MATH0024 – Modeling with PDEs

Laplace/Poisson equation

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Outline

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Laplace/Poisson equation

Laplace operator

$$\triangle_{\boldsymbol{x}} = \text{div}_{\boldsymbol{x}} \boldsymbol{\nabla}_{\boldsymbol{x}}.$$

- Cartesian coordinates ($m = 3$): $\triangle_{\boldsymbol{x}} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.
- Cylindrical coordinates ($m = 3$): $\triangle_{\boldsymbol{x}} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$.
- Spherical coordinates ($m = 3$): $\triangle_{\boldsymbol{x}} u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{1}{(r \sin(\chi))^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \chi^2} + \frac{1}{r^2 \tan(\chi)} \frac{\partial u}{\partial \chi}$.

Laplace and Poisson equations

- $$\begin{cases} \text{Laplace equation} \\ \triangle_{\boldsymbol{x}} u = 0. \end{cases} \qquad \begin{cases} \text{Poisson equation} \\ \triangle_{\boldsymbol{x}} u = f. \end{cases}$$

  The Laplace equation is called **homogeneous** because its right-hand side is zero. By contrast, if $f \neq 0$, the Poisson equation is called **inhomogeneous** because its right-hand side is nonzero.

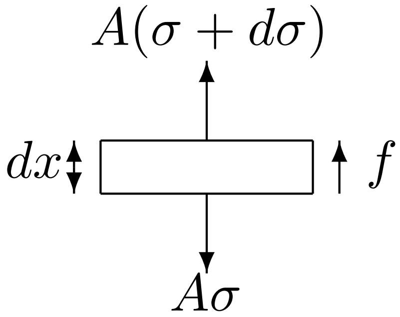
Areas of application in mechanics and physics

- The **Laplace/Poisson equation** is the mathematical prototype of **equilibrium problems**, such as those arising in stationary Darcy flow, stationary heat conduction, electrostatics, elasticity, . . .

1D elasticity

- System of PDEs governing the static behavior of a linear elastic bar:

$$\begin{cases} -A\dfrac{d\sigma}{dx} = f & \text{(equilibrium equation)} \\ \sigma = E\epsilon & \text{(constitutive equation)} \\ \epsilon = \dfrac{du}{dx} & \text{(strain-displacement relationship)} \end{cases}$$



Here, u is the displacement, ϵ the strain, σ the stress, A the cross section of the bar, E the Young's modulus, and f the external force per unit length.

- Inserting the constitutive equation into the equilibrium equation, we obtain

$$-EA\frac{d\epsilon}{dx} = f.$$

Combining this result with the strain-displacement relationship, we obtain

$$\boxed{-EA\tfrac{d^2u}{dx^2} = f.}$$

3D elasticity

- System of PDEs governing the static behavior of a homogeneous isotropic linear elastic solid:

$$\begin{cases} \mathbf{div}_{\boldsymbol{x}}\boldsymbol{\sigma} + \boldsymbol{f}_{\mathsf{v}} = \mathbf{0} & \text{(equilibrium equation)} \\ \boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon} & \text{(constitutive equation)} \\ \boldsymbol{\epsilon} = \dfrac{1}{2}\left(\mathbf{D}_{\boldsymbol{x}}\boldsymbol{u} + \mathbf{D}_{\boldsymbol{x}}\boldsymbol{u}^{\mathrm{T}}\right) & \text{(strain-displacement relationship)} \end{cases}, \quad \begin{cases} \boldsymbol{u}\text{: displacement,} \\ \boldsymbol{\epsilon}\text{: strain,} \\ \boldsymbol{\sigma}\text{: stress,} \\ \boldsymbol{f}_{\mathsf{v}}\text{: volume force.} \end{cases}$$

  Here, λ and μ are the Lamé parameters, which are related to the Young's modulus E and Poisson coefficient ν through $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$.

- Inserting the constitutive equation into the equilibrium equation, we obtain

$$\mathbf{div}_{\boldsymbol{x}}\left(\lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}\right) + \boldsymbol{f}_{\mathsf{v}} = \mathbf{0}.$$

  Using the properties $\mathbf{div}_{\boldsymbol{x}}(\varphi \boldsymbol{A}) = \boldsymbol{A}(\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi) + \varphi \mathbf{div}_{\boldsymbol{x}}\boldsymbol{A}$ and $\boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \mathbf{div}_{\boldsymbol{x}}(\mathbf{D}_{\boldsymbol{x}}\boldsymbol{a}^{\mathrm{T}})$ and the definition $\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \mathrm{tr}(\mathbf{D}_{\boldsymbol{x}}\boldsymbol{a})$ (Lecture 1 Part B), we obtain

$$\mathbf{div}_{\boldsymbol{x}}\left(\mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I}\right) = \boldsymbol{\nabla}_{\boldsymbol{x}}\left(\mathrm{tr}(\boldsymbol{\epsilon})\right) = \boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{u} \quad \text{and} \quad \mathbf{div}_{\boldsymbol{x}}\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{div}_{\boldsymbol{x}}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{u}\right).$$

  Combining these results, we obtain

$$\boxed{-(\lambda + \mu)\boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{u} - \mu \mathbf{div}_{\boldsymbol{x}}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{u} = \boldsymbol{f}_{\mathsf{v}}.}$$

Stationary heat conduction with heat source

- For an open bounded subset Ω of $\mathbb{R}^3$ with a sufficiently smooth boundary $\partial\Omega$, the stationary conservation of energy reads as follows:

$$-\int_{\partial\Omega} \boldsymbol{q} \cdot d\boldsymbol{S} + \int_{\Omega} r dV = 0, \qquad \begin{cases} r\text{: heat source } \big([r] = \text{J m}^{-3}\,\text{s}^{-1}\big), \\ \boldsymbol{q}\text{: heat flux } \big([\boldsymbol{q}] = \text{J m}^{-2}\,\text{s}^{-1}\big). \end{cases}$$

