
MATH0024 – Modeling with PDEs

Classifications of PDEs

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- Motivation.
- From linear to nonlinear.
- Model PDEs.
- Classical versus generalized solutions.
- Ellipticity.
- References.

- An important ingredient of a systematic theory of partial differential equations (PDEs) is a **classification scheme** which identifies classes of equations with common properties. The “type” of an equation determines the nature of boundary and initial conditions which may be imposed, the type of physical phenomena that can be modeled by the equation, the nature of numerical methods which can be used to approximate the solution, and so forth.
- Common classification schemes include the following ones:
 - ◆ Order.
 - ◆ Linear, semilinear, quasilinear, versus fully nonlinear PDEs.
 - ◆ Constant versus nonconstant coefficients.
 - ◆ Scalar PDE versus system of PDEs.
 - ◆ Area of application in mechanics and physics.
 - ◆ Similarities with Laplace/Poisson, heat, transport, wave, and other model PDEs.
 - ◆ Ellipticity.
 - ◆ ...

- A partial differential equation of **order** k is an equation of the form

$$f(\mathbf{x}, \{\partial_{\mathbf{x}}^{\alpha} u\}_{|\alpha| \leq k}) = 0,$$

relating a function u of the variable \mathbf{x} in \mathbb{R}^m and its derivatives up to order k . Thus, the order of a PDE is the order of the highest derivative appearing in the equation.

- The PDE is called **linear** if f is a linear function of $\{\partial_{\mathbf{x}}^{\alpha} u\}_{|\alpha| \leq k}$, that is, if it can be written as

$$\sum_{|\alpha| \leq k} a_{\alpha}(\mathbf{x}) \partial_{\mathbf{x}}^{\alpha} u = b(\mathbf{x}).$$

- More general than the linear PDEs are the **quasilinear** PDEs, that is, those wherein f is a linear function of $\{\partial_{\mathbf{x}}^{\alpha} u\}_{|\alpha|=k}$ and which can therefore be written as

$$\sum_{|\alpha| \leq k} a_{\alpha}(\mathbf{x}, \{\partial_{\mathbf{x}}^{\alpha} u\}_{|\alpha| \leq k-1}) \partial_{\mathbf{x}}^{\alpha} u = b_{\alpha}(\mathbf{x}, \{\partial_{\mathbf{x}}^{\alpha} u\}_{|\alpha| \leq k-1}).$$

Thus, in quasilinear equations, the derivatives of the highest order appear only linearly, with coefficients that may depend on lower-order derivatives. A PDE is called **semilinear** if it is quasilinear and the coefficients of the highest order depend only on \mathbf{x} , but not on the solution.

- A nonlinear PDE that is not quasilinear is also called **fully nonlinear**.

- The transport equation is an example of a first-order linear PDE:

$$\frac{\partial u}{\partial t} + \operatorname{div}_{\mathbf{x}}(au) = 0.$$

- The Laplace/Poisson, heat, and wave equations are examples of second-order linear PDEs:

$$\Delta_{\mathbf{x}}u = f, \quad \frac{\partial u}{\partial t} - \Delta_{\mathbf{x}}u = f, \quad \frac{\partial^2 u}{\partial t^2} - \Delta_{\mathbf{x}}u = f.$$

- The Burgers equation is an example of a first-order quasilinear PDE:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0.$$

- An example of a first-order fully nonlinear PDE is given by

$$\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 = f.$$

- Laplace/Poisson equation:

$$\Delta_{\mathbf{x}}u = f.$$

- Heat equation:

$$\frac{\partial u}{\partial t} - \Delta_{\mathbf{x}}u = f.$$

- Transport equation:

$$\frac{\partial u}{\partial t} + \operatorname{div}_{\mathbf{x}}(au) = 0.$$

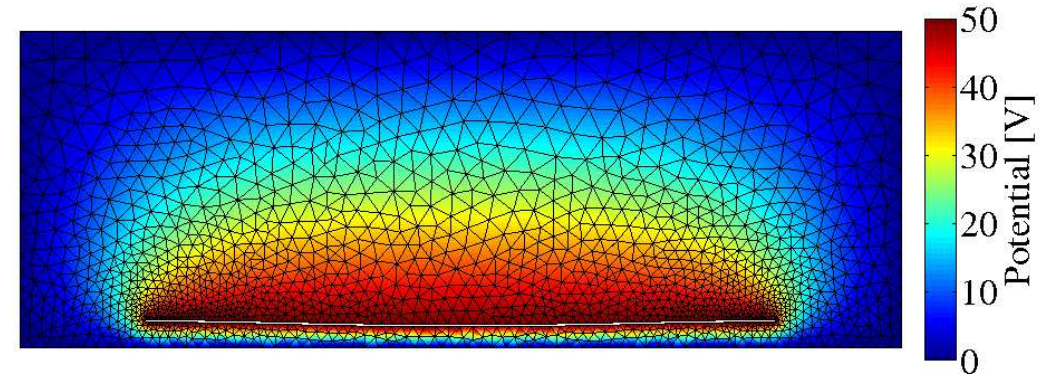
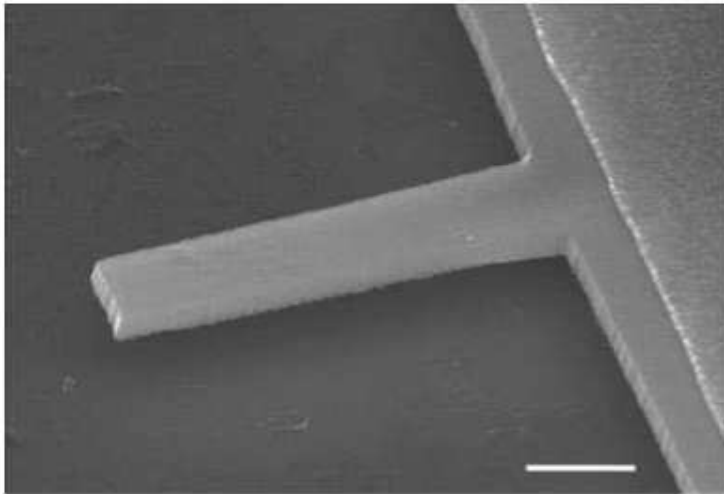
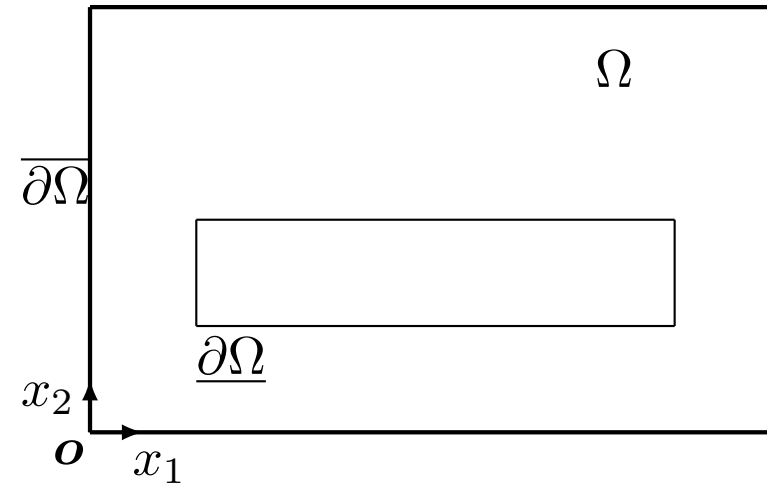
- Wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \Delta_{\mathbf{x}}u = f.$$

- These PDEs are of fundamental importance not only because of their applications but also because they serve as model problems in the study of PDEs.

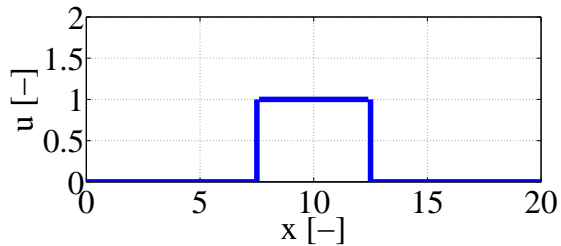
■ Laplace/Poisson equation:

$$\begin{cases} \Delta_{\mathbf{x}} \phi = 0 & \text{in } \Omega, \\ \Phi = \Phi_0 & \text{on } \underline{\partial\Omega}, \\ \Phi = 0 & \text{on } \overline{\partial\Omega}. \end{cases}$$

