MATH0024 – Modeling with PDEs

Notations and review of background
Intrinsic formulations in physics and mechanics

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Notations

Notations

General notations

- A lowercase letter, for example, a, is a scalar.
- A boldface lowercase letter, for example, $\boldsymbol{a} = (a_1, \ldots, a_m)$, is a vector.
- A boldface uppercase letter, for example, $\boldsymbol{A}$, is a linear mapping (“application linéaire” in French).
- An uppercase letter between square brackets, for example, $[A]$, is a matrix.

Notations for matrices

- We denote by A_{ij} the (i, j)-th entry of the matrix $[A]$.
- tr$[A]$ = trace of the matrix $[A]$.
- det$[A]$ = determinant of the matrix $[A]$.
- $[A]^{\mathrm{T}}$ = transpose of the matrix $[A]$.

“Big-oh” and “little-oh” notation

- If $f(h)$ and $g(h)$ are two functions of h, then we say that
$$f(h) = O\big(g(h)\big) \quad \text{as } h \to 0$$
if there is a constant c such that
$$\frac{f(h)}{g(h)} < c \quad \text{for all } h \text{ sufficiently small.}$$
This means that $f(h)$ decays to zero at least as fast as the function $g(h)$ does.

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$$f(h) = o\big(g(h)\big) \quad \text{as } h \to 0$$
if
$$\frac{f(h)}{g(h)} \to 0 \quad \text{as } h \to 0.$$
This is slightly **stronger** and means that $f(h)$ decays to zero faster than $g(h)$.

- Examples:

$$2h^3 = O(h^2) \qquad \text{as } h \to 0 \qquad \text{because } \frac{2h^3}{h^2} = 2h < 1 \text{ for all } h < \frac{1}{2}.$$

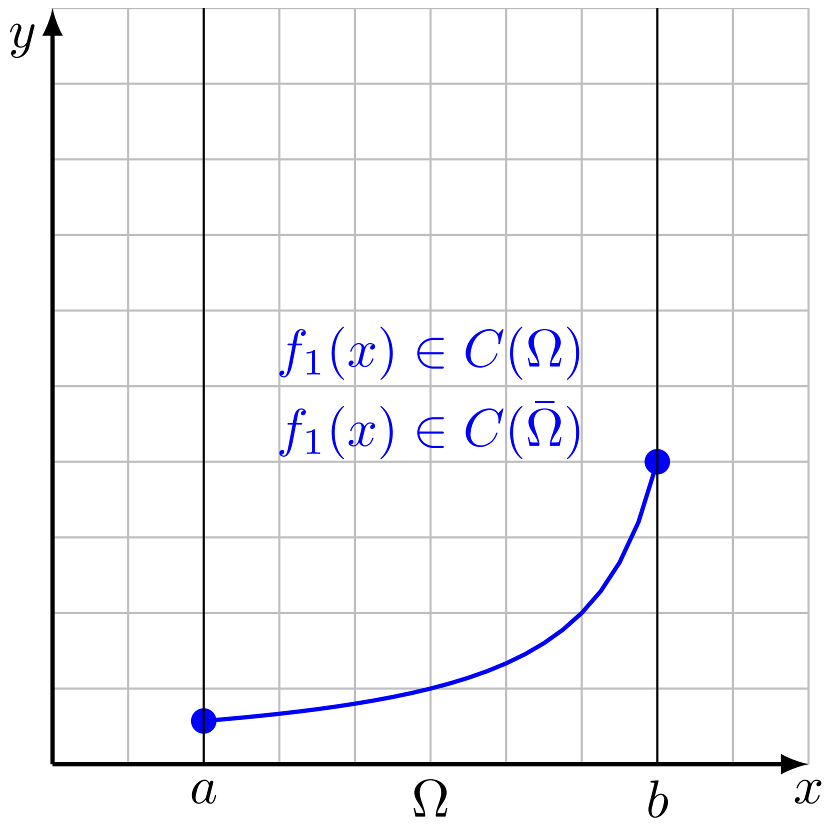
$$2h^3 = o(h^2) \qquad \text{as } h \to 0 \qquad \text{because } \frac{2h^3}{h^2} = 2h \to 0 \text{ as } h \to 0.$$

