MATH0024 – Modeling with PDEs

Wave equation

Maarten Arnst and Romain Boman

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Outline

- Wave equation:
  - Wave operator.
  - Physical examples.
  - Vibrating string.
- Wave equation on all of space:
  - Fundamental solution.
  - Superposition formulae.
  - Properties of solutions.
- Wave equation on a portion of space:
  - Separation of variables.
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Wave equation

Wave operator

$$\frac{\partial^2}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} = \frac{\partial^2}{\partial t^2} - c^2 \mathsf{div}_{\boldsymbol{x}} \boldsymbol{\nabla}_{\boldsymbol{x}}.$$

Here, c is a fixed constant that has the dimension of a velocity; in fact, we shall see that it corresponds to the speed of propagation of waves determined by the wave equation.

Wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = f.$$

Areas of application in mechanics and physics

- The **wave equation** is the **mathematical prototype** of models of **wave propagation phenomena**. The equation can represent waves in acoustics, elastodynamics, electrodynamics, ...

Acoustics

- System of PDEs governing the dynamical behavior of a homogeneous inviscid acoustic fluid:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \rho_0 \text{div}_{\boldsymbol{x}} \boldsymbol{v} = q & \text{(linearized conservation of mass)}, \\ \textbf{div}_{\boldsymbol{x}} \boldsymbol{\sigma} = \rho_0 \dfrac{\partial \boldsymbol{v}}{\partial t} & \text{(linearized conservation of momentum)}, \\ \boldsymbol{\sigma} = -p\boldsymbol{I} \quad \text{with} \quad p = c_0^2 \rho & \text{(inviscid acoustic constitutive equation)}. \end{cases}$$

  Here, ρ, p, and $\boldsymbol{\sigma}$ are the disturbances of the mass density, the pressure, and the stress tensor about the equilibrium values, ρ_0 the mass density at equilibrium, c_0 the speed of sound in the fluid at equilibrium, q the mass source, and $\boldsymbol{v}$ the velocity.

- Inserting the constitutive equation into the conservation equations, we obtain

$$\begin{cases} \dfrac{\partial p}{\partial t} + \rho_0 c_0^2 \text{div}_{\boldsymbol{x}} \boldsymbol{v} = c_0^2 q, \\ -\boldsymbol{\nabla}_{\boldsymbol{x}} p = \rho_0 \dfrac{\partial \boldsymbol{v}}{\partial t}. \end{cases}$$

- Combining these results, we obtain the wave equation

$$\boxed{\frac{\partial^2 p}{\partial t^2} - c_0^2 \triangle_{\boldsymbol{x}} p = c_0^2 \frac{\partial q}{\partial t}.}$$

Elastic wave propagation

- System of PDEs governing the dynamical behavior of a homogeneous isotropic linear elastic solid:

$$\begin{cases} \mathbf{div}_{\boldsymbol{x}}\boldsymbol{\sigma} + \boldsymbol{f}_{\mathsf{v}} = \rho \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} & \text{(equilibrium equation)}, \\ \boldsymbol{\sigma} = \lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon} & \text{(constitutive equation)}, \\ \boldsymbol{\epsilon} = \dfrac{1}{2}\left(\mathbf{D}_{\boldsymbol{x}}\boldsymbol{u} + \mathbf{D}_{\boldsymbol{x}}\boldsymbol{u}^{\mathrm{T}}\right) & \text{(strain-displacement relationship)}. \end{cases}$$

Here, λ and μ are the Lamé parameters, which are related to the Young's modulus E and Poisson coefficient ν through $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$.

- Inserting the constitutive equation into the equilibrium equation, we obtain

$$\mathbf{div}_{\boldsymbol{x}}\left(\lambda \mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}\right) + \boldsymbol{f}_{\mathsf{v}} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}.$$

Using the properties $\mathbf{div}_{\boldsymbol{x}}(\varphi\boldsymbol{A}) = \boldsymbol{A}(\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi) + \varphi\mathbf{div}_{\boldsymbol{x}}\boldsymbol{A}$ and $\boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \mathbf{div}_{\boldsymbol{x}}(\mathbf{D}_{\boldsymbol{x}}\boldsymbol{a}^{\mathrm{T}})$ and the definition $\mathrm{div}_{\boldsymbol{x}}\boldsymbol{a} = \mathrm{tr}(\mathbf{D}_{\boldsymbol{x}}\boldsymbol{a})$ (Lecture 1 Part B), we obtain

$$\mathbf{div}_{\boldsymbol{x}}\left(\mathrm{tr}(\boldsymbol{\epsilon})\boldsymbol{I}\right) = \boldsymbol{\nabla}_{\boldsymbol{x}}\left(\mathrm{tr}(\boldsymbol{\epsilon})\right) = \boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{u} \quad \text{and} \quad \mathbf{div}_{\boldsymbol{x}}\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{div}_{\boldsymbol{x}}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{u} + \boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{u}\right).$$

Combining these results, we obtain the **Navier equation**

$$\boxed{(\lambda + \mu)\boldsymbol{\nabla}_{\boldsymbol{x}}\mathrm{div}_{\boldsymbol{x}}\boldsymbol{u} + \mu\mathbf{div}_{\boldsymbol{x}}\mathbf{D}_{\boldsymbol{x}}\boldsymbol{u} + \boldsymbol{f}_{\mathsf{v}} = \rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2}.}$$