  Owing to Stokes's theorem (Lecture 1 Part B), we have, $\int_{\partial\Omega} \boldsymbol{q} \cdot d\boldsymbol{S} = \int_{\Omega} \text{div}_{\boldsymbol{x}} \boldsymbol{q} dV$, hence,

$$-\text{div}_{\boldsymbol{x}} \boldsymbol{q} + r = 0.$$

- Fourier's law for heat conduction indicates that $\boldsymbol{q}$ is proportional to the gradient $\boldsymbol{\nabla}_{\boldsymbol{x}} T$ but points oppositely because the heat flux is from regions of higher to regions of lower temperature:

$$\boldsymbol{q} = -k \boldsymbol{\nabla}_{\boldsymbol{x}} T, \qquad k\text{: the thermal conductivity } \big([k] = \text{J m}^{-1}\,\text{K}^{-1}\,\text{s}^{-1}\big).$$

- Combining the aforementioned results, we obtain

$$\boxed{-\text{div}_{\boldsymbol{x}}(k \boldsymbol{\nabla}_{\boldsymbol{x}} T) = r.}$$

Laplace/Poisson equation on all of space

Motivating example: superposition formula for solution to linear problem

- Let us consider the linear problem

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Let us assume that this linear problem is well posed in that $[A]$ is invertible.

- The solution is obtained as $\boldsymbol{x} = [A]^{-1}\boldsymbol{y}$. Alternatively, after solving the linear problems

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

the solution is also obtained as follows:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = y_1 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + y_2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + y_3 \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

Thus, we obtain a **representation** of the solution $\boldsymbol{x}$ to $[A]\boldsymbol{x} = \boldsymbol{y}$ **as a superposition** of solutions $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ to the linear problems $[A]\boldsymbol{u} = \boldsymbol{i}_1$, $[A]\boldsymbol{v} = \boldsymbol{i}_2$, and $[A]\boldsymbol{w} = \boldsymbol{i}_3$, where $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, and $\boldsymbol{i}_3$ are unit vectors with 1 on the first row, the second row, and the third row, respectively.

Notion of fundamental solution

Notion of fundamental solution of PDE

- For **linear ODEs**, the notion of **impulse response function** allows a superposition formula to be established for determining a solution for a general right-hand side (cfr. L 1 Part B).

- For **linear PDEs**, a similar notion, although it is called **fundamental solution** in the case of PDEs, is introduced to allow a superposition formula to be established for determining a solution for a general right-hand side.

Superposition formula for solution to Poisson equation

- Let us consider the Poisson equation

$$\triangle_{\boldsymbol{x}} u = f, \qquad \boldsymbol{x} \in \mathbb{R}^m,$$

- After determining a fundamental solution E, that is, a solution that solves the Poisson equation for a Dirac impulse δ centered at $\mathbf{0}$ on the right-hand side,

$$\triangle_{\boldsymbol{x}} E = \delta, \qquad \boldsymbol{x} \in \mathbb{R}^m,$$

  a solution to the Poisson equation with general right-hand side f is obtained as follows:

$$u = E \star f.$$

  Thus, the fundamental solution allows a **superposition formula** to be established for determining a solution for a general right-hand side.

Notion of fundamental solution

Superposition formula for solution to Poisson equation (continued)

- As in the case of the solution of linear ODEs, one way of justifying this superposition formula is through Fourier analysis. In fact, with the Fourier transform, which, for an integrable or square-integrable function f from $\mathbb{R}^m$ into $\mathbb{R}$, would read as

$$\hat{f}(\boldsymbol{\xi}) = \mathcal{F}f(\boldsymbol{\xi}) = \int_{\mathbb{R}^m} \exp(-i\boldsymbol{\xi}\cdot\boldsymbol{x})f(\boldsymbol{x})d\boldsymbol{x},$$

where we call $\boldsymbol{\xi}$ the wavenumber if $\boldsymbol{x}$ is a spatial position, we have

Poisson equation $\triangle_{\boldsymbol{x}} u = f$ —FT→ **Algebraic equation** $-\|\boldsymbol{\xi}\|^2 \hat{u}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})$

$$\widehat{E}(\boldsymbol{\xi}) = \frac{-1}{\|\boldsymbol{\xi}\|^2}$$

Fundamental solution $\triangle_{\boldsymbol{x}} E = \delta$ ←inv FT— $-\|\boldsymbol{\xi}\|^2 \widehat{E}(\boldsymbol{\xi}) = 1$

Spatial-domain solution $u = E \star f$ ←inv FT— **Wavenumber-domain solution** $\hat{u}(\boldsymbol{\xi}) = \widehat{E}(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})$

Superposition formula for solution to Poisson equation (continued)

- However, mathematically, it is not so easy to make the previous justification fully rigorous:
  - In 1D ($m = 1$), the function $\frac{-1}{|\xi|^2}$ is not locally (square-)integrable near $\xi = 0$.
  - In 2D ($m = 2$), the function $\frac{-1}{\|\boldsymbol{\xi}\|^2}$ is not locally (square-)integrable near $\boldsymbol{\xi} = \mathbf{0}$.
  - In 3D ($m = 3$), the function $\frac{-1}{\|\boldsymbol{\xi}\|^2}$ is locally near $\boldsymbol{\xi} = \mathbf{0}$, but not globally, (square-)integrable.