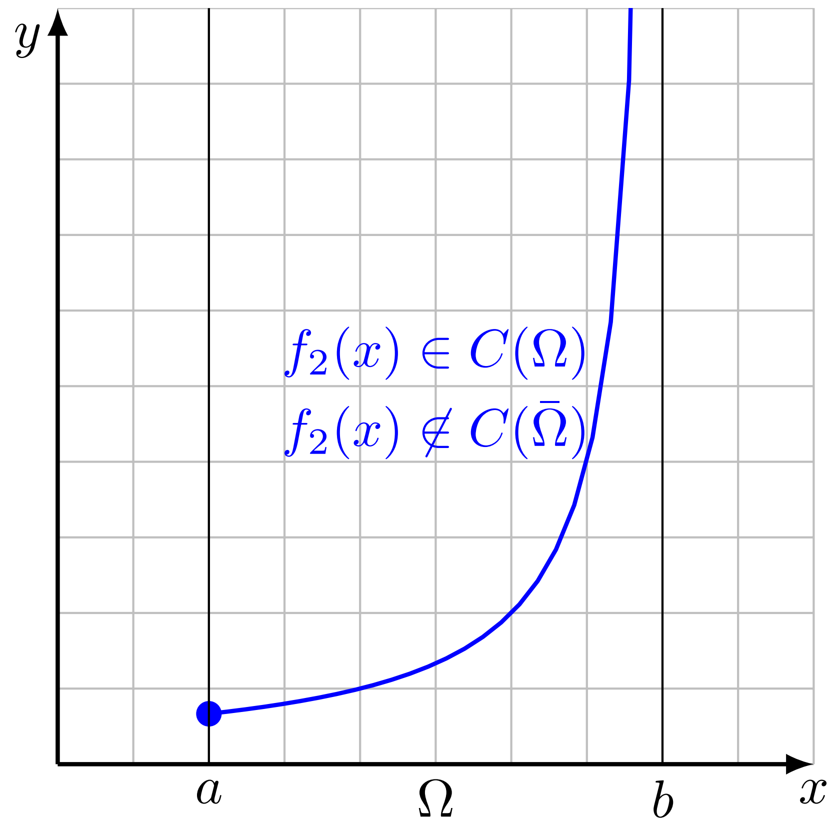
Notations for derivatives

- Let $f : \Omega \subset \mathbb{R}^m \to \mathbb{R} : \boldsymbol{x} \mapsto f(\boldsymbol{x})$ be a function from $\Omega \subset \mathbb{R}^m$ into $\mathbb{R}$.
- We denote by $\frac{\partial f}{\partial x_j}(\boldsymbol{x})$ the j-th partial derivative of f evaluated at $\boldsymbol{x}$.
- We sometimes denote $\frac{\partial f}{\partial x_j}$ by f_{x_j} or $\partial_{x_j} f$.
- Similarly, $\frac{\partial^2 f}{\partial x_i \partial x_j} = f_{x_i x_j} = \partial_{x_i}\partial_{x_j} f$, $\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} = f_{x_i x_j x_k} = \partial_{x_i}\partial_{x_j}\partial_{x_k} f$, and so forth.
- Multi-index notation:
  - We call a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$, where each component α_j is a nonnegative integer, a multi-index of order $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_m$.
  - For a multi-index $\boldsymbol{\alpha}$, we define
  $$\partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_m}^{\alpha_m} f.$$
  - Example:
  $$\partial_{(x_1,x_2,x_3)}^{(1,2,1)} f = \frac{\partial^4}{\partial x_1 \partial x_2^2 \partial x_3} f.$$

Notations for function spaces

- Let Ω be an open subset of $\mathbb{R}^n$. Let $\partial\Omega$ denote the boundary of Ω.
- $C(\Omega) = \{f : \Omega \to \mathbb{R} \ : \ f$ is continuous$\}$ space of continuous functions.
- $C(\overline{\Omega}) = \{f \in C(\Omega) \ : \ f$ admits a continuous extension to $\overline{\Omega} = \Omega \cup \partial\Omega\}$.





Notations for function spaces (continued)

- $C^k(\Omega) = \{f : \Omega \to \mathbb{R} \ : \ \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f$ is continuous, $|\boldsymbol{\alpha}| \leq k\}$ space of k times continuously differentiable functions.

- $C^k(\overline{\Omega}) = \{f \in C^k(\Omega) \ : \ \partial_{\boldsymbol{x}}^{\boldsymbol{\alpha}} f$ admits a continuous extension to $\overline{\Omega} = \Omega \cup \partial\Omega$, $|\boldsymbol{\alpha}| \leq k\}$.

- $L^1(\Omega) = \{f : \Omega \to \mathbb{R} \ : \ \int_\Omega |f(\boldsymbol{x})| d\boldsymbol{x} < +\infty\}$ space of integrable functions.

- $L^2(\Omega) = \{f : \Omega \to \mathbb{R} \ : \ \int_\Omega |f(\boldsymbol{x})|^2 d\boldsymbol{x} < +\infty\}$ space of square-integrable functions.

- Although we will avoid the use of these notations for function spaces as much as possible, we list them here because they are encountered often in the literature.

Review of vector calculus

Vector calculus

This is not a lecture but rather a summary of key elements of vector calculus. For a more complete treatment of vector calculus, please refer to MATH0007 Analyse Mathématique II (F. Bastin).

Vectors

- Let us consider the **m-dimensional Euclidean vector space** $\mathbb{R}^m$.

- For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^m$, the (Euclidean) **inner product** is the scalar denoted by $\boldsymbol{a} \cdot \boldsymbol{b}$.

- We denote by $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ an **orthonormal basis** for $\mathbb{R}^m$, that is, a basis such that $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, $1 \leq i, j \leq m$, where δ_{ij} is the **Kronecker delta** equal to 1 if $i = j$ and 0 otherwise.

- Given an orthonormal basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ for $\mathbb{R}^m$, we have that

  any **vector** $\boldsymbol{a}$ in $\mathbb{R}^m$ can be represented by a **column matrix** $\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$

  of its components a_j such that $\boldsymbol{a} = \sum_{j=1}^m a_j \boldsymbol{e}_j$ with $a_j = \boldsymbol{a} \cdot \boldsymbol{e}_j,\ 1 \leq j \leq m$.

- For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the inner product $\boldsymbol{a} \cdot \boldsymbol{b}$ is the scalar $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{j=1}^m a_j b_j$.

- If $m = 3$, for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is the vector $\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\boldsymbol{e}_1 + (a_3 b_1 - a_1 b_3)\boldsymbol{e}_2 + (a_1 b_2 - a_2 b_1)\boldsymbol{e}_3$.

Linear mappings

- A **linear mapping** $\boldsymbol{A}$ from $\mathbb{R}^m$ into $\mathbb{R}^m$ is a function that maps any vector $\boldsymbol{a}$ in $\mathbb{R}^m$ onto a vector $\boldsymbol{b} = \boldsymbol{A}(\boldsymbol{a})$ in $\mathbb{R}^m$ in a manner that satisfies additivity ($\boldsymbol{A}(\boldsymbol{a}_1 + \boldsymbol{a}_2) = \boldsymbol{A}(\boldsymbol{a}_1) + \boldsymbol{A}(\boldsymbol{a}_2)$, $\forall \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbb{R}^m$) and homogeneity ($\boldsymbol{A}(\alpha \boldsymbol{a}) = \alpha \boldsymbol{A}(\boldsymbol{a}), \forall \alpha \in \mathbb{R},\ \forall \boldsymbol{a} \in \mathbb{R}^m$) properties.