Elastic wave propagation (continued)

- Using the property $\boldsymbol{\nabla}_{\boldsymbol{x}} \text{div}_{\boldsymbol{x}} \boldsymbol{a} = \textbf{div}_{\boldsymbol{x}} \mathbf{D}_{\boldsymbol{x}} \boldsymbol{a} + \textbf{curl}_{\boldsymbol{x}} \textbf{curl}_{\boldsymbol{x}} \boldsymbol{a}$, we can write this equivalently as

$$(\lambda + 2\mu) \boldsymbol{\nabla}_{\boldsymbol{x}} \text{div}_{\boldsymbol{x}} \boldsymbol{u} - \mu \textbf{curl}_{\boldsymbol{x}} \textbf{curl}_{\boldsymbol{x}} \boldsymbol{u} + \boldsymbol{f}_{\text{v}} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}.$$

- The Helmholtz theorem indicates that in 3D, any sufficiently regular vector field can be decomposed into a curl-free and a divergence-free component. Here, the Helmholtz decomposition of $\boldsymbol{u}$ reads as

$$\boldsymbol{u} = \boldsymbol{\nabla}_{\boldsymbol{x}} \phi + \textbf{curl}_{\boldsymbol{x}} \boldsymbol{\psi} \qquad \text{with} \qquad \text{div}_{\boldsymbol{x}} \boldsymbol{\psi} = 0.$$

  With $\textbf{curl}_x \boldsymbol{\nabla}_{\boldsymbol{x}} \varphi = 0$, $\text{div}_{\boldsymbol{x}} \textbf{curl}_x \boldsymbol{a} = 0$, and $\boldsymbol{\nabla}_{\boldsymbol{x}} \text{div}_{\boldsymbol{x}} \boldsymbol{a} = \textbf{div}_{\boldsymbol{x}} \mathbf{D}_{\boldsymbol{x}} \boldsymbol{a} + \textbf{curl}_{\boldsymbol{x}} \textbf{curl}_{\boldsymbol{x}} \boldsymbol{a}$, we obtain

$$\text{div}_{\boldsymbol{x}} \boldsymbol{u} = \underbrace{\text{div}_{\boldsymbol{x}} \boldsymbol{\nabla}_{\boldsymbol{x}} \phi}_{=\triangle_{\boldsymbol{x}} \phi} + \underbrace{\text{div}_{\boldsymbol{x}} \textbf{curl}_x \boldsymbol{\psi}}_{=0} \quad \text{and} \quad \textbf{curl}_x \boldsymbol{u} = \underbrace{\textbf{curl}_x \boldsymbol{\nabla}_{\boldsymbol{x}} \phi}_{=0} + \underbrace{\boldsymbol{\nabla}_{\boldsymbol{x}} \text{div}_{\boldsymbol{x}} \boldsymbol{\psi}}_{=0} - \textbf{div}_{\boldsymbol{x}} \mathbf{D}_{\boldsymbol{x}} \boldsymbol{\psi}.$$

  Combining these results, we can write the Navier equation equivalently as

$$\boldsymbol{\nabla}_{\boldsymbol{x}} \left((\lambda + 2\mu) \triangle_{\boldsymbol{x}} \phi - \rho \frac{\partial^2 \phi}{\partial t^2} \right) + \textbf{curl}_x \left(\mu \textbf{div}_{\boldsymbol{x}} \mathbf{D}_{\boldsymbol{x}} \boldsymbol{\psi} - \rho \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} \right) + \boldsymbol{f}_{\text{v}} = 0.$$

- Thus, in regions of space where there are no external volume forces, that is, $\boldsymbol{f}_{\text{v}} = \boldsymbol{0}$, the displacement is the sum of a curl-free component which derives from a scalar potential ϕ that satisfies

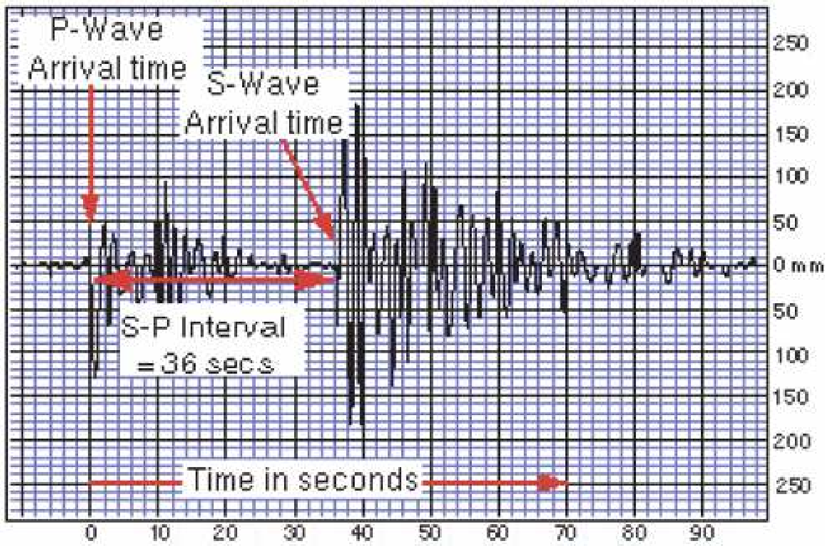
$$\boxed{\frac{\partial^2 \phi}{\partial t^2} - c_{\text{p}}^2 \triangle_{\boldsymbol{x}} \phi = 0 \text{ with } c_{\text{p}} = \sqrt{\frac{\lambda + 2\mu}{\rho}},}$$

  and a divergence-free component which derives from a vector potential $\boldsymbol{\psi}$ that satisfies

$$\boxed{\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} - c_{\text{s}}^2 \textbf{div}_{\boldsymbol{x}} \mathbf{D}_{\boldsymbol{x}} \boldsymbol{\psi} = 0 \text{ with } c_{\text{s}} = \sqrt{\frac{\mu}{\rho}}.}$$