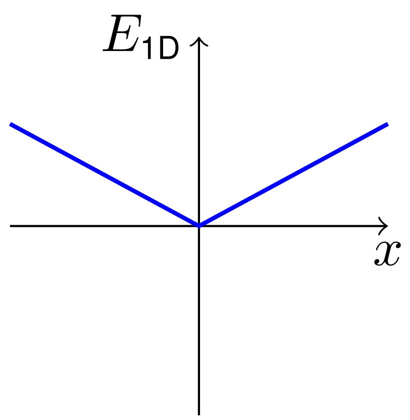
- Mathematicians have introduced a generalized notion of Fourier transform, in addition to regularization techniques, which allows the previous justification to be made fully rigorous. However, this generalized notion of Fourier transform is outside the scope of this theoretical lecture.

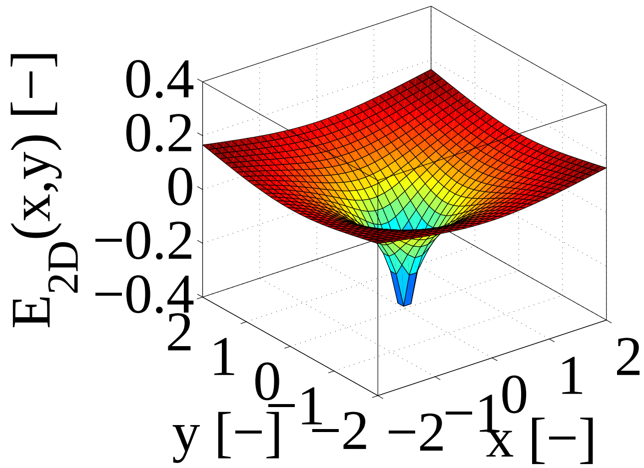
- In the following, we will limit ourselves to looking at expressions of fundamental solutions in 1D, 2D, and 3D and to providing some intuition for the particular form that these expressions take.

Fundamental solution in 1D, 2D, and 3D

Expressions for fundamental solutions in 1D, 2D, and 3D

- In 1D ($m = 1$), the function $E_{\text{1D}}(x) = \frac{1}{2}\,|x|$ is a fundamental solution.
- In 2D ($m = 2$), the function $E_{\text{2D}}(\boldsymbol{x}) = \frac{1}{2\pi}\log(\|\boldsymbol{x}\|)$ is a fundamental solution.
- In 3D ($m = 3$), the function $E_{\text{3D}}(\boldsymbol{x}) = \frac{-1}{4\pi}\frac{1}{\|\boldsymbol{x}\|}$ is a fundamental solution.





- We can observe that in 1D ($m = 1$), the first derivative of the fundamental solution E_{1D} is discontinuous at the origin. Further, in 2D ($m = 2$) and 3D ($m = 3$), the fundamental solutions E_{2D} and E_{3D} are singular at the origin. Consequently, these fundamental solutions are not classical solutions! Instead, these **fundamental solutions are generalized solutions**.

Dirac impulse and theory of distributions

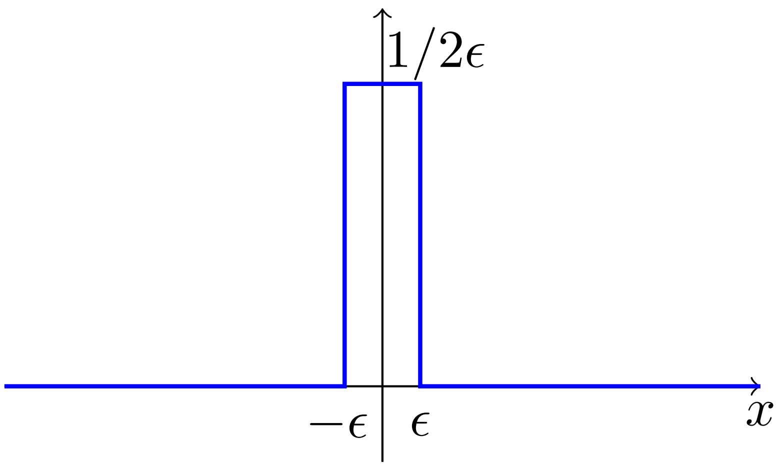
- To clarify what is meant by "generalized solution," we must take a closer look at the Dirac impulse.
- The **Dirac impulse** δ is not a function in the usual sense but rather an idealization of a sharply peaked function Φ_ϵ that is nonzero only on an interval $]-\epsilon, \epsilon[$ near the origin and has the property

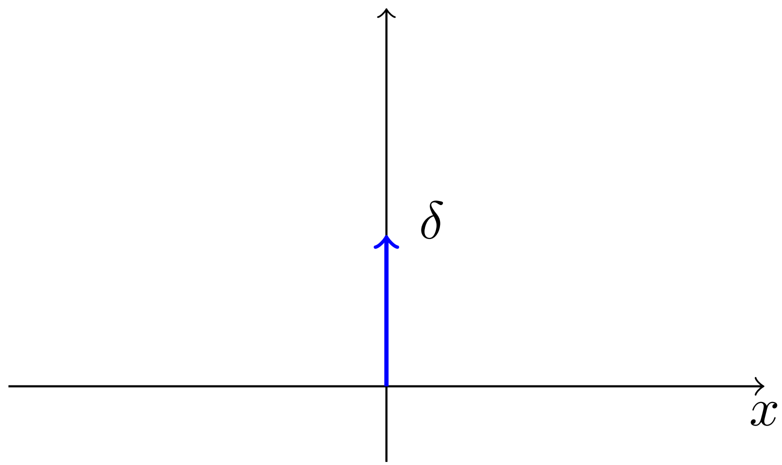
$$\int_{-\infty}^{\infty} \Phi_\epsilon(x)dx = \int_{-\epsilon}^{\epsilon} \Phi_\epsilon(x)dx = 1.$$

For example, we may consider:

$$\Phi_\epsilon(x) = \begin{cases} 1/2\epsilon & \text{if } -\epsilon \leq x \leq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