- Given an orthonormal basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ for $\mathbb{R}^m$, we have that any **linear mapping** $\boldsymbol{A}$ from $\mathbb{R}^m$ into $\mathbb{R}^m$ can be represented by a **matrix** $\begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mm} \end{bmatrix}$ of its components a_{ij} such that $a_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A}(\boldsymbol{e}_j),\ 1 \leq i, j \leq m$.

- We have for two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and a linear mapping $\boldsymbol{A}$ with $\boldsymbol{b} = \boldsymbol{A}(\boldsymbol{a})$ that

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mm} \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}.$$

Linear mappings (continued)

- The **sum** of two linear mappings $\boldsymbol{A}$ and $\boldsymbol{B}$ is the linear mapping $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ with components $c_{ij} = a_{ij} + b_{ij},\ 1 \leq i, j \leq m$.
- We denote by $\boldsymbol{A} : \boldsymbol{B}$ the **inner product** of two linear mappings $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^{m} a_{ij} b_{ij}$.
- The **composition** of two linear mappings $\boldsymbol{A}$ and $\boldsymbol{B}$ is the linear mapping $\boldsymbol{C}$ such that $\boldsymbol{C}(\boldsymbol{a}) = \boldsymbol{A}(\boldsymbol{B}(\boldsymbol{a})),\ \forall \boldsymbol{a} \in \mathbb{R}^m$.
- The **transpose** of a linear mapping $\boldsymbol{A}$ is the linear mapping $\boldsymbol{A}^{\mathrm{T}}$ such that $\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{A}(\boldsymbol{b}),\ \forall \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$.
- The **inverse** of a linear mapping $\boldsymbol{A}$ (if it exists) is the linear mapping $\boldsymbol{A}^{-1}$ such that $\boldsymbol{a} = \boldsymbol{A}^{-1}(\boldsymbol{b})$, $\boldsymbol{b} = \boldsymbol{A}(\boldsymbol{a})$, $\forall \boldsymbol{a} \in \mathbb{R}^m$. Thus, the inverse satisfies $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the **identity linear mapping**.
- The **trace** of a linear mapping tr$(\boldsymbol{A})$ is defined by tr$(\boldsymbol{A}) = \boldsymbol{A} : \boldsymbol{I} = \sum_{j=1}^{m} a_{jj}$. We have that tr$(\boldsymbol{A}\boldsymbol{B}) = \boldsymbol{A} : \boldsymbol{B}^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} : \boldsymbol{B}$.
- The composition, transpose, inverse, and trace can be made explicit in terms of the components of the linear mapping.

Differential operators

- We consider scalar-, vector-, and linear-mapping-valued functions φ, $\boldsymbol{a}$, and $\boldsymbol{A}$ from $\mathbb{R}^m$ into $\mathbb{R}$, $\mathbb{R}^m$, and the space of linear mappings from $\mathbb{R}^m$ into $\mathbb{R}^m$, respectively, that is,

$$\boldsymbol{x} \mapsto \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \mapsto \boldsymbol{a}(\boldsymbol{x}), \qquad \boldsymbol{x} \mapsto \boldsymbol{A}(\boldsymbol{x}).$$

- The **gradient** of φ with respect to $\boldsymbol{x}$ at $\boldsymbol{x}$ (if it exists) is the vector $\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi(\boldsymbol{x})$ such that

$$\underbrace{\lim_{h\to 0} \frac{\varphi(\boldsymbol{x}+h\boldsymbol{y}) - \varphi(\boldsymbol{x})}{h}}_{\text{directional derivative of } \varphi \text{ at } \boldsymbol{x} \text{ in direction } \boldsymbol{y}} = \underbrace{\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi(\boldsymbol{x})}_{\text{gradient of } \varphi \text{ w.r.t. } \boldsymbol{x} \text{ at } \boldsymbol{x}} \cdot \underbrace{\boldsymbol{y}}_{\text{direction } \boldsymbol{y}}, \qquad \forall \boldsymbol{y} \in \mathbb{R}^m.$$

- The **gradient** of $\boldsymbol{a}$ with respect to $\boldsymbol{x}$ at $\boldsymbol{x}$ (if it exists) is the linear mapping $\mathbf{D}_{\boldsymbol{x}}\boldsymbol{a}(\boldsymbol{x})$ such that

$$\underbrace{\lim_{h\to 0} \frac{\boldsymbol{a}(\boldsymbol{x}+h\boldsymbol{y}) - \boldsymbol{a}(\boldsymbol{x})}{h}}_{\text{directional derivative of } \boldsymbol{a} \text{ at } \boldsymbol{x} \text{ in direction } \boldsymbol{y}} = \underbrace{\left(\mathbf{D}_{\boldsymbol{x}}\boldsymbol{a}(\boldsymbol{x})\right)}_{\text{gradient of } \boldsymbol{a} \text{ w.r.t. } \boldsymbol{x} \text{ at } \boldsymbol{x}} \underbrace{(\boldsymbol{y})}_{\text{direction } \boldsymbol{y}}, \qquad \forall \boldsymbol{y} \in \mathbb{R}^m.$$

- The **divergence** of $\boldsymbol{a}$ with respect to $\boldsymbol{x}$ (if it exists) is the scalar $\text{div}_{\boldsymbol{x}}\boldsymbol{a}$ such that

$$\text{div}_{\boldsymbol{x}}\boldsymbol{a} = \text{tr}(\mathbf{D}_{\boldsymbol{x}}\boldsymbol{a}).$$

- The **divergence** of $\boldsymbol{A}$ with respect to $\boldsymbol{x}$ (if it exists) is the vector $\textbf{div}_{\boldsymbol{x}}\boldsymbol{A}$ such that

$$\textbf{div}_{\boldsymbol{x}}\boldsymbol{A} \cdot \boldsymbol{b} = \text{div}_{\boldsymbol{x}}(\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{b})), \qquad \forall \boldsymbol{b} \in \mathbb{R}^m.$$