Elastic wave propagation (continued)

- We have seen that two types of waves can propagate through a homogeneous isotropic linear elastic solid, namely, waves with a speed of propagation of $c_{\text{p}} = \sqrt{\lambda + 2\mu/\rho}$ (also called P-waves) and waves with a speed of propagation of $c_{\text{s}} = \sqrt{\mu/\rho}$ (also called S waves).
- One aspect of analyzing seismic records involves determining the arrival time of P and S waves.



Given c_{p} and c_{s}, the length of the time interval between the arrival of the P and S waves after an earthquake allows the distance between the seismic recording station and the epicenter of the earthquake to be estimated. Repeating such an analysis for multiple seismic recording stations allows the location of the epicenter of the earthquake to be estimated.

Electromagnetic wave propagation

- In regions of space free of charges and currents, Maxwell's equations read as

$$\begin{cases} \text{div}_{\boldsymbol{x}}\boldsymbol{E} = 0, \\ \textbf{curl}_{\boldsymbol{x}}\boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}, \\ \text{div}_{\boldsymbol{x}}\boldsymbol{B} = 0, \\ c^2\textbf{curl}_{\boldsymbol{x}}\boldsymbol{B} = \dfrac{\partial \boldsymbol{E}}{\partial t}. \end{cases}$$

Taking the curl of the second and fourth equations, and using the first and third equation, we obtain

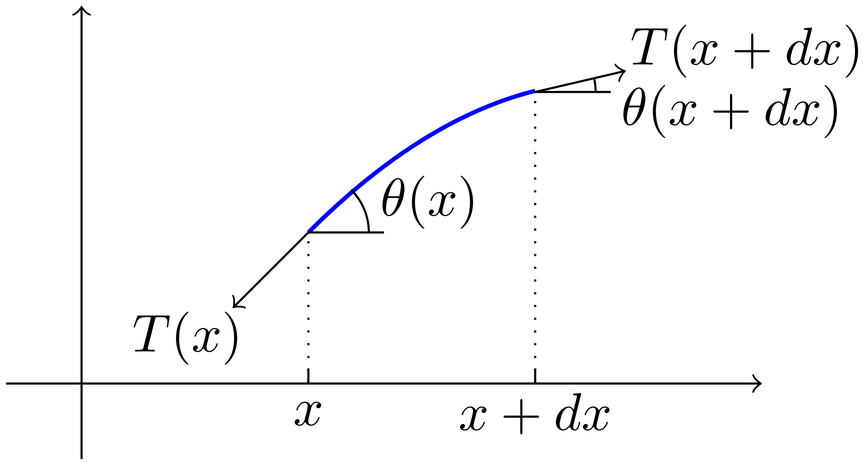
$$\underbrace{\textbf{curl}_{\boldsymbol{x}}\textbf{curl}_{\boldsymbol{x}}\boldsymbol{E}}_{=-\textbf{div}_{\boldsymbol{x}}\textbf{D}_{\boldsymbol{x}}\boldsymbol{E}} = -\frac{\partial}{\partial t}\underbrace{\textbf{curl}_{\boldsymbol{x}}\boldsymbol{B}}_{=\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}} \quad \text{and} \quad c^2\underbrace{\textbf{curl}_{\boldsymbol{x}}\textbf{curl}_{\boldsymbol{x}}\boldsymbol{B}}_{=-\textbf{div}_{\boldsymbol{x}}\textbf{D}_{\boldsymbol{x}}\boldsymbol{B}} = \frac{\partial}{\partial t}\underbrace{\textbf{curl}_{\boldsymbol{x}}\boldsymbol{E}}_{=-\frac{\partial \boldsymbol{B}}{\partial t}}$$

- Thus, in regions of space where there are no charges and no currents, we have

$$\boxed{\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2\textbf{div}_{\boldsymbol{x}}\textbf{D}_{\boldsymbol{x}}\boldsymbol{E} = 0,}$$

$$\boxed{\frac{\partial^2 \boldsymbol{B}}{\partial t^2} - c^2\textbf{div}_{\boldsymbol{x}}\textbf{D}_{\boldsymbol{x}}\boldsymbol{B} = 0.}$$

Vibrating string model



T: tensile force,
u: displacement,
θ: angle,
ρ: mass density.