- The exact shape of Φ_ϵ is not important, but Φ_ϵ must attain a height that is $O(1/\epsilon)$ in order for the integral to be 1. We can think of the Dirac impulse as a sort of limit of such functions as $\epsilon \to 0$:





Fundamental solution in 1D, 2D, and 3D

Dirac impulse and theory of distributions (continued)

- Calculations (summation, multiplication by a function, convolution, differentiation, Fourier transform, ...) using Dirac impulses (and other generalized functions) can be made rigorous mathematically by using the theory of distributions, of which we need to know only the following:
  - The theory of distributions provides mathematically rigorous definitions of generalized functions. For example, the Dirac impulse is defined as follows ($m = 1$):

    $$\int_{\mathbb{R}} \delta(x)\varphi(x)dx = \varphi(0)$$

    for all smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with closed and bounded support.
  - The theory of distributions makes it possible to differentiate certain functions whose derivatives need not exist in the usual sense, as well as to take the Fourier transform of certain functions whose Fourier transform need not exist in the usual sense. For example, differentiation is defined using the formula of partial integration as follows ($m = 1$):

    $$\int_{\mathbb{R}} T'(x)\varphi(x)dx = -\int_{\mathbb{R}} T(x)\frac{d\varphi}{dx}(x)dx$$

    for all smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with closed and bounded support.
- The theory of distributions provides a modern framework to define generalized solutions to PDEs that need not admit solutions in the classical sense.

Fundamental solution in 1D, 2D, and 3D

Justification of expression for fundamental solution in 1D

- To justify that in 1D ($m = 1$), the function $E_{1D}(x) = \frac{1}{2}|x|$ is a fundamental solution, we must show

$$\int_{\mathbb{R}} E''_{1D}(x)\varphi(x)dx = \int_{\mathbb{R}} \delta(x)\varphi(x)dx = \varphi(0)$$

for all smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with closed and bounded support.

- For the first derivative of E_{1D} in the sense of the distributions, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} E'_{1D}(x)\varphi(x)dx &= -\int_{\mathbb{R}} E_{1D}(x)\frac{d\varphi}{dx}(x)dx \qquad \text{(definition)} \\
&= -\int_{-\infty}^{0} E_{1D}(x)\frac{d\varphi}{dx}(x)dx - \int_{0}^{+\infty} E_{1D}(x)\frac{d\varphi}{dx}(x)dx \\
&= \int_{-\infty}^{0} \frac{dE_{1D}}{dx}(x)\varphi(x)dx - [E_{1D}(x)\varphi(x)]_{-\infty}^{0} + \int_{0}^{+\infty} \frac{dE_{1D}}{dx}(x)\varphi(x)dx - [E_{1D}(x)\varphi(x)]_{0}^{+\infty} \\
&= \int_{\mathbb{R}} \frac{dE_{1D}}{dx}(x)\varphi(x)dx.
\end{aligned}$$

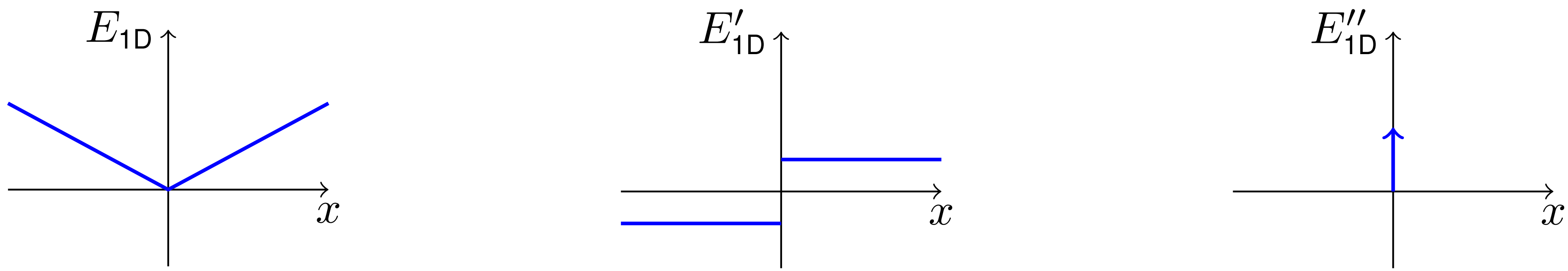
In the partial integration going from the second to the third line, the boundary terms at infinity vanish since φ has bounded support and those at 0 vanish since $E_{1D}(0) = 0$. For the continuous function E_{1D}, the derivative in the sense of the distributions coincides with the derivative in the usual sense, namely, $E'_{1D}(x) = \partial_x E_{1D}(x)$ is equal to -1/2 if $x < 0$ and 1/2 if $x > 0$.

Justification of expression for fundamental solution in 1D (continued)

- For the second derivative of E_{1D} in the sense of the distributions, we obtain

$$\begin{aligned}\int_{\mathbb{R}} E''_{\text{1D}}(x)\varphi(x)dx &= -\int_{\mathbb{R}} E'_{\text{1D}}(x)\frac{d\varphi}{dx}(x)dx && \text{(definition)}\\ &= -\int_{-\infty}^{0}\frac{1}{2}(-1)\frac{d\varphi}{dx}(x)dx - \int_{0}^{\infty}\frac{1}{2}(1)\frac{d\varphi}{dx}(x)dx\\ &= \frac{1}{2}\varphi(0)+\frac{1}{2}\varphi(0) && \text{(partial integration)}\\ &= \int_{\mathbb{R}}\delta(x)\varphi(x)dx\end{aligned}$$

For the discontinuous function E'_{1D}, the derivative in the sense of the distributions is different from the derivative in the usual sense, namely, $E''_{\text{1D}}(x) = \delta(x)$ versus $\partial_x^2 E_{\text{1D}}(x) = 0$ if $x < 0$ and $x > 0$. When a discontinuous function is differentiated in the sense of the distributions, a Dirac impulse appears at the discontinuity.