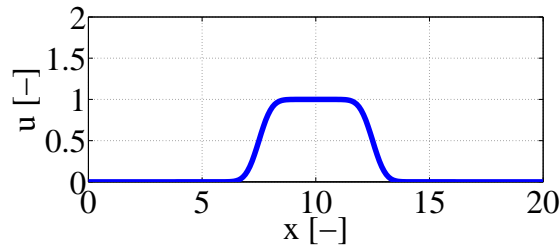


■ Heat equation:

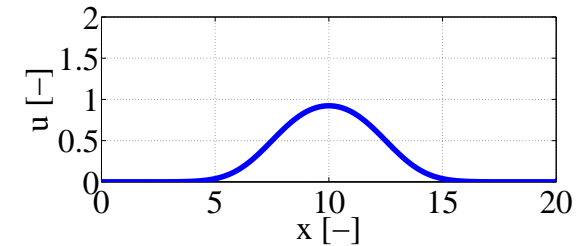
$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R} \times \mathbb{R}, \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$



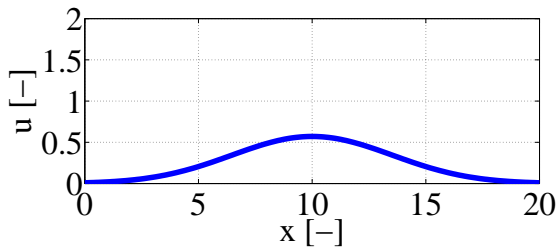
$t = 0.$



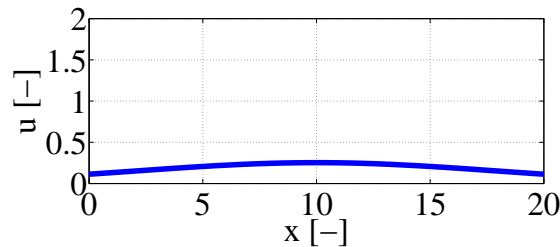
$t = 0.1.$



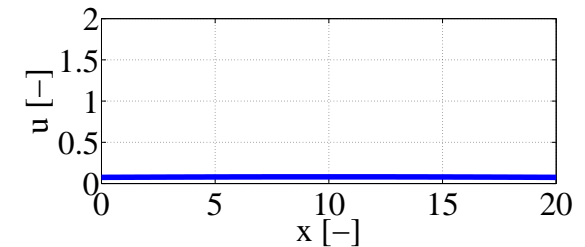
$t = 1.$



$t = 5.$



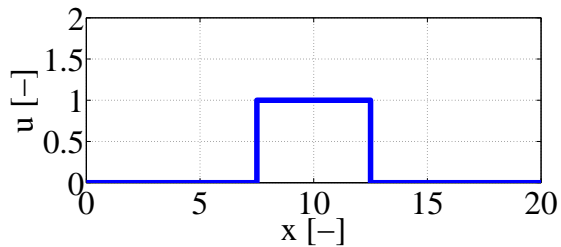
$t = 30.$



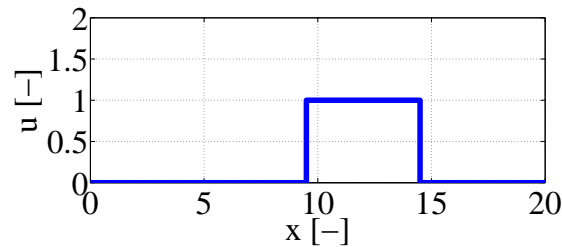
$t = 300.$

■ Transport equation:

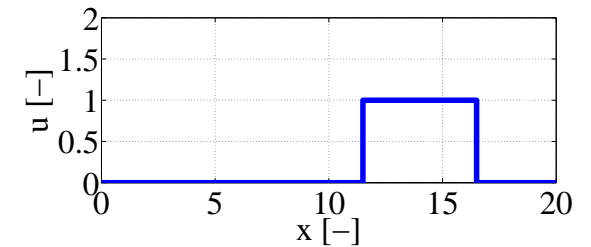
$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 & \text{in } \mathbb{R} \times \mathbb{R}, \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$