Differential operators (continued)

- The **Curl** of $\boldsymbol{a}$ with respect to $\boldsymbol{x}$ (if it exists) is the linear mapping $\textbf{Curl}_{\boldsymbol{x}}\boldsymbol{a}$ such that
$$\textbf{Curl}_{\boldsymbol{x}}\boldsymbol{a} = \textbf{D}_{\boldsymbol{x}}\boldsymbol{a} - \textbf{D}_{\boldsymbol{x}}\boldsymbol{a}^{\mathrm{T}}.$$
If $m = 3$, we can associate to $\textbf{Curl}_{\boldsymbol{x}}\boldsymbol{a}$ the vector $\textbf{curl}_{\boldsymbol{x}}\boldsymbol{a}$ such that
$$\textbf{curl}_{\boldsymbol{x}}\boldsymbol{a} \times \boldsymbol{b} = \textbf{Curl}_{\boldsymbol{x}}\boldsymbol{a}(\boldsymbol{b}), \qquad \forall \boldsymbol{b} \in \mathbb{R}^m.$$

- The **Laplacian** of φ with respect to $\boldsymbol{x}$ (if it exists) is the scalar $\triangle_{\boldsymbol{x}}\varphi$ such that
$$\triangle_{\boldsymbol{x}}\varphi = \mathrm{div}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi.$$

Differential operators (properties)

- $\textbf{curl}_{\boldsymbol{x}}\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi = \boldsymbol{0}$.
- $\mathrm{div}_{\boldsymbol{x}}\textbf{curl}_{\boldsymbol{x}}\boldsymbol{a} = 0$.
- $\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot \textbf{curl}_{\boldsymbol{x}}\boldsymbol{a} - \boldsymbol{a} \cdot \textbf{curl}_{\boldsymbol{x}}\boldsymbol{b}$.
- $\boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \textbf{div}_{\boldsymbol{x}}(\textbf{D}_{\boldsymbol{x}}\boldsymbol{a}^{\mathrm{T}})$.
- $\boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \textbf{div}_{\boldsymbol{x}}\textbf{D}_{\boldsymbol{x}}\boldsymbol{a} + \textbf{curl}_{\boldsymbol{x}}\textbf{curl}_{\boldsymbol{x}}\boldsymbol{a}$.
- $\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{A}^{\mathrm{T}}(\boldsymbol{a})) = \boldsymbol{A} : \textbf{D}_{\boldsymbol{x}}\boldsymbol{a} + \boldsymbol{a} \cdot \textbf{div}_{\boldsymbol{x}}\boldsymbol{A}$.
- $\mathrm{div}_{\boldsymbol{x}}(\varphi\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}\varphi + \varphi\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a}$.
- $\textbf{div}_{\boldsymbol{x}}(\varphi\boldsymbol{A}) = \boldsymbol{A}(\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi) + \varphi\textbf{div}_{\boldsymbol{x}}\boldsymbol{A}$.

Coordinate system

- We consider again scalar-, vector-, and linear-mapping-valued functions φ, $\boldsymbol{a}$, and $\boldsymbol{A}$ from $\mathbb{R}^m$ into $\mathbb{R}$, $\mathbb{R}^m$, and the space of linear mappings from $\mathbb{R}^m$ into $\mathbb{R}^m$, respectively, that is,

$$\boldsymbol{x} \mapsto \varphi(\boldsymbol{x}), \qquad \boldsymbol{x} \mapsto \boldsymbol{a}(\boldsymbol{x}), \qquad \boldsymbol{x} \mapsto \boldsymbol{A}(\boldsymbol{x}).$$

- A **coordinate system** is a one-to-one correspondence between vectors $\boldsymbol{x}$ in $\mathbb{R}^m$ ("position") and m-tuples $(\xi_1, \ldots, \xi_m)$ in $\mathbb{R}^m$ ("coordinates"):

$$(\xi_1, \ldots, \xi_m) \mapsto \boldsymbol{x}(\xi_1, \ldots, \xi_m).$$

- If a Cartesian coordinate system is used, its basis vectors are most often reused for the representation of vectors ($\boldsymbol{a}$), linear mappings ($\boldsymbol{A}$), and differential operators. However, if a curvilinear coordinate system is used, basis vectors are sometimes redefined locally for use for the representation of vectors ($\boldsymbol{a}$), linear mappings ($\boldsymbol{A}$), and differential operators.

- A coordinate system also allows us to define volume, surface, and line integrals.

Volume, surface, and line integrals

- For a volume V parameterized as

$$\begin{cases} x_1 = x_1(\xi_1,\ldots,\xi_m) \\ \vdots \\ x_m = x_m(\xi_1,\ldots,\xi_m) \end{cases}, \qquad \underline{\xi}_m \le \xi_m \le \overline{\xi}_m, \ldots, \underline{\xi}_1 \le \xi_1 \le \overline{\xi}_1,$$

the **volume integral** of a scalar-valued function φ over the volume V is given by

$$\int_V \varphi dV = \int_{\underline{\xi}_m}^{\overline{\xi}_m} \ldots \int_{\underline{\xi}_1}^{\overline{\xi}_1} \varphi\big(\boldsymbol{x}(\xi_1,\ldots,\xi_m)\big) \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \cdots & \frac{\partial x_1}{\partial \xi_m} \\ \vdots & & \vdots \\ \frac{\partial x_m}{\partial \xi_1} & \cdots & \frac{\partial x_m}{\partial \xi_m} \end{vmatrix} d\xi_1 \ldots d\xi_m.$$