Fundamental solution in 1D, 2D, and 3D

Justification of expressions for fundamental solutions in 2D and 3D

- By using the theory of distributions in a similar fashion, it can be shown that E_{2D} and E_{3D} are fundamental solutions to the Poisson equation in 2D ($m = 2$) and 3D ($m = 3$).
  However, these proofs are slightly complicated because of the singularity at $\mathbf{0}$. Here, we limit ourselves to providing some intuition for the form of these fundamental solutions.

- Let $r = \|\boldsymbol{x}\|$. Because of radial symmetry, it follows from the expressions of the Laplacian operator in cylindrical and spherical coordinates, respectively, that the fundamental solutions must satisfy
  - in 2D ($m = 2$): $\frac{d^2 E_{\text{2D}}}{dr^2} + \frac{1}{r}\frac{dE_{\text{2D}}}{dr} = 0$, hence, $\frac{\partial_r^2 E_{\text{2D}}}{\partial_r E_{\text{2D}}} = \frac{-1}{r}$, for $\boldsymbol{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$.
  - in 3D ($m = 3$): $\frac{d^2 E_{\text{3D}}}{dr^2} + \frac{2}{r}\frac{dE_{\text{3D}}}{dr} = 0$, hence, $\frac{\partial_r^2 E_{\text{3D}}}{\partial_r E_{\text{3D}}} = \frac{-2}{r}$, for $\boldsymbol{x} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$.

- By integrating, we obtain
  - in 2D ($m = 2$): $\log(\partial_r E_{\text{2D}}) = -1\log(r) + \log(c)$, hence, $\partial_r E_{\text{2D}} = c\, r^{-1}$, for $\boldsymbol{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$.
  - in 3D ($m = 3$): $\log(\partial_r E_{\text{3D}}) = -2\log(r) + \log(c)$, hence, $\partial_r E_{\text{3D}} = c\, r^{-2}$, for $\boldsymbol{x} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$.

- Finally, by integrating once more, we obtain
  - in 2D ($m = 2$): $E_{\text{2D}}(r) = a + b\log(r)$ for $\boldsymbol{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$.
  - in 3D ($m = 3$): $E_{\text{3D}}(r) = a + b\, r^{-1}$ for $\boldsymbol{x} \in \mathbb{R}^3 \backslash \{\mathbf{0}\}$.

Fundamental solution in 1D, 2D, and 3D

Physical example relevant to 3D electrostatics

- Coulomb's law for the force on a point charge q at location $\boldsymbol{x}$ due to a point charge $\tilde{q}$ at location $\tilde{\boldsymbol{x}}$:
$$\boldsymbol{f} = \frac{1}{4\pi\epsilon_0} q\, \tilde{q}\, \frac{\boldsymbol{x} - \tilde{\boldsymbol{x}}}{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|^3}.$$

- Coulomb's law for the electric field at location $\boldsymbol{x}$ due to a charge density ρ over $\mathbb{R}^3$:
$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho(\tilde{\boldsymbol{x}}) \frac{\boldsymbol{x} - \tilde{\boldsymbol{x}}}{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|^3} dV_{\tilde{\boldsymbol{x}}}.$$

- Because
$$\frac{\boldsymbol{x} - \tilde{\boldsymbol{x}}}{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|^3} = -\boldsymbol{\nabla}_{\boldsymbol{x}} \left(\frac{1}{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|} \right),$$
we obtain, with the definition of the electrical potential $\boldsymbol{E} = -\boldsymbol{\nabla}_{\boldsymbol{x}} \Phi$, that
$$\Phi(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho(\tilde{\boldsymbol{x}}) \frac{1}{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|} dV_{\tilde{\boldsymbol{x}}}.$$

- In a manner that is consistent with Gauss's law from electrostatics,
$$\triangle_{\boldsymbol{x}} \Phi = -\rho/\epsilon_0, \quad \boldsymbol{x} \in \mathbb{R}^3,$$
we can understand this electrical potential as a convolution between E_{3D} and $-\rho/\epsilon_0$, that is,
$$\boxed{\Phi = E_{\text{3D}} \star (-\rho/\epsilon_0).}$$

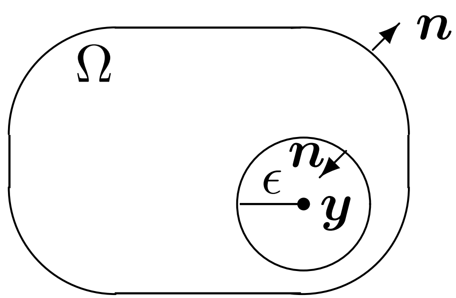
Integral representation theorem in 3D

- Let Ω be an open bounded subset of $\mathbb{R}^3$ with a sufficiently smooth boundary.
  Let u be a sufficiently regular function on Ω (specifically, let u be in $C^2(\overline{\Omega})$).
  Then, at any location $\boldsymbol{y}$ in the interior of Ω, we have the representation

$$u(\boldsymbol{y}) = \int_{\partial\Omega} \Big(u(\boldsymbol{x}) \boldsymbol{\nabla}_{\boldsymbol{x}} E_{\text{3D}}(\boldsymbol{x}-\boldsymbol{y}) - E_{\text{3D}}(\boldsymbol{x}-\boldsymbol{y}) \boldsymbol{\nabla}_{\boldsymbol{x}} u(\boldsymbol{x}) \Big) \cdot d\boldsymbol{S}_{\boldsymbol{x}} + \int_{\Omega} E_{\text{3D}}(\boldsymbol{x}-\boldsymbol{y}) \triangle_{\boldsymbol{x}} u(\boldsymbol{x}) dV_{\boldsymbol{x}}.$$

Proof of integral representation theorem in 3D

- The challenge is the singularity of E_{3D} at $\mathbf{0}$. This singularity is handled by excising a small ball around the singularity and then considering a limit as this ball becomes smaller and smaller.