- The dynamical equilibrium of the string reads as follows:

$$T\big(x+dx,t\big)\sin\big(\theta(x+dx)\big) - T\big(x,t\big)\sin\big(\theta(x,t)\big) = \rho dx \frac{\partial^2 u}{\partial t^2}.$$

- Let the vibration be of small amplitude so that the angle can be assumed to be small and the tensile force can be assumed to be constant. Then, we obtain

$$\underbrace{T\big(x+dx,t\big)}_{=T_0}\underbrace{\sin\big(\theta(x+dx)\big)}_{=\frac{\partial u}{\partial x}(x+dx,t)} - \underbrace{T\big(x,t\big)}_{=T_0}\underbrace{\sin\big(\theta(x,t)\big)}_{=\frac{\partial u}{\partial x}(x)} = \rho dx \frac{\partial^2 u}{\partial t^2}.$$

Hence, dividing both sides by dx and taking the limit as $dx \to 0$, we find

$$\boxed{\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \text{ with } c = \sqrt{\frac{T_0}{\rho}}.}$$

Initial-value problem

- Let us consider the initial-value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R}\times]0,+\infty[, \\ u = g & \text{on } \mathbb{R}\times\{t=0\}, \\ \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R}\times\{t=0\}. \end{cases}$$

- By the change of variables $\xi = x + ct$ and $\eta = x - ct$, the PDE can be written as $\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$, whose solution is the sum $u = \phi(\xi) + \psi(\eta)$ with sufficiently regular functions ϕ and ψ; hence,

$$u(x,t) = \phi(x+ct) + \psi(x-ct).$$

- The functions ϕ and ψ can be obtained by inspection from the initial conditions: requiring that

$$g(x) = \phi(x) + \psi(x) \quad \text{and} \quad w(x) = c\phi'(x) - c\psi'(x),$$

we obtain by integration of $g'(x) + \frac{w(x)}{c} = 2\phi'(x)$ and $g'(x) - \frac{w(x)}{c} = 2\psi'(x)$ that

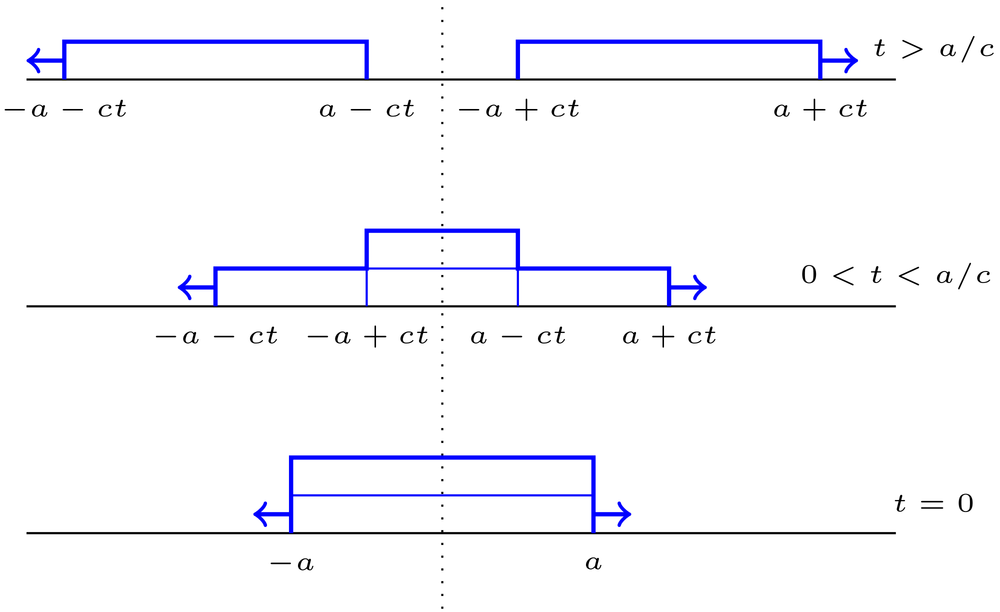
$$\phi(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_0^x w(z)dz + k \quad \text{and} \quad \psi(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_0^x w(z)dz + k.$$

- Combining these results, we obtain the **D'Alembert representation**

$$\boxed{u(x,t) = \tfrac{1}{2}\big(g(x+ct) + g(x-ct)\big) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} w(z)dz.}$$

Initial-value problem (continued)

- As an example, let us consider the motion of an infinite vibrating string that is initially deformed into the shape of a single rectangular pulse and then let go from rest:

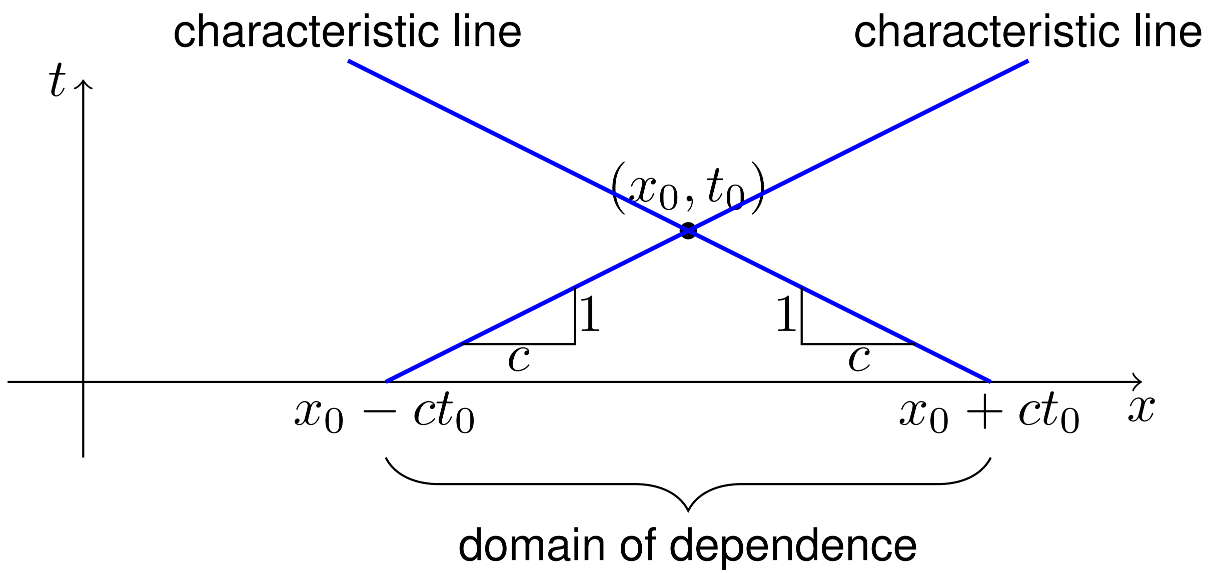


$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases} & \text{on } \mathbb{R} \times \{t=0\}, \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{on } \mathbb{R} \times \{t=0\}. \end{cases}$$

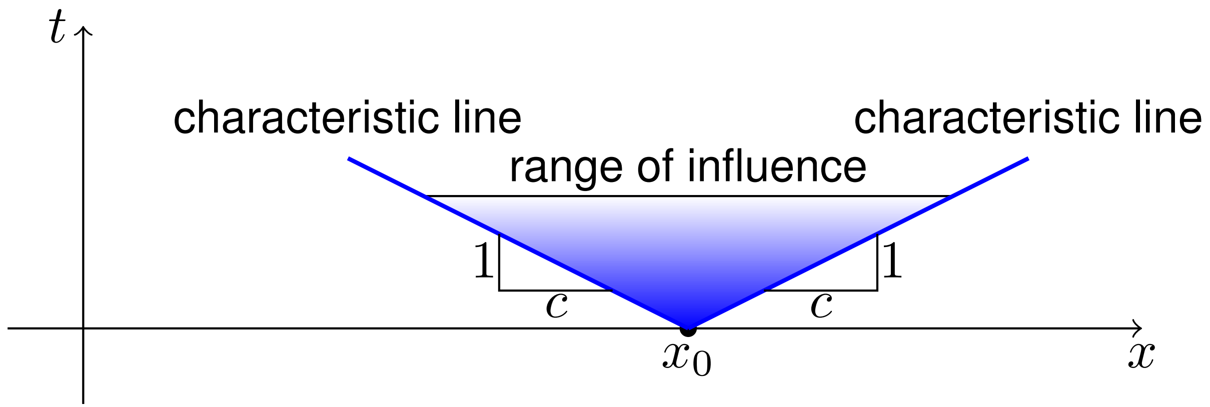
- The D'Alembert representation and this example show that:
  - The initial conditions split into two parts, one moving to the right and the other moving to the left, both with a **finite speed of propagation** of c.
  - The regularity of the solution is determined by the regularity of the initial conditions; in general, **the solution is not smoother than the initial conditions**.
  - The solution at location x and time t depends only on the initial data in the interval $[x - ct, x + ct]$. Thus, the finite speed of propagation gives rise to a **finite domain of dependence**.

Domain of dependence and range of influence

- The solution at location x_0 and time t_0 depends only on the initial data in the interval $[x_0 - ct_0, x_0 + ct_0]$, also called the **domain of dependence**:

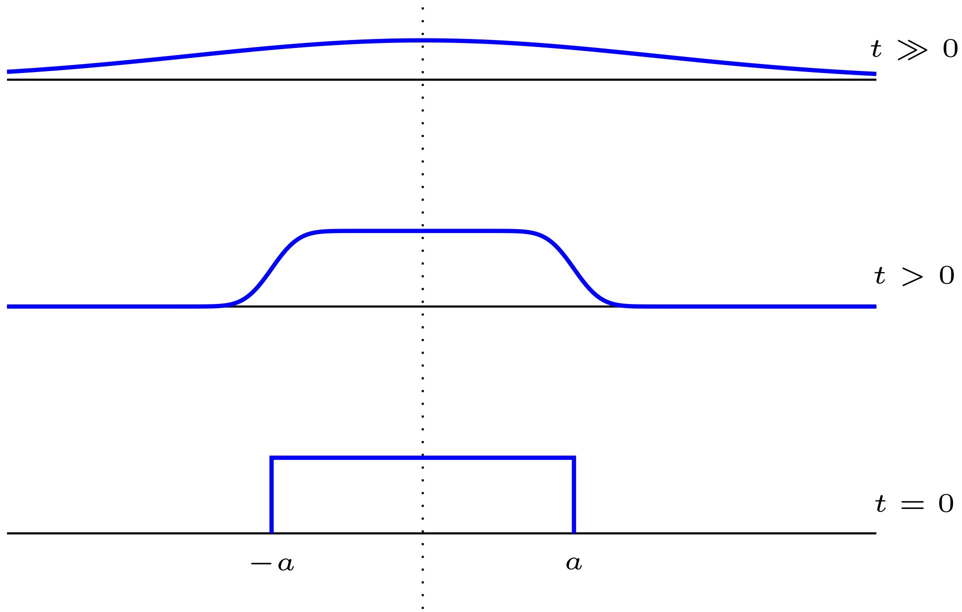


- Conversely, the initial data at location x_0 only affect the solution in the **range of influence**:

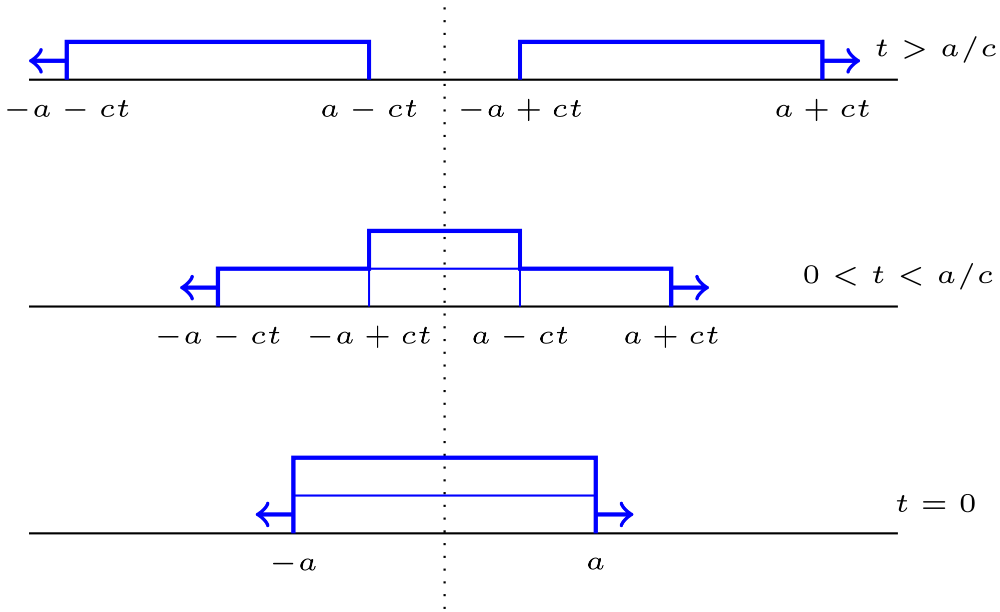


Heat equation versus wave equation

Heat equation:



Wave equation:



$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases} & \text{on } \mathbb{R}\times\{t=0\}. \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R}\times]0,+\infty[, \\ u(x,0) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases} & \text{on } \mathbb{R}\times\{t=0\}, \\ \dfrac{\partial u}{\partial t}(x,0) = 0 & \text{on } \mathbb{R}\times\{t=0\}. \end{cases}$$

Infinite speed of propagation.
Smoothening.
Infinite domain of dependence.

Finite speed of propagation.
No smoothening.
Finite domain of dependence.

Wave equation on all of space

Notion of fundamental solution for wave equation

- A **fundamental solution** Φ for the wave equation is a solution that solves the wave equation for a Dirac impulse $\delta_{(\mathbf{0},0)}$ centered at $(\boldsymbol{x},t) = (\mathbf{0},0)$ on the right-hand side,
$$\frac{\partial^2 \Phi}{\partial t^2}(\boldsymbol{x},t) - c^2 \triangle_{\boldsymbol{x}} \Phi(\boldsymbol{x},t) = \delta_{(\mathbf{0},0)}(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in \mathbb{R}^m \times \mathbb{R}.$$
- As in Lectures 2 and 6, the theory of distributions must be used here to define mathematically fully rigorously the derivatives and the Dirac impulse involved in this equation.

Retarded Green's function

- It can be shown that the so called **retarded Green's function** Φ such that
$$\Phi(\boldsymbol{x},t) = \begin{cases} 0 & \text{for } t < 0, \\ \Phi_t(\boldsymbol{x}) & \text{for } t \geq 0, \end{cases}$$
where Φ_t is defined such that
$$\int_{\mathbb{R}^m} \Phi_t(\boldsymbol{x}) \varphi(\boldsymbol{x}) dV_{\boldsymbol{x}} = \begin{cases} \dfrac{t}{2} \displaystyle\int_{-1}^{1} \varphi(cty) dy & \text{in 1D (m=1)}, \\ \dfrac{t}{2\pi} \displaystyle\int_{\|\boldsymbol{y}\| \leq 1} \varphi(ct\boldsymbol{y}) \sqrt{\frac{1}{1 + \|\boldsymbol{y}\|^2}} dV_{\boldsymbol{y}} & \text{in 2D (m=2)}, \\ \dfrac{t}{4\pi} \displaystyle\int_{\|\boldsymbol{y}\| = 1} \varphi(ct\boldsymbol{y}) dS_{\boldsymbol{y}} & \text{in 3D (m=3)}, \end{cases}$$
for all smooth functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ with closed and bounded support, is a fundamental solution for the wave equation. This fundamental solution is a **generalized solution**.

Retarded Green's function in 1D ($m = 1$)

- Let us recall from the previous slide that in 1D ($m = 1$), we have

$$\int_{\mathbb{R}} \Phi_t(x)\varphi(x)dx = \frac{t}{2}\int_{-1}^{1} \varphi(cty)dy = \frac{1}{2c}\int_{-ct}^{ct} \varphi(x)dx$$

for all smooth functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ with closed and bounded support for $t \geq 0$.