- For a surface S parameterized as

$$\begin{cases} x_1 = x_1(\xi_1,\ldots,\xi_{m-1}) \\ \vdots \\ x_m = x_m(\xi_1,\ldots,\xi_{m-1}) \end{cases}, \qquad \underline{\xi}_{m-1} \le \xi_{m-1} \le \overline{\xi}_{m-1}, \ldots, \underline{\xi}_1 \le \xi_1 \le \overline{\xi}_1,$$

Volume, surface, and line integrals (continued)

- the **surface integral** of a vector-valued function $\boldsymbol{a}$ over the surface S is given by

$$\int_S \boldsymbol{a}\cdot d\boldsymbol{S} = \int_{\underline{\xi}_{m-1}}^{\overline{\xi}_{m-1}} \dots \int_{\underline{\xi}_1}^{\overline{\xi}_1} \begin{vmatrix} \frac{\partial x_1}{\partial \xi_1} & \dots & \frac{\partial x_1}{\partial \xi_{m-1}} & a_1\big(\boldsymbol{x}(\xi_1,\dots,\xi_{m-1})\big) \\ \vdots & & \vdots & \vdots \\ \frac{\partial x_m}{\partial \xi_1} & \dots & \frac{\partial x_m}{\partial \xi_{m-1}} & a_m\big(\boldsymbol{x}(\xi_1,\dots,\xi_{m-1})\big) \end{vmatrix} d\xi_1 \dots d\xi_{m-1}.$$

  If $m = 3$, then the surface integral reads, equivalently, as follows:

$$\int_S \boldsymbol{a}\cdot d\boldsymbol{S} = \int_{\underline{\xi}_2}^{\overline{\xi}_2} \int_{\underline{\xi}_1}^{\overline{\xi}_1} \boldsymbol{a}\big(\boldsymbol{x}(\xi_1,\dots,\xi_{m-1})\big)\cdot\left(\frac{\partial \boldsymbol{x}}{\partial \xi_1}\times\frac{\partial \boldsymbol{x}}{\partial \xi_2}\right) d\xi_1 d\xi_2.$$

- For a curve C parameterized as

$$\begin{cases} x_1 = x_1(\xi) \\ \vdots \\ x_m = x_m(\xi) \end{cases}, \qquad \underline{\xi} \le \xi \le \overline{\xi},$$

  the **line integral** of a vector-valued function $\boldsymbol{a}$ over the curve C is given by

$$\int_C \boldsymbol{a}\cdot d\boldsymbol{\ell} = \int_{\underline{\xi}}^{\overline{\xi}} \sum_{j=1}^{m} a_j\big(\boldsymbol{x}(\xi)\big)\frac{dx_j}{d\xi} d\xi = \int_{\underline{\xi}}^{\overline{\xi}} \boldsymbol{a}\big(\boldsymbol{x}(\xi)\big)\cdot\frac{d\boldsymbol{x}}{d\xi} d\xi.$$

Volume, surface, and line integrals (properties)

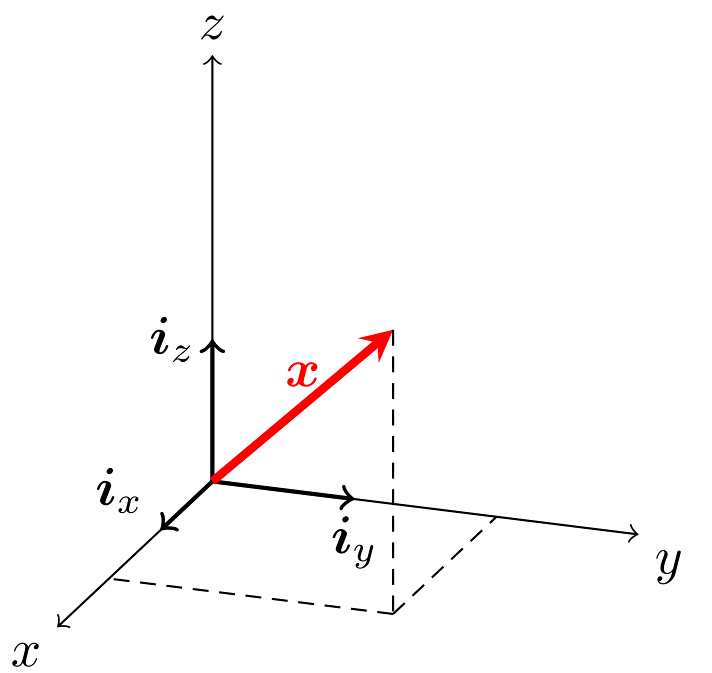
- **Stokes's theorem for a volume**: Let Ω be a bounded open subset of $\mathbb{R}^m$ with $m \geq 2$ with a sufficiently regular boundary $\partial\Omega$. Let $\boldsymbol{a}$ be a sufficiently regular function from Ω into $\mathbb{R}^m$ (specifically, let $\boldsymbol{a}$ be in $C(\overline{\Omega}, \mathbb{R}^m) \cap C^1(\Omega, \mathbb{R}^m)$). Then, we have

$$\int_{\Omega} \text{div}_{\boldsymbol{x}} \boldsymbol{a} \, dV = \int_{\partial\Omega} \boldsymbol{a} \cdot d\boldsymbol{S}.$$

- **Green's identities**: Let Ω be a bounded open subset of $\mathbb{R}^m$ with $m \geq 2$ with a sufficiently regular boundary $\partial\Omega$. Let φ and ψ be a sufficiently regular function from Ω into $\mathbb{R}$ (specifically, let φ and ψ be in $C^1(\overline{\Omega}) \cap C^2(\Omega)$). Then, we have