- The proof proceeds by applying the second Green's identity (Lecture 1 Part B) with $\varphi = u$ and $\psi = E_{\text{3D}}(\cdot - \boldsymbol{y})$ in $\Omega \backslash \overline{B_\epsilon(\boldsymbol{y})}$, where $\overline{B_\epsilon(\boldsymbol{y})} = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x} - \boldsymbol{y}\| \leq \epsilon\}$ is the closed ball centered at $\boldsymbol{y}$ with radius ϵ,

$$\int_{\Omega \backslash \overline{B_\epsilon(\boldsymbol{y})}} E_{\text{3D}}(\boldsymbol{x}-\boldsymbol{y}) \triangle_{\boldsymbol{x}} u(\boldsymbol{x}) dV_{\boldsymbol{x}} = \int_{\partial\Omega \cup \partial B_\epsilon(\boldsymbol{y})} \Big(E_{\text{3D}}(\boldsymbol{x}-\boldsymbol{y}) \boldsymbol{\nabla}_{\boldsymbol{x}} u(\boldsymbol{x}) - u(\boldsymbol{x}) \boldsymbol{\nabla}_{\boldsymbol{x}} E_{\text{3D}}(\boldsymbol{x}-\boldsymbol{y}) \Big) \cdot d\boldsymbol{S}_{\boldsymbol{x}}.$$

Proof of integral representation theorem in 3D (continued)

- We excise this small ball around the singularity because the second Green's identity does not hold for functions with singularities. Further, we require the boundary of Ω to be sufficiently smooth because the second Green's identity does not hold for domains with nonsmooth boundaries.

- Let $\boldsymbol{n}(\boldsymbol{x})$ denote the unit outward normal vector to the surface $\partial\Omega \cup \partial B_\epsilon(\boldsymbol{y})$ at $\boldsymbol{x}$. Because u is in $C^2(\overline{\Omega})$, we have that $\boldsymbol{\nabla}_{\boldsymbol{x}} u(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})$ is bounded. Further, the area of the surface $\partial B_\epsilon(\boldsymbol{y})$ is $4\pi\epsilon^2$. Finally, $E_{\text{3D}}(\boldsymbol{x} - \boldsymbol{y})$ is equal to $-1/(4\pi\epsilon)$ on the surface $\partial B_\epsilon(\boldsymbol{y})$. It follows that

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\boldsymbol{y})} E_{\text{3D}}(\boldsymbol{x} - \boldsymbol{y}) \boldsymbol{\nabla}_{\boldsymbol{x}} u(\boldsymbol{x}) \cdot dS_{\boldsymbol{x}} = 0.$$

- We have that $\boldsymbol{\nabla}_{\boldsymbol{x}} E_{\text{3D}}(\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{n}(\boldsymbol{x}) = -1/(4\pi\epsilon^2)$ on the surface $\partial B_\epsilon(\boldsymbol{y})$. It follows that

$$\lim_{\epsilon \to 0} \int_{\partial B_\epsilon(\boldsymbol{y})} u(\boldsymbol{x}) \boldsymbol{\nabla}_{\boldsymbol{x}} E_{\text{3D}}(\boldsymbol{x} - \boldsymbol{y}) \cdot d\boldsymbol{S}_{\boldsymbol{x}} = \lim_{\epsilon \to 0} -\frac{1}{4\pi\epsilon^2} \int_{\partial B_\epsilon(\boldsymbol{y})} u(\boldsymbol{x}) dS_{\boldsymbol{x}} = -u(\boldsymbol{y}).$$

- Since u is in $C^2(\overline{\Omega})$, $\triangle_{\boldsymbol{x}} u(\boldsymbol{x})$ is bounded. Further, $E_{\text{3D}}(\cdot - \boldsymbol{y})$ is locally integrable near $\boldsymbol{y}$. Thus,

$$\lim_{\epsilon \to 0} \int_{\Omega \setminus \overline{B_\epsilon(\boldsymbol{y})}} E_{\text{3D}}(\boldsymbol{x} - \boldsymbol{y}) \triangle_{\boldsymbol{x}} u(\boldsymbol{x}) dV_{\boldsymbol{x}} = \int_{\Omega} E_{\text{3D}}(\boldsymbol{x} - \boldsymbol{y}) \triangle_{\boldsymbol{x}} u(\boldsymbol{x}) dV_{\boldsymbol{x}}.$$

- Finally, by letting $\epsilon \to 0$ in the aforementioned Green's identity and combining all the aforementioned results, we find the integral representation theorem, as asserted.

Properties of harmonic functions

Notion of harmonic function

- Let Ω be an open bounded subset of $\mathbb{R}^m$ with a sufficiently smooth boundary. Then, a sufficiently regular function u on Ω is **harmonic** if it satisfies the Laplace equation $\triangle_{\boldsymbol{x}} u = 0$ in Ω.

- The notion of fundamental solution and the integral representation theorem allow certain properties of harmonic functions to be proved, some of which are listed below.

Mean-value property in 3D

- For a harmonic function u within an open bounded subset Ω of $\mathbb{R}^3$, we have, at any location $\boldsymbol{y}$ in the interior of Ω and for any radius $r > 0$ small enough so that $\overline{B_r(\boldsymbol{y})}$ is included in Ω, the representation

$$u(\boldsymbol{y}) = \frac{1}{4\pi r^2} \int_{\partial B_r(\boldsymbol{y})} u(\boldsymbol{x}) dS_{\boldsymbol{x}}.$$

- Thus, the mean-value property in 3D asserts that the value taken by u at $\boldsymbol{y}$ is equal to the average of u over a sphere $\partial B_r(\boldsymbol{y})$ around $\boldsymbol{y}$.