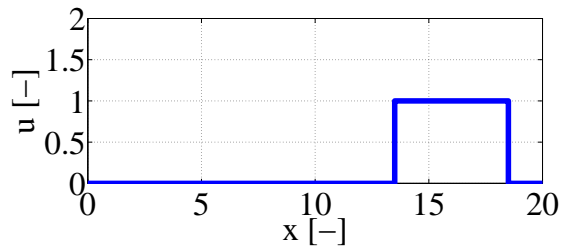
$t = 0.$



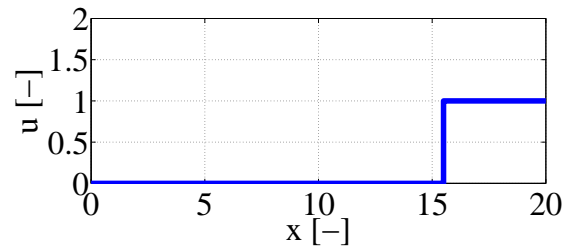
$t = 2.$



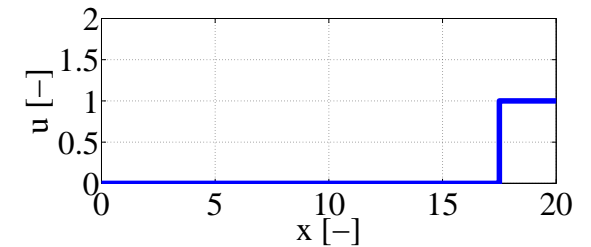
$t = 4.$



$t = 6.$



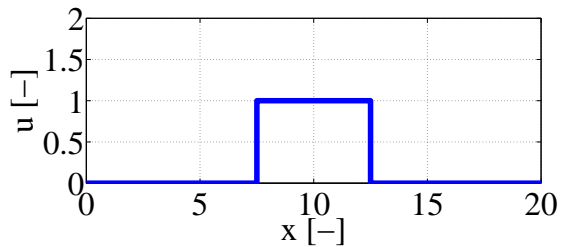
$t = 8.$



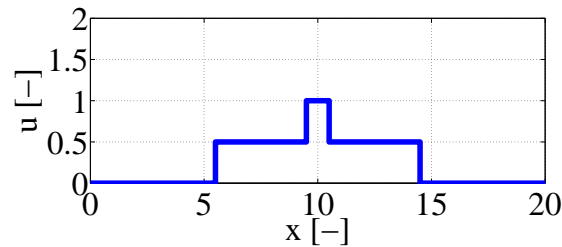
$t = 10.$

■ Wave equation:

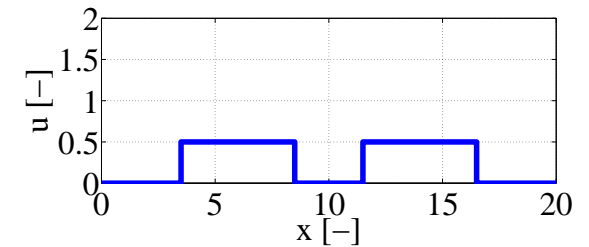
$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R} \times \mathbb{R}, \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R} \times \{0\}, \\ \frac{\partial u}{\partial t}(x, 0) = 0 & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$