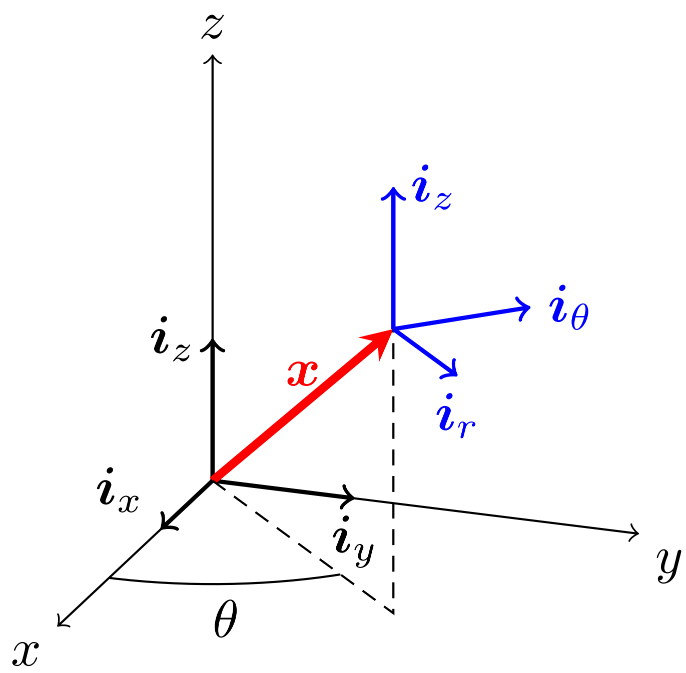
$$\begin{cases} \displaystyle\int_{\partial\Omega} \psi \boldsymbol{\nabla}_{\boldsymbol{x}} \varphi \cdot d\boldsymbol{S} = \int_{\Omega} (\psi \triangle_{\boldsymbol{x}} \varphi + \boldsymbol{\nabla}_{\boldsymbol{x}} \psi \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} \varphi) dV, \\ \displaystyle\int_{\partial\Omega} (\psi \boldsymbol{\nabla}_{\boldsymbol{x}} \varphi - \varphi \boldsymbol{\nabla}_{\boldsymbol{x}} \psi) \cdot d\boldsymbol{S} = \int_{\Omega} (\psi \triangle_{\boldsymbol{x}} \varphi - \varphi \triangle_{\boldsymbol{x}} \psi) dV. \end{cases}$$

Cartesian coordinates



- $\boldsymbol{x} = x\boldsymbol{i}_x + y\boldsymbol{i}_y + z\boldsymbol{i}_z$.
  Coordinates x, y, and z.
  Orthonormal basis $\boldsymbol{i}_x$, $\boldsymbol{i}_y$, and $\boldsymbol{i}_z$.
- $\boldsymbol{a} = a_x\boldsymbol{i}_x + a_y\boldsymbol{i}_y + a_z\boldsymbol{i}_z$.
- $\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi = \frac{\partial\varphi}{\partial x}\boldsymbol{i}_x + \frac{\partial\varphi}{\partial y}\boldsymbol{i}_y + \frac{\partial\varphi}{\partial z}\boldsymbol{i}_z$.
- $\text{div}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$.
- $\textbf{curl}_{\boldsymbol{x}}\boldsymbol{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\boldsymbol{i}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\boldsymbol{i}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\boldsymbol{i}_z$.
- $\triangle_{\boldsymbol{x}}\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}$.
- $dV = dxdydz$.

Cylindrical coordinates



- $\boldsymbol{x} = r\cos(\theta)\boldsymbol{i}_x + r\sin(\theta)\boldsymbol{i}_y + z\boldsymbol{i}_z$.
  Coordinates r, θ, and z.
  Orthonormal basis $\boldsymbol{i}_x$, $\boldsymbol{i}_y$, and $\boldsymbol{i}_z$.

- $\boldsymbol{a} = a_r\boldsymbol{i}_r(\theta) + a_\theta\boldsymbol{i}_\theta(\theta) + a_z\boldsymbol{i}_z$.
  $\boldsymbol{i}_r(\theta) = \cos(\theta)\boldsymbol{i}_x + \sin(\theta)\boldsymbol{i}_y$.
  $\boldsymbol{i}_\theta(\theta) = -\sin(\theta)\boldsymbol{i}_x + \cos(\theta)\boldsymbol{i}_y$.

- $\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi = \frac{\partial\varphi}{\partial r}\boldsymbol{i}_r(\theta) + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\boldsymbol{i}_\theta(\theta) + \frac{\partial\varphi}{\partial z}\boldsymbol{i}_z$.

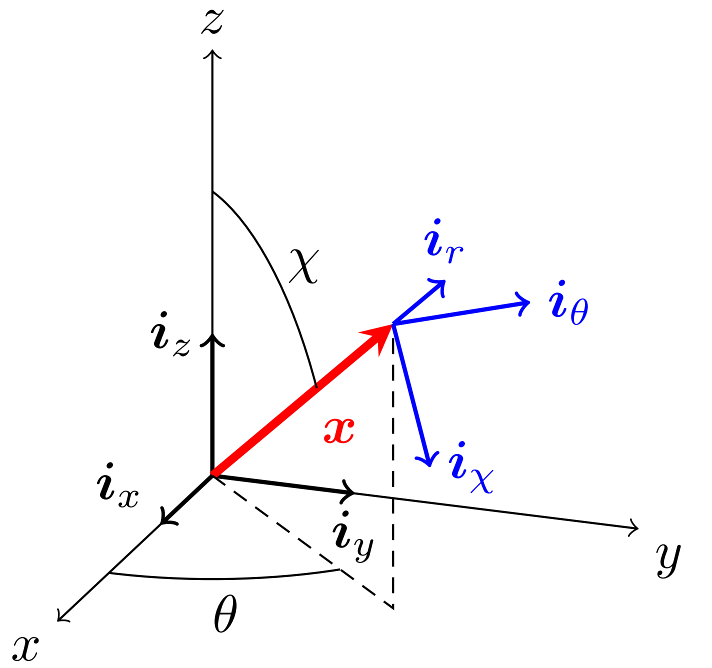
- $\text{div}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial a_r}{\partial r} + \frac{a_r}{r} + \frac{1}{r}\frac{\partial a_\theta}{\partial\theta} + \frac{\partial a_z}{\partial z}$.

