integral $\int_1^{+\infty} f(x)dx$ is finite.

Convergence of function series

- Let $\{f_k\}_{k=1}^{+\infty}$ be a sequence of functions f_k from a subset Ω of $\mathbb{R}^d$ into $\mathbb{R}$. Then, the function series $\sum_{k=1}^{+\infty} f_k$ is said to **converge pointwise** to a function s from Ω into $\mathbb{R}$ if the sequence of partial sums $s_\ell = \sum_{k=1}^{\ell} f_k$ convergence pointwise to s, that is,

$$\lim_{\ell \to +\infty} s_\ell(\boldsymbol{x}) = s(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Omega.$$

- Let $\{f_k\}_{k=1}^{+\infty}$ be a sequence of functions f_k from a subset Ω of $\mathbb{R}^d$ into $\mathbb{R}$. Then, the function series $\sum_{k=1}^{+\infty} f_k$ is said to **converge uniformly** to a function s from Ω into $\mathbb{R}$ if the sequence of partial sums $s_\ell = \sum_{k=1}^{\ell} f_k$ converges uniformly to s, that is,

$$\forall \epsilon > 0, \quad \exists \tilde{\ell} \in \mathbb{N} \quad : \quad |s_\ell(\boldsymbol{x}) - s(\boldsymbol{x})| \leq \epsilon, \quad \forall \boldsymbol{x} \in \Omega, \quad \ell \geq \tilde{\ell}.$$

Clearly, every uniformly convergent series is also pointwise convergent.

- Let $\{f_k\}_{k=1}^{+\infty}$ be a sequence of square-integrable functions from a subset Ω of $\mathbb{R}^d$ into $\mathbb{R}$. Then, the function series $\sum_{k=1}^{+\infty} f_k$ is said to **converge in the norm of the square-integrable functions** to a square-integrable function s from Ω into $\mathbb{R}$ if the sequence of partial sums $s_\ell = \sum_{k=1}^{\ell} f_k$ converges in the norm of the square-integrable functions to s, that is,

$$\lim_{\ell \to +\infty} \sqrt{\int_\Omega |s_\ell(\boldsymbol{x}) - s(\boldsymbol{x})|^2 d\boldsymbol{x}} = 0.$$

Review of series and function series

Cauchy criterion for uniform convergence of function series

■ Let $\{f_k\}_{k=1}^{+\infty}$ be a sequence of functions f_k from a subset Ω of $\mathbb{R}^d$ into $\mathbb{R}$. Then, $\sum_{k=1}^{+\infty} f_k$ converges uniformly if and only if the sequence of partial sums $s_\ell = \sum_{k=1}^{\ell} f_k$ is such that

$$\forall \epsilon > 0, \quad \exists \tilde{\ell} \in \mathbb{N} \quad : \quad |s_{\ell_1}(\boldsymbol{x}) - s_{\ell_2}(\boldsymbol{x})| \leq \epsilon, \quad \forall \boldsymbol{x} \in \Omega, \quad \ell_1, \ell_2 \geq \tilde{\ell}.$$

Weierstrass's m-test for uniform convergence of function series

■ For function series, there is a very convenient test for uniform convergence, due to Weierstrass.

■ Let $\{f_k\}_{k=1}^{+\infty}$ be a sequence of functions f_k from a subset Ω of $\mathbb{R}^d$ into $\mathbb{R}$ such that $|f_k(\boldsymbol{x})| \leq m_k$ for all $\boldsymbol{x}$ in Ω. Then, the **Weierstrass's m-test** asserts that the function series $\sum_{k=1}^{+\infty} f_k$ converges uniformly if the series $\sum_{k=1}^{+\infty} m_k$ converges.

Review of series and function series

Convergence, continuity, and differentiability

- A key problem which arises is to determine whether important properties of functions are preserved under limit operations. For example, if the functions f_k in a function series $\sum_{k=1}^{+\infty} f_k$ are continuous or differentiable, is the same true for the sum of the function series s (if it exists)?

  Results from earlier courses in the engineering curriculum include the following ones.

- Let $\{f_k\}_{k=1}^{+\infty}$ be a sequence of continuous functions from a subset Ω of $\mathbb{R}^d$ into $\mathbb{R}$. If the function series $\sum_{k=1}^{+\infty} f_k$ converges uniformly to s, then s is **continuous**.