- The proof is outside the scope of this theoretical lecture.

Properties of harmonic functions

Smoothness property (Laplacian is "regularizing")

- For a harmonic function u within Ω, all derivatives exist and are continuous.
- The interesting point is that the algebraic structure of the Laplace equation leads to the existence and continuity of all derivatives of a harmonic function, even those that do not appear in the PDE.
- The proof is outside the scope of this theoretical lecture.

Maximum property

- Let Ω be a connected open bounded subset of $\mathbb{R}^m$ with a sufficiently smooth boundary. For a harmonic function u within Ω, we have that if $\max_{\boldsymbol{x}\in\Omega} u(\boldsymbol{x}) = a < +\infty$, then either $u(\boldsymbol{x}) < a$ for all $\boldsymbol{x}$ in Ω or $u(\boldsymbol{x}) = a$ for all $\boldsymbol{x}$ in Ω.
- Thus, the maximum property asserts that a harmonic function must attain its maximum on the boundary and cannot attain its maximum in the interior of a connected set unless it is constant.
- For example, for a stationary heat conduction problem, this is physically obvious: the steady-state temperature won't exceed what is imposed at boundaries if there is no heat source.
- The proof is outside the scope of this theoretical lecture.

Boundary value problems involving Laplace/Poisson equation

Dirichlet and Neumann problems

Notion of boundary value problem

- So far, we assumed that our PDE was posed on the whole of space. We can also consider a PDE posed on a subregion of space. Then, the PDE is typically completed by boundary conditions specified on the boundary of this subregion of space, thus obtaining a **boundary value problem**.

Notion of "well-posed" problem

- A problem involving a PDE is called **"well-posed"** if

  (i) the problem has a solution,
  (ii) this solution is unique,
  (iii) the solution depends continuously on the data given in the problem.

- The last condition is particularly important in applications: it is desirable that the solution changes only a little when the conditions specifying the problem change only a little.

- The choice of boundary conditions in a boundary value problem (BVP) is very important. Finding which are good boundary conditions, that is, those that lead to a well-posed problem, is an important aspect of the mathematical theory of PDEs.

Dirichlet problem

- Let Ω be an open subset of $\mathbb{R}^m$. For given functions f on Ω and g on $\partial\Omega$, the **Dirichlet problem** is the BVP that consists in finding a function u on $\overline{\Omega}$ satisfying
$$\begin{cases} \triangle_{\boldsymbol{x}} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$
- The interpretation of the **Dirichlet boundary condition** $u = g$ on $\partial\Omega$ depends on the application. E.g., in stationary heat conduction, it would impose the temperature on the boundary. And, in electrostatics, it would impose the electrical potential on the boundary.

Neumann problem

- Let Ω be an open subset of $\mathbb{R}^m$. For given functions f on Ω and g on $\partial\Omega$, the **Neumann problem** is the BVP that consists in finding a function u on $\overline{\Omega}$ satisfying
$$\begin{cases} \triangle_{\boldsymbol{x}} u = f & \text{in } \Omega, \\ \boldsymbol{\nabla}_{\boldsymbol{x}} u \cdot \boldsymbol{n} = g & \text{on } \partial\Omega. \end{cases}$$
- The interpretation of the **Neumann boundary condition** $\boldsymbol{\nabla}_{\boldsymbol{x}} u \cdot \boldsymbol{n} = g$ on $\partial\Omega$ depends on the application. E.g., in stationary heat conduction, it imposes the heat flux through the boundary.

Mixed problem

- Let Ω be an open subset of $\mathbb{R}^m$. For given functions f on Ω, g on $\partial\Omega_{\mathsf{D}}$, and h on $\partial\Omega_{\mathsf{N}}$, where $\partial\Omega_{\mathsf{D}}$ and $\partial\Omega_{\mathsf{N}}$ form a partitioning of the boundary $\partial\Omega$ such that $\partial\Omega_{\mathsf{D}} \cup \partial\Omega_{\mathsf{N}} = \partial\Omega$ and $\partial\Omega_{\mathsf{D}} \cap \partial\Omega_{\mathsf{N}} = \emptyset$, the **mixed problem** is the BVP that consists in finding a function u on $\overline{\Omega}$ satisfying
$$\begin{cases} \triangle_{\boldsymbol{x}} u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega_{\mathsf{D}}, \\ \boldsymbol{\nabla}_{\boldsymbol{x}} u \cdot \boldsymbol{n} = h & \text{on } \partial\Omega_{\mathsf{N}}. \end{cases}$$

Uniqueness of solution to Dirichlet problem

- Let Ω be an open bounded subset of $\mathbb{R}^m$ with sufficiently smooth boundary. Let f and g be sufficiently regular functions on Ω and $\partial\Omega$. Then, there is at most one solution to Dirichlet's problem.
- The proof follows immediately from the aforementioned maximum property. In fact, if u and v are two solutions to the Dirichlet problem, then the function $w = \pm(u - v)$ satisfies $\triangle_{\boldsymbol{x}} w = 0$ in Ω and $w = 0$ on $\partial\Omega$; hence, by the maximum property, $w = 0$ in $\overline{\Omega}$, so that $u = v$.
- It follows that Dirichlet and Neumann boundary conditions cannot in general be imposed simultaneously everywhere on the boundary. In fact, if Dirichlet and Neumann boundary conditions are imposed simultaneously everywhere on the boundary, they will in general be incompatible, so that the resulting BVP will not have a solution.