We can observe that in 1D ($m = 1$), Φ_t is defined implicitly by how it acts on "test functions." The same held in 2D ($m = 2$) and 3D ($m = 3$) on the previous slide.

- In 1D ($m = 1$), we can identify Φ_t with the function

$$\Phi_t(x) = \frac{1}{2c}1_{[-ct,ct]}(x) \qquad \text{for } t \geq 0,$$

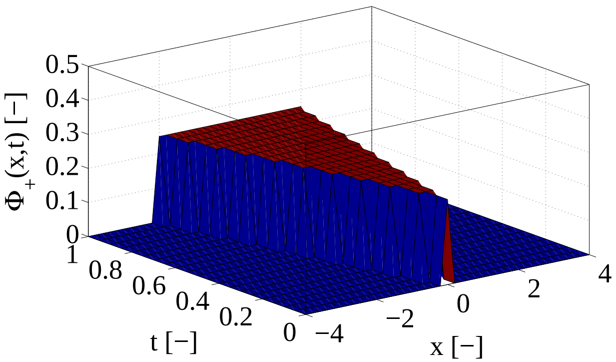
where $1_{[-ct,ct]}$ is the indicator function of the interval $[-ct, ct]$, that is,

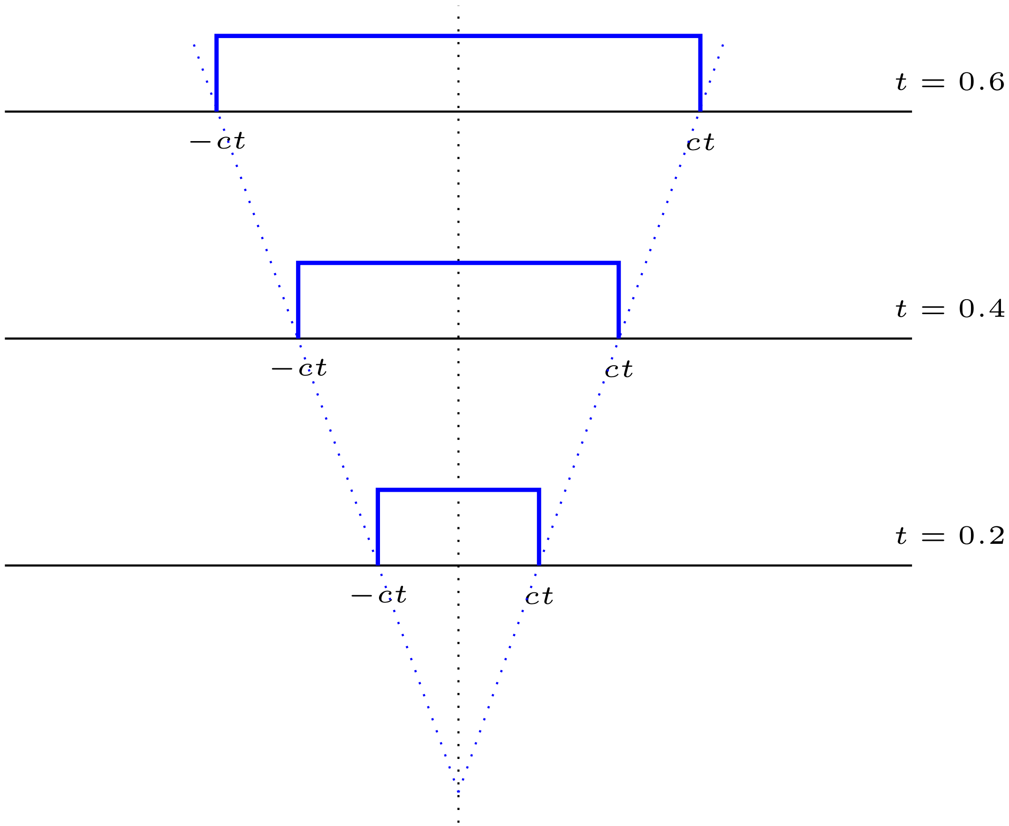
$$1_{[-ct,ct]} = \begin{cases} 1 & \text{if } x \text{ is in } [-ct, ct], \\ 0 & \text{otherwise.} \end{cases}$$

We can observe that in 1D ($m = 1$), Φ_t is discontinuous along the characteristic lines $x = ct$ and $x = -ct$, thus confirming that it is a generalized solution.

Retarded Green's function in 1D ($m = 1$) (continued)

- The following figures show the retarded Green's function in 1D ($m = 1$):





- We can observe that at any time $t > 0$, the retarded Green's function has a bounded support. This is consistent with the property of the wave equation entailing a finite speed of propagation.

- We can observe that in 1D ($m = 1$), the retarded Green's function is an expanding rectangular pulse that widens at the speed of propagation c.