- Let $\{f_k\}_{k=1}^{+\infty}$ be a sequence of differentiable functions from a closed interval $[a, b]$ of $\mathbb{R}$ into $\mathbb{R}$ and let the series $\sum_{k=1}^{+\infty} f_k(x_0)$ converge at least at a point x_0 in $[a, b]$. Then, we have that if the function series $\sum_{k=1}^{+\infty} \frac{df_k}{dx}$ converges uniformly, then the function series $\sum_{k=1}^{+\infty} f_k$ converges uniformly to a function s such that $\frac{ds}{dx} = \sum_{k=1}^{+\infty} \frac{df_k}{dx}$, that is, we can **differentiate term by term**.

Fourier series

- Let f be an a-periodic function from $\mathbb{R}$ into $\mathbb{C}$ whose restriction to the bounded open interval $]0, a[$ is integrable. Then, one can associate with f the following **Fourier series**:

$$\sum_{k=-\infty}^{+\infty} c_k \exp\left(ik\frac{2\pi}{a}x\right), \quad \text{where} \quad c_k = \frac{1}{a}\int_0^a f(x)\exp\left(-ik\frac{2\pi}{a}x\right)dx.$$

- Let f be an a-periodic function from $\mathbb{R}$ into $\mathbb{C}$ whose restriction to the bounded open interval $]0, a[$ is square-integrable. Then, the Fourier series associated with f **converges in the norm of the square-integrable functions** to f, that is,

$$\lim_{\ell\to+\infty} \sqrt{\int_0^a \left| f(x) - \sum_{k=-\ell}^{\ell} c_k \exp\left(ik\frac{2\pi}{a}x\right)\right|^2 dx} = 0.$$

- Let f be an a-periodic function from $\mathbb{R}$ into $\mathbb{C}$ whose restriction to the bounded open interval $]0, a[$ is integrable. If at a point x_0 in $]0, a[$, the left and right limits of f at x_0, as well as the left and right derivatives of f at x_0, exist, then the Fourier series associated with f **converges pointwise** at x_0,

$$\lim_{\ell\to+\infty} \sum_{k=-\ell}^{\ell} c_k \exp\left(ik\frac{2\pi}{a}x_0\right) = \frac{1}{2}\big(f(x_{0+}) + f(x_{0-})\big),$$

and therefore $\lim_{\ell\to+\infty} \sum_{k=-\ell}^{\ell} c_k \exp\left(ik\frac{2\pi}{a}x_0\right) = f(x_0)$ if f is continuous at x_0.

Fourier series (continued)

- Let f be a continuous a-periodic function from $\mathbb{R}$ into $\mathbb{C}$ whose restriction to the bounded closed interval $[0, a]$ is differentiable except perhaps at a finite number of points. Let $\frac{df}{dx}$ be piecewise continuous on $[0, a]$. Then, the Fourier series associated with f **converges uniformly** to f on $[0, a]$, and the Fourier series of $\frac{df}{dx}$ is obtained by **term-by-term differentiation** of that of f.

- Let $]0, a[$ be a bounded open interval, and let f be a square-integrable function from $]0, a[$ into $\mathbb{C}$. Then, one can associate with f the following **Fourier sine series**:

$$\sum_{k=1}^{+\infty} c_k \sin\left(k\frac{2\pi}{2a}x\right), \quad \text{where} \quad c_k = \frac{2}{a}\int_0^a f(x)\sin\left(k\frac{2\pi}{2a}x\right)dx.$$

This Fourier sine series converges to f in the norm of the square-integrable functions, that is,

$$\lim_{\ell \to +\infty}\sqrt{\int_0^a \left|f(x) - \sum_{k=1}^{\ell} c_k \sin\left(k\frac{2\pi}{2a}x\right)\right|^2 dx} = 0.$$

This result can be proved by extending f to $]-a, a[$ so as to obtain an odd function, then extending this function to $\mathbb{R}$ with period $2a$, and finally constructing a Fourier series with period $2a$.