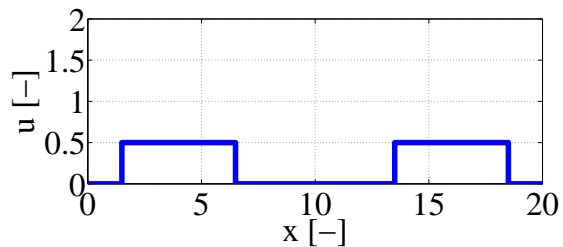
$t = 0.$



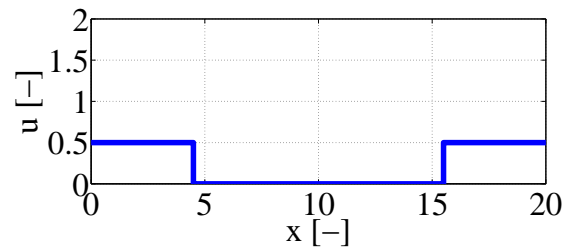
$t = 2.$



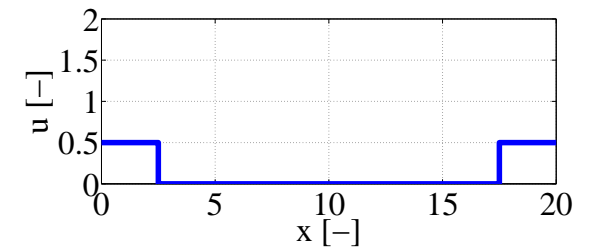
$t = 4.$



$t = 6.$



$t = 8.$



$t = 10.$

Classical versus generalized solutions

- For a PDE of order k , we say that a solution is a **classical solution** if it is at least k times continuously differentiable. For a classical solution, at least all the derivatives which appear in the statement of the PDE exist and are continuous, although certain higher derivatives may not exist.

- It is of interest to devise proper notions of **generalized solution**:
 - ◆ By allowing for solutions that need not be continuously differentiable or continuous, a **greater range of physical phenomena** can be studied. For example, the formation and propagation of **shocks** can be studied by using generalized solutions of first-order nonlinear PDEs.

 - ◆ As in the case of linear ODEs, generalized solutions allow certain linear PDEs to be viewed as **convolution filters** and various interesting properties to be deduced therefrom.

 - ◆ **Variational formulation**. Functional analysis. **Finite element methods**.

- Much of the material in this course will be concerned with generalized solutions.

- A linear PDE with constant coefficients of order k is called **elliptic** if (i) the coefficients of the k -th order partial derivatives do not vanish and (ii) there does not exist a “meaningful” linear coordinate transformation that makes one or more coefficients of the k -th order partial derivatives vanish.
- As an example, let us consider the Laplace/Poisson equation in 2D ($m = 2$):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f.$$

- ◆ We can observe that the coefficients of $\partial_{xx}^2 u$ and $\partial_{yy}^2 u$ do not vanish.
- ◆ Let us consider a linear coordinate transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} A_{\tilde{x}x} & A_{\tilde{x}y} \\ A_{\tilde{y}x} & A_{\tilde{y}y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let us assume that this linear coordinate transformation is “meaningful” in that the matrix $[A]$ is nonsingular so that \tilde{x}, \tilde{y} is a valid coordinate system in \mathbb{R}^2 . Then, by the chain rule, we have

$$(A_{\tilde{x}x}^2 + A_{\tilde{x}y}^2) \frac{\partial^2 u}{\partial \tilde{x}^2} + (A_{\tilde{y}x}^2 + A_{\tilde{y}y}^2) \frac{\partial^2 u}{\partial \tilde{y}^2} + \text{mixed and lower order terms} = f.$$

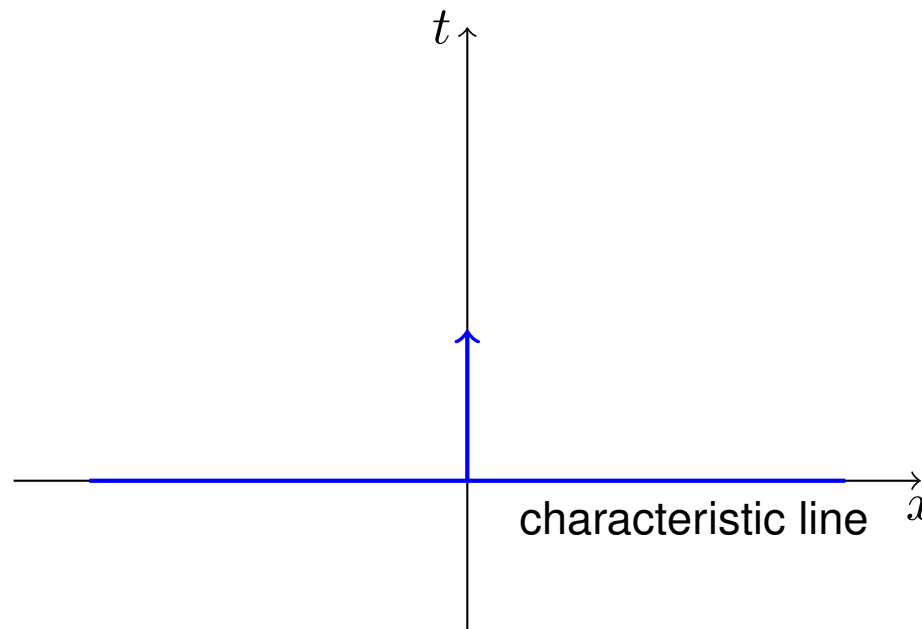
We can observe that the coefficients of $\partial_{\tilde{x}\tilde{x}}^2 u$ and $\partial_{\tilde{y}\tilde{y}}^2 u$ still do not vanish. Indeed, if $A_{\tilde{x}x}$ and $A_{\tilde{x}y}$, or, likewise, $A_{\tilde{y}x}$ and $A_{\tilde{y}y}$, would vanish simultaneously, then the matrix of the coordinate transformation would be singular, which would violate our assumption.