- $\textbf{curl}_{\boldsymbol{x}}\boldsymbol{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial\theta} - \frac{\partial a_\theta}{\partial z}\right)\boldsymbol{i}_r(\theta) + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\boldsymbol{i}_\theta(\theta) + \left(\frac{\partial a_\theta}{\partial r} + \frac{a_\theta}{r} - \frac{1}{r}\frac{\partial a_r}{\partial\theta}\right)\boldsymbol{i}_z$.

- $\triangle_{\boldsymbol{x}}\varphi = \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2}$.

- $dV = r dr d\theta dz$.

Spherical coordinates



- $\boldsymbol{x} = r\sin(\chi)\cos(\theta)\boldsymbol{i}_x + r\sin(\chi)\sin(\theta)\boldsymbol{i}_y + r\cos(\chi)\boldsymbol{i}_z.$
  Coordinates r, θ, and χ.
  Orthonormal basis $\boldsymbol{i}_x$, $\boldsymbol{i}_y$, and $\boldsymbol{i}_z$.

- $\boldsymbol{a} = a_r\boldsymbol{i}_r(\theta,\chi) + a_\theta\boldsymbol{i}_\theta(\theta) + a_\chi\boldsymbol{i}_\chi(\theta,\chi).$
  $\boldsymbol{i}_r(\theta,\chi) = \sin(\chi)\cos(\theta)\boldsymbol{i}_x + \sin(\chi)\sin(\theta)\boldsymbol{i}_y + \cos(\chi)\boldsymbol{i}_z.$
  $\boldsymbol{i}_\theta(\theta) = -\sin(\theta)\boldsymbol{i}_x + \cos(\theta)\boldsymbol{i}_y.$
  $\boldsymbol{i}_\chi(\theta,\chi) = \cos(\chi)\cos(\theta)\boldsymbol{i}_x + \cos(\chi)\sin(\theta)\boldsymbol{i}_y - \sin(\chi)\boldsymbol{i}_z.$

- $\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi = \frac{\partial\varphi}{\partial r}\boldsymbol{i}_r(\theta,\chi) + \frac{1}{r\sin(\chi)}\frac{\partial\varphi}{\partial\theta}\boldsymbol{i}_\theta(\theta) + \frac{1}{r}\frac{\partial\varphi}{\partial\chi}\boldsymbol{i}_\chi.$

- $\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \frac{\partial a_r}{\partial r} + 2\frac{a_r}{r} + \frac{1}{r\sin(\chi)}\frac{\partial a_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial a_\chi}{\partial\chi} + \frac{\cot(\chi)}{r}a_\chi.$

- $\mathbf{curl}_{\boldsymbol{x}}\boldsymbol{a} = \left(\frac{1}{r}\frac{\partial a_\theta}{\partial\chi} + \frac{\cot(\chi)}{r}a_\theta - \frac{1}{r\sin(\chi)}\frac{\partial a_\chi}{\partial\theta}\right)\boldsymbol{i}_r(\theta,\chi) + \left(\frac{\partial a_\chi}{\partial r} + \frac{a_\chi}{r} - \frac{1}{r}\frac{\partial a_r}{\partial\chi}\right)\boldsymbol{i}_\theta(\theta)$
  $+\left(\frac{1}{r\sin(\chi)}\frac{\partial a_r}{\partial\chi} - \frac{\partial a_\theta}{\partial r} - \frac{a_\theta}{r}\right)\boldsymbol{i}_\chi(\theta,\chi).$

- $\triangle_{\boldsymbol{x}}\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin(\chi)^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{1}{r^2\sin(\chi)}\frac{\partial}{\partial\chi}\left(\sin(\chi)\frac{\partial\varphi}{\partial\chi}\right).$

- $dV = r^2\sin(\chi)dr d\theta d\chi.$

Review of Fourier analysis

Fourier analysis

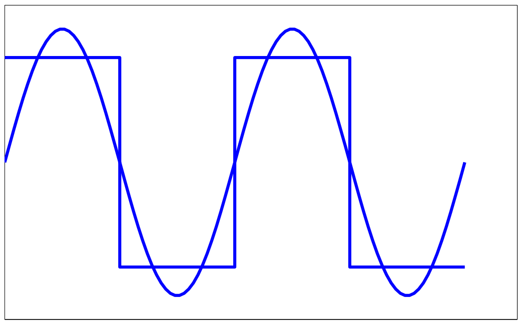
This is not a lecture but rather a summary of key elements of Fourier analysis. For a more complete treatment of Fourier analysis, please refer to MATH0007 Analyse Mathématique II (F. Bastin) and SYST0002 Modélisation et analyse des systèmes (R. Sepulchre).

- This slide recalls the Fourier series of a periodic function.
- Let $f : \mathbb{R} \to \mathbb{R}$ be a nonconstant **periodic function** that has period a and is square-integrable on $[-a/2, a/2]$, that is, $\int_{-a/2}^{a/2} |f(t)|^2 dt < +\infty$. Then, the **Fourier series** of f reads as follows:

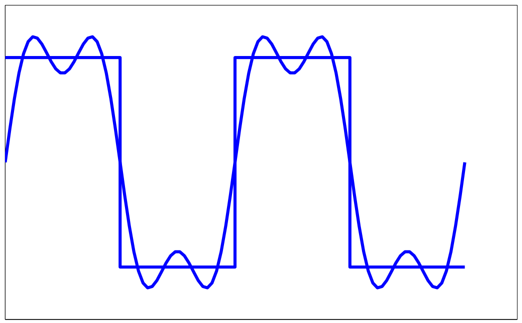
$$\begin{cases} f(t) = \displaystyle\sum_{k=-\infty}^{+\infty} f_k \exp\left(ik\frac{2\pi}{a}t\right), \\ f_k = \displaystyle\frac{1}{a}\int_{-a/2}^{a/2} f(t)\exp\left(-ik\frac{2\pi}{a}t\right) dt. \end{cases}$$