Convergence and separation of variables

Convergence and separation of variables

Example: Laplace's equation in a square (continued)

- Previously, we obtained the following function series:

$$u(x,y) = \sum_{k=1}^{+\infty} \frac{f_k}{\sinh(k\pi)} \sinh\big(k\pi(1-x)\big) \sqrt{2}\sin(k\pi y) \quad \text{with} \quad f_k = \int_0^1 f(y)\sqrt{2}\sin(k\pi y)dy.$$

- Let f be an integrable function from $]0,1[$ into $\mathbb{R}$. Then, we can prove the following properties:

    - We have that $\sqrt{2}|\sin(k\pi x)| \le \sqrt{2}$. Further, because f is an integrable function from $]0,1[$ into $\mathbb{R}$, we have that $|f_k| \le \sqrt{2}\int_0^1 |f(y)|dy$. Finally, we have that

$$\left|\frac{\sinh\big(k\pi(1-x)\big)}{\sinh(k\pi)}\right| = \left|\frac{\exp\big(k\pi(1-x)\big) - \exp\big(-k\pi(1-x)\big)}{\exp(k\pi) - \exp(-k\pi)}\right|$$

$$= \left|\exp(-k\pi x)\frac{1 - \exp\big(-2k\pi(1-x)\big)}{1 - \exp(-2k\pi)}\right|$$

$$\le \exp(-k\pi x)\frac{1}{1 - \exp(-2\pi)}$$

Thus, the terms in the function series are dominated by a constant times $\exp(-k\pi x)$.

Example: Laplace's equation in a square (continued)

◆ By the **integral test**, $\sum_{k=1}^{+\infty} \exp(-k\pi x_0)$ converges for $x_0 > 0$. Therefore, by **Weierstrass's m-test**, the function series for $u(x, y)$ **converges uniformly** for $x \geq x_0 > 0$.

◆ Because the terms in the function series for $u(x, y)$ are continuous, $u(x, y)$ is itself **continuous** for $x \geq x_0 > 0$.

◆ Because the terms in the function series for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x^2}$, and $\frac{\partial^2 u}{\partial y^2}$ are dominated by a constant times $(k\pi)^2 \exp(-k\pi x)$, it follows similarly from the integral test and Weierstrass's m-test that these function series converge uniformly for $x \geq x_0 > 0$. It follows that these partial derivatives may be obtained by **term-by-term differentiation**.

◆ Because each term in the function series for $u(x, y)$ satisfies Laplace's equation, the same is true for $u(x, y)$ itself for $x \geq x_0 > 0$.

Example: Laplace's equation in a square (continued)

■ Let f, moreover, be a continuous function from $[0, 1]$ into $\mathbb{R}$ with $f(0) = f(1) = 0$, whose derivative $\frac{df}{dy}$ exists except perhaps at a finite number of points and is piecewise continuous on $[0, 1]$. Then, we can prove the following additional properties:

◆ The **Fourier sine series** of f converges to f uniformly, that is, by the **Cauchy criterion**,
$$\forall \epsilon > 0, \quad \exists \tilde{\ell} \in \mathbb{N} \quad : \quad \left| s_{\ell_1}(0, y) - s_{\ell_2}(0, y) \right| \leq \epsilon, \quad 0 \leq y \leq 1, \quad \ell_1, \ell_2 \geq \tilde{\ell}.$$
where $s_\ell(x, y) = \sum_{k=1}^{\ell} \frac{f_k}{\sinh(k\pi)} \sinh\left(k\pi(1 - x)\right) \sqrt{2} \sin(k\pi y)$.

Because each term in the function series for $u(x, y)$ satisfies Laplace's equation and vanishes on $y = 0$, $x = 1$, and $y = 1$, we have that $s_k - s_\ell$ satisfies Laplace's equation and vanishes on $y = 0$, $x = 1$, and $y = 1$.

Then, by the **maximum property of harmonic functions**, which we discussed in Lecture 2, $\left| s_k(0, y) - s_\ell(0, y) \right| \leq \epsilon$ everywhere in the domain. Thus, by the Cauchy criterion, the function series for $u(x, y)$ **converges uniformly** for $0 \leq x \leq 1$, $0 \leq y \leq 1$.