- ◆ As a conclusion, the **Laplace/Poisson equation** is an example of an **elliptic** PDE.

- For a linear PDE with constant coefficients of order k , a surface is called **characteristic** if the normal to this surface is a direction in which one or more k -th order partial derivatives vanish. Thus, an elliptic PDE is a PDE for which there exist no characteristic surfaces.
- As an example, let us consider the heat equation in 1D ($m = 1$):

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f.$$

The **heat equation** is **not elliptic** because the coefficient of the partial derivative $\partial_{tt}^2 u$ vanishes. For the heat equation, the lines $t = cte$ are characteristic lines.



- Let us consider the wave equation in 1D ($m = 1$):

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f.$$

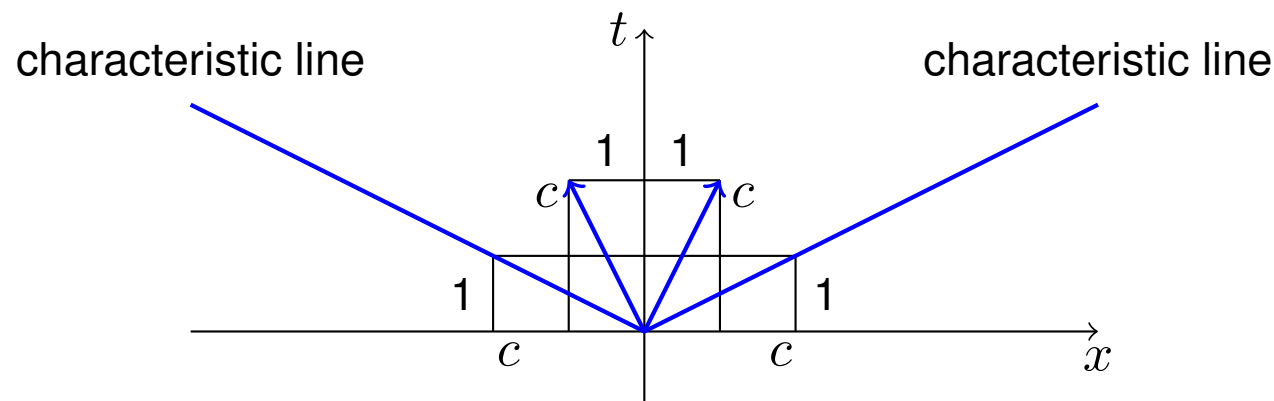
- For a linear coordinate transformation

$$\begin{bmatrix} x \\ t \end{bmatrix} \mapsto \begin{bmatrix} \tilde{x} \\ \tilde{t} \end{bmatrix} = \begin{bmatrix} A_{\tilde{x}x} & A_{\tilde{x}t} \\ A_{\tilde{t}x} & A_{\tilde{t}t} \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}.$$

we have, by the chain rule,

$$(A_{\tilde{x}t}^2 - c^2 A_{\tilde{x}x}^2) \frac{\partial^2 u}{\partial \tilde{x}^2} + (A_{\tilde{t}t}^2 - c^2 A_{\tilde{t}x}^2) \frac{\partial^2 u}{\partial \tilde{t}^2} + \text{mixed and lower order terms} = f.$$

- The **wave equation** is **not elliptic** because a coordinate transformation with $A_{\tilde{x}t} = \pm c A_{\tilde{x}x}$ results in a zero coefficient of $\partial_{\tilde{x}\tilde{x}}^2 u$. If the coordinate transformation is orthogonal, the condition $A_{\tilde{x}t} = \pm c A_{\tilde{x}x}$ corresponds to the \tilde{x} -axis being along $(x = x, t = \pm c x)$. The lines $x = \pm ct + \text{constant}$ are characteristic lines.