Green's function

- Green's function is for a BVP on a bounded subset of space the analogue of a fundamental solution for a PDE on all of space.
- Because a BVP on a bounded domain is considered, translation invariance is lost; hence, it is no longer sufficient to determine only the response to a Dirac impulse centered at the origin.
- Let Ω be an open bounded subset of $\mathbb{R}^m$ with sufficiently smooth boundary. The **Green's function** is the function G on $\overline{\Omega} \times \Omega$ such that

$$\begin{cases} \triangle_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}) & \text{in } \Omega, \\ G(\boldsymbol{x}, \boldsymbol{y}) = 0 & \text{on } \partial\Omega, \end{cases}$$

  where $\delta(\cdot - \boldsymbol{y})$ is the Dirac impulse centered at $\boldsymbol{y}$.

Integral representation theorem using Green's function

- Let Ω be an open bounded subset of $\mathbb{R}^m$ with sufficiently smooth boundary. Let u be a sufficiently regular function on Ω. Then, at any location $\boldsymbol{y}$ in the interior of Ω, we have the representation

$$u(\boldsymbol{y}) = \int_{\partial\Omega} u(\boldsymbol{x}) \boldsymbol{\nabla}_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) \cdot d\boldsymbol{S}_{\boldsymbol{x}} + \int_{\Omega} G(\boldsymbol{x}, \boldsymbol{y}) \triangle_{\boldsymbol{x}} u(\boldsymbol{x}) dV_{\boldsymbol{x}}.$$

- The proof is omitted. Instead, let us build some intuition using an example in 1D.

Example of Green's function in 1D

- The **Green's function** for the open interval $]0,1[$ is the function G on $[0,1]\times]0,1[$ such that

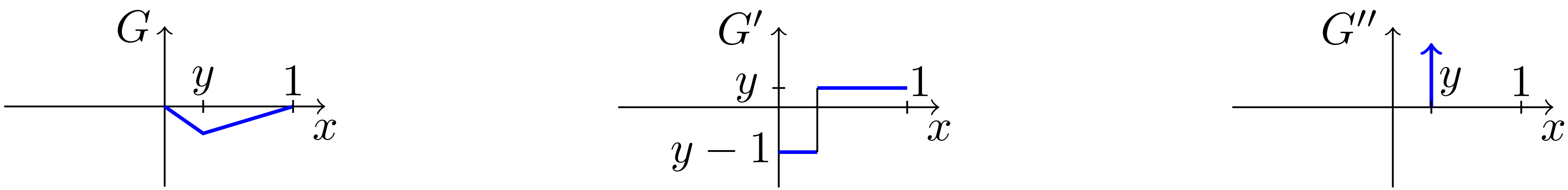
$$\begin{cases} G''(x,y) = \delta(x-y) & \text{in }]0,1[, \\ G(0,y) = G(1,y) = 0, & \text{on } \{x=0\} \text{ and } \{x=1\}. \end{cases}$$

- This Green's function has the following expression:

$$\begin{cases} G(x,y) = x(y-1) & \text{if } 0 \le x \le y \le 1, \\ G(x,y) = y(x-1) & \text{if } 0 \le y \le x \le 1. \end{cases}$$

- Indeed, $G(0,y) = 0(y-1) = 0$ and $G(1,y) = y(1-1) = 0$; further, we have

$$\begin{aligned} \int_0^1 G''(x,y)\varphi(x)dx &= -\int_0^y (y-1)\varphi'(x)dx - \int_y^1 y\varphi'(x)dx \\ &= -(y-1)\varphi(y) + y\varphi(y) \\ &= \int_0^1 \delta(x-y)\varphi(x)dx. \end{aligned}$$



Example of integral representation theorem in 1D

- Let us consider the Dirichlet problem

$$\begin{cases} \dfrac{d^2u}{dx^2} = f & \text{in }]0,1[, \\ u(0) = u(1) = 0 & \text{on } \{x=0\} \text{ and } \{x=1\}. \end{cases}$$

- Based on calculus, we seek a solution of the form

$$u(x) = a + bx + \int_0^x g(y)dy, \qquad g(y) = \int_0^y f(z)dz.$$

- Using partial integration, we obtain

$$\int_0^x g(y)dy = x\int_0^x f(y)dy - \int_0^x yf(y)dy = \int_0^x (x-y)f(y)dy.$$

- We can determine the constants a and b by enforcing the boundary conditions. The condition $u(0) = 0$ implies that $a = 0$, and $u(1) = 0$ implies that $b = -\int_0^1 (1-y)f(y)dy$. Hence,

$$u(x) = \int_0^1 x(y-1)f(y)dy + \int_0^x (x-y)f(y)dy,$$

or, more compactly,

$$u(x) = \int_0^1 G(y,x)f(y)dy,$$

as asserted by the integral representation theorem.

Summary and conclusion

- A fundamental solution solves a PDE on all of space for a Dirac impulse on the right-hand side.

- For the Laplace/Poisson equation, we considered expressions of fundamental solutions in 1D, 2D, and 3D. In 1D, the derivative of the fundamental solution E_{1D} is discontinuous at the origin. In 2D and 3D, the fundamental solutions E_{2D} and E_{3D} have a singularity at the origin.

- A fundamental solution allows a superposition formula to be established for determining a solution to a PDE on all of space for a general right-hand side.

- For the Laplace/Poisson equation, the notion of fundamental solution allowed us to prove interesting theorems and properties of solutions, such as the integral representation theorem, the mean-value property, the smoothness property, and the maximum property.

- Green's function is for a BVP on a bounded subset of space the analogue of a fundamental solution for a PDE on all of space.

References

Suggested reading material

- P. Olver. Introduction to Partial Differential Equations. Springer, 2014. Chapter 6. Chapter 7. Sections 12.1 and 12.3.

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