◆ Because each term in the function series for $u(x, y)$ is continuous, $u(x, y)$ is itself **continuous** for $0 \leq x \leq 1$, $0 \leq y \leq 1$. Putting $x = 0$, we find $u(0, y) = \sum_{k=1}^{+\infty} f_k \sqrt{2} \sin(k\pi y) = f(y)$.

Sturm-Liouville problem

■ Let $[a, b]$ be a bounded closed interval of $\mathbb{R}$. Let p be in $C^1([a, b])$ with $p(x) \geq \alpha > 0$ for $a \leq x \leq b$, let q be in $C(]a, b[)$, and let f be in $L^2(]a, b[)$. Then, the Dirichlet problem

$$\begin{cases} -\dfrac{d}{dx}\left(p\dfrac{du}{dx}\right) + qu = f & \text{in }]a, b[, \\[2mm] u(a) = u(b) = 0 & \text{at } x = a \text{ and } x = b, \end{cases}$$

is an example of a **Sturm-Liouville problem**.

■ Let $[a, b]$ be a bounded closed interval of $\mathbb{R}$. Let p be in $C^1([a, b])$ with $p(x) \geq \alpha > 0$ for $a \leq x \leq b$ and let q be in $C(]a, b[)$. Then, the eigenproblem

$$\begin{cases} -\dfrac{d}{dx}\left(p\dfrac{d\varphi_k}{dx}\right) + q\varphi_k = \lambda_k \varphi_k & \text{in }]a, b[, \\[2mm] \varphi_k(a) = \varphi_k(b) = 0 & \text{at } x = a \text{ and } x = b, \end{cases}$$

is an example of a **Sturm-Liouville spectral problem**.

■ In the preceding slides, we found that for $[a, b] = [0, 1]$, $p = 1$, and $q = 0$, the solution to this eigenproblem is a sequence of eigenvalues $\lambda_k = k^2\pi^2$ with corresponding eigenfunctions $\varphi_k = \sqrt{2}\sin(k\pi x)$, which form an orthonormal basis for $L^2(]0, 1[)$.

■ The study (well-posedness, completeness of the eigenfunctions,...) of similar eigenproblems (under different types of regularity properties imposed on p and q, under different types of boundary conditions,...) is the subject of **Sturm-Liouville theory**. The study of eigenproblems involving more general partial differential operators is part of the subject of **spectral theory**.

Summary and conclusion

- Separation of variables refers to a family of solution methods that share the property that a solution is sought in the form of a (series of) product(s) of functions of fewer independent variables.

- This (series of) product(s) of functions of fewer independent variables is often constructed by using eigenfunctions obtained by solving an eigenproblem. A particular advantage of eigenfunctions is that their use often transforms the problem under consideration into a system of subsidiary problems that involve fewer independent variables ("diagonalization").

- The method of separation of variables depends critically on the ability to "separate" the original PDE or BVP (harder to solve) in a useful way into subsidiary PDEs or BVPs involving fewer independent variable (easier to solve).

  It is not in general possible to "separate" PDEs or BVPs into PDEs or BVPs with fewer independent variables. BVPs that can be "separated" into BVPs with fewer independent variables are typically BVPs defined on simple geometries, such as squares, disks, or cylinders.

  Thus, as a method for solving BVPs, the range of applicability of separation of variables is typically limited to BVPs defined on simple geometries, such as squares, disks, or cylinders.

- To make results obtained by using the method of separation of variables mathematically fully rigorous, it is typically required to show that the function series converges in an appropriate sense and that this series can be differentiated by term-by-term differentiation. This is not optional!

Suggested reading material:

- F. Bastin. MATH0007 Analyse Mathématique II. ULg. Lecture notes.

- E. Delhez. MATH0002 Analyse Mathématique. ULg. Lecture notes.

- P. Olver. Introduction to Partial Differential Equations. Springer, 2014. Chapter 4.

Additional references also consulted to prepare this lecture:

- H. Brezis. Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2010.

- C. Gasquet and P. Witomski. Analyse de Fourier et applications. Masson, 1990.

- G. Golub and C. Van Loan. Matrix computations. The Johns Hopkins University Press, 1996.

- W. Rudin. Principles of mathematical analysis. McGraw-Hill, 1964.

- H. Weinberger. A first course in partial differential equations. Dover Publications, 1965.