- It is a standard exercise in changes of bases to see that the condition $A_{\tilde{x}t} = \pm c A_{\tilde{x}x}$ corresponds to the \tilde{x} -axis being along $(x = x, t = \pm c x)$.

- For this purpose, let us consider the orthogonal coordinate transformation

$$\begin{bmatrix} \tilde{x} \\ \tilde{t} \end{bmatrix} = \frac{1}{\sqrt{1+c^2}} \begin{bmatrix} 1 & \pm c \\ \mp c & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}.$$

- Let $\mathbf{i}_x, \mathbf{i}_t, \mathbf{i}_{\tilde{x}}$, and $\mathbf{i}_{\tilde{t}}$ be basis vectors along the x, t, \tilde{x} , and \tilde{t} axes. Let them be related as

$$\begin{cases} \mathbf{i}_x = B_{x\tilde{x}}\mathbf{i}_{\tilde{x}} + B_{x\tilde{t}}\mathbf{i}_{\tilde{t}}, \\ \mathbf{i}_t = B_{t\tilde{x}}\mathbf{i}_{\tilde{x}} + B_{t\tilde{t}}\mathbf{i}_{\tilde{t}}. \end{cases}$$

- Injecting this relationship in $x\mathbf{i}_x + t\mathbf{i}_t$, we obtain $x\mathbf{i}_x + t\mathbf{i}_t = (B_{x\tilde{x}}x + B_{t\tilde{x}}t)\mathbf{i}_{\tilde{x}} + (B_{x\tilde{t}}x + B_{t\tilde{t}}t)\mathbf{i}_{\tilde{t}}$; hence, by the orthogonality of the coor. transf., $\tilde{x} = B_{x\tilde{x}}x + B_{t\tilde{x}}t$ and $\tilde{t} = B_{x\tilde{t}}x + B_{t\tilde{t}}t$.

- By identification, we obtain $B_{x\tilde{x}} = \frac{1}{\sqrt{1+c^2}}$, $B_{t\tilde{x}} = \frac{\pm c}{\sqrt{1+c^2}}$, $B_{x\tilde{t}} = \frac{\mp c}{\sqrt{1+c^2}}$, and $B_{t\tilde{t}} = \frac{1}{\sqrt{1+c^2}}$; thus

$$\begin{cases} \mathbf{i}_x = \frac{1}{\sqrt{1+c^2}}\mathbf{i}_{\tilde{x}} + \frac{\mp c}{\sqrt{1+c^2}}\mathbf{i}_{\tilde{t}}, \\ \mathbf{i}_t = \frac{\pm c}{\sqrt{1+c^2}}\mathbf{i}_{\tilde{x}} + \frac{1}{\sqrt{1+c^2}}\mathbf{i}_{\tilde{t}}, \end{cases} \quad \text{and therefore} \quad \begin{cases} \mathbf{i}_{\tilde{x}} = \frac{1}{\sqrt{1+c^2}}\mathbf{i}_x + \frac{\pm c}{\sqrt{1+c^2}}\mathbf{i}_t, \\ \mathbf{i}_{\tilde{t}} = \frac{\mp c}{\sqrt{1+c^2}}\mathbf{i}_x + \frac{1}{\sqrt{1+c^2}}\mathbf{i}_t. \end{cases}$$

Suggested reading material

- P. Olver. Introduction to Partial Differential Equations. Springer, 2014. Chapter 1.

Additional references also consulted to prepare this lecture

- L. Evans. Partial differential equations. American Mathematical Society, 2010.
- G. Folland. Introduction to partial differential equations. Princeton University Press, 1995.
- M. Renardy and R. Rogers. An introduction to partial differential equations. Springer, 2004.