- It has the following approximation property:

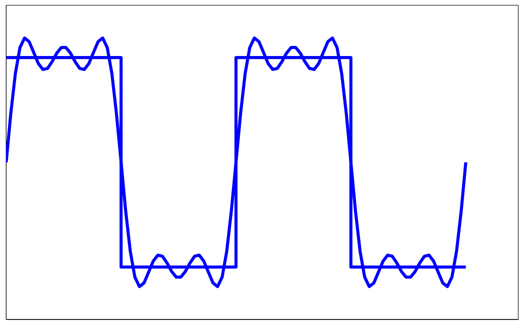
$$\lim_{n\to+\infty}\int_{-a/2}^{a/2}\left| f(t) - \sum_{k=-n}^{n} f_k \exp\left(ik\frac{2\pi}{a}t\right)\right|^2 dt = 0.$$



$n = 1.$



$n = 3.$



$n = 5.$

- This slide recalls the Fourier transform of a function (which need not be periodic).

- Let $f : \mathbb{R} \to \mathbb{R}$ be an **integrable function**, that is, $\int_{\mathbb{R}} |f(t)| dt < +\infty$. Then, the **Fourier transform** (FT) $\hat{f}$ of f is the bounded, continuous function $\hat{f}$ from $\mathbb{R}$ into $\mathbb{C}$ such that

$$\hat{f}(\omega) = \mathcal{F}f(\omega) = \int_{\mathbb{R}} \exp(-i\omega t) f(t) dt.$$

The Fourier transform of an integrable function is not necessarily integrable itself.

- Let $f : \mathbb{R} \to \mathbb{R}$ be a **square-integrable function** , that is, $\int_{\mathbb{R}} |f(t)|^2 dt < +\infty$. Then, the **Fourier transform** $\hat{f}$ of f is the square-integrable function $\hat{f}$ from $\mathbb{R}$ into $\mathbb{C}$ such that

$$\begin{cases} \hat{f}(\omega) = \mathcal{F}f(\omega) = \int_{\mathbb{R}} \exp(-i\omega t) f(t) dt, \\ f(t) = \mathcal{F}^{-1}\hat{f}(t) = \dfrac{1}{2\pi} \int_{\mathbb{R}} \exp(i\omega t) \hat{f}(\omega) d\omega. \end{cases}$$

- This slide recalls properties of the Fourier transform of a function (which need not be periodic).

- The Fourier transform interchanges **differentiation** and multiplication by a monomial:

$$d^k\hat{f}/d\omega^k = \widehat{(-it)^k f} \qquad \text{and} \qquad \widehat{d^k f/dt^k} = (i\omega)^k \hat{f}.$$

- The Fourier transform interchanges **convolution** and multiplication of functions. This means that if

$$(f \star g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds = \int_{\mathbb{R}} f(s)g(t-s)ds,$$

where $\star$ denotes the convolution operation, then

$$\widehat{f \star g}(\omega) = \hat{f}(\omega)\,\hat{g}(\omega).$$

- Lastly, we recall the application of Fourier analysis to linear ordinary differential equations (ODEs).
- **Ordinary Differential Equation (ODE)**:

$$\sum_{k=0}^{q} b_k \frac{d^k u_f}{dt^k}(t) = f(t), \quad t \in \mathbb{R}, \quad b_q \neq 0, \quad q \geq 1.$$

- **Algebraic equation** obtained by FT (if it exists):

$$\sum_{k=0}^{q} b_k (i\omega)^k \hat{u}_f(\omega) = \hat{f}(\omega), \quad \omega \in \mathbb{R}.$$

- **Frequency Response Function (FRF)**:

$$\hat{u}_f(\omega) = \hat{h}(\omega)\hat{f}(\omega) \qquad \text{where} \qquad \hat{h}(\omega) = \frac{1}{p(i\omega)} = \frac{1}{\sum_{k=0}^{q} b_k (i\omega)^k}.$$

  If $1/p$ has no poles on the imaginary axis, $\hat{h} : \mathbb{R} \to \mathbb{C}$ is a bounded, square-integrable function.
- **Impulse response function**:

$$h = \mathcal{F}^{-1}(\hat{h}).$$

  If $1/p$ has no poles on the imaginary axis, $h : \mathbb{R} \to \mathbb{R}$ is an integrable, square-integrable, and bounded function that decays rapidly at infinity and is continuous (except perhaps at the origin).
- **Generalized solution** :

$$u_f = h \star f, \quad \text{that is,} \quad u_f(t) = \int_{\mathbb{R}} h(s) f(t-s) ds, \quad \text{(using convolution that makes sense).}$$

References

Suggested reading material

- F. Bastin. MATH0007 Analyse Mathématique II. ULg. Lecture notes.
- E. Delhez. MATH0002 Analyse Mathématique. ULg. Lecture notes.
- E. Delhez. MATH0013 Algèbre. ULg. Lecture notes.
- R. Sepulchre. SYST0002 Modélisation et analyse des systèmes. ULg. Lecture notes.

Additional references also consulted to prepare this review

- D. Aubry. Mécanique des milieux continus. Ecole Centrale Paris. Lecture notes.
- C. Gasquet and P. Witomski. Analyse de Fourier et applications. Masson, 1990.
- J. Hladik and P. Hladik. Le calcul tensoriel en physique. Dunod, 1999.
- R. LeVeque. Finite-difference methods for ordinary and partial differential equations. SIAM, 2007.
- A. Lichnerowicz. Elements of tensor calculus. John Wiley & Sons, 1962.
- C. Semay and B. Silvestre-Brac. Introduction au calcul tensoriel. Dunod, 2009.