Retarded Green's function in 1D ($m = 1$) (continued)

- The retarded Green's function being a fundamental solution can be proven through Fourier analysis.
- A fundamental solution Φ for the wave equation must satisfy
$$\frac{\partial^2 \Phi}{\partial t^2}(x,t) - c^2 \frac{\partial^2 \Phi}{\partial x^2}(x,t) = \delta_{(0,0)}(x,t), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}.$$
- Taking the Fourier transform of this equation with respect to the x variable, we obtain
$$\frac{\partial^2 \widehat{\Phi}}{\partial t^2}(\xi,t) + c^2 |\xi|^2 \widehat{\Phi}(\xi,t) = \delta_0(t), \qquad (\xi,t) \in \mathbb{R} \times \mathbb{R}.$$
- This equation is solved by the function
$$\widehat{\Phi}(\xi,t) = H(t) \frac{\sin(ct|\xi|)}{c|\xi|},$$
where H is the Heavyside function such that $H(t) = 1$ if $t \geq 0$ and $H(t) = 0$ otherwise. Indeed, the second derivative of $\widehat{\Phi}(\xi,t)$ w.r.t. t in the sense of the distributions read as
$$\frac{\partial^2 \widehat{\Phi}}{\partial t^2}(\xi,t) = -H(t)c|\xi| \sin(ct|\xi|) + \delta_0(t)\cos(ct|\xi|) = -H(t)c|\xi| \sin(ct|\xi|) + \delta_0(t).$$
- Because the functions $\frac{1}{2a} 1_{[-a,a]}(x)$ and $\frac{\sin(a\xi)}{a\xi}$ form a Fourier transform pair, we obtain
$$\Phi(x,t) = H(t) \frac{1}{2c} 1_{[-ct,ct]}(x).$$

Superposition formulae

- A fundamental solution allows **superposition formulae** to be established for determining a solution to the wave equation for a general right-hand side:

  - To facilitate the formulation of these superposition formulae, let us introduce Ψ_t such that
  $$\int_{\mathbb{R}^m} \Psi_t(\boldsymbol{x})\varphi(\boldsymbol{x})dV_{\boldsymbol{x}} = \frac{d}{dt}\int_{\mathbb{R}^m} \Phi_t(\boldsymbol{x})\varphi(\boldsymbol{x})dV_{\boldsymbol{x}}$$
  for all smooth functions $\varphi : \mathbb{R}^m \to \mathbb{R}$ with closed and bounded support. Hence, just like Φ_t, we can observe that Ψ_t is defined implicitly by how it acts on “test functions.”

  - Initial-value problem:
  Given sufficiently regular functions g and w from $\mathbb{R}^m$ into $\mathbb{R}$, the problem
  $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = 0 & \text{in } \mathbb{R}^m \times]0, +\infty[, \\ u = g \quad \text{and} \quad \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R}^m \times \{t = 0\}, \end{cases}$$
  is solved by the sum of the convolution of Ψ_t with g and the convolution of Φ_t with w:
  $$u(\boldsymbol{x}, t) = \Psi_t \star g(\boldsymbol{x}) + \Phi_t \star w(\boldsymbol{x}).$$

Superposition formulae (continued)

- Inhomogeneous problem:
  Given a sufficiently regular function f from $\mathbb{R}^m \times]0, +\infty[$ into $\mathbb{R}$, the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = f & \text{in } \mathbb{R}^m \times]0, +\infty[, \\ u = 0 \quad \text{and} \quad \dfrac{\partial u}{\partial t} = 0 & \text{on } \mathbb{R}^m \times \{t = 0\}, \end{cases}$$

  is solved by the convolution of Φ with f:

$$u(\boldsymbol{x}, t) = \Phi \star f(\boldsymbol{x}, t).$$

- Inhomogeneous problem with general initial data:
  The previous results can be combined to assert that given sufficiently regular functions f from $\mathbb{R}^m \times]0, +\infty[$ into $\mathbb{R}$ and g and w from $\mathbb{R}^m$ into $\mathbb{R}$, the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = f & \text{in } \mathbb{R}^m \times]0, +\infty[, \\ u = g \quad \text{and} \quad \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R}^m \times \{t = 0\}, \end{cases}$$

  is solved by the convolution of Φ with f:

$$u(\boldsymbol{x}, t) = \Phi \star f(\boldsymbol{x}, t) + \Psi_t \star g(\boldsymbol{x}) + \Phi_t \star w(\boldsymbol{x}).$$

Superposition formulae in 1D ($m = 1$)

- In 1D ($m = 1$), the aforementioned superposition formulae take the following form:
  - Initial-value problem:
    Given sufficiently regular functions g and w, the solution $u(x,t) = \Psi_t \star g(x) + \Phi_t \star w(x)$ to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R} \times]0,+\infty[, \\ u = g \quad \text{and} \quad \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

    can be written equivalently in the form of the **D'Alembert representation**

$$\boxed{u(x,t) = \tfrac{1}{2}\Big(g(x+ct) + g(x-ct)\Big) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} w(y)dy.}$$

  - Inhomogeneous problem:
    Given a sufficiently regular function f, the solution $u(x,t) = \Phi \star f(x,t)$ to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = f & \text{in } \mathbb{R} \times]0,+\infty[, \\ u = 0 \quad \text{and} \quad \dfrac{\partial u}{\partial t} = 0 & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

    can be written equivalently as follows:

$$\boxed{u(x,t) = \int_0^t \tfrac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(z,s)dzds.}$$

Superposition formulae in 1D ($m = 1$) (continued)

- These representations follow from developing the convolutions in the superposition formulae:
  - For $\Phi_t(x) = \frac{1}{2c} 1_{[-ct,ct]}(x)$, we have $\Psi_t(x) = \frac{1}{2}\big(\delta(x-ct) + \delta(x+ct)\big)$ because

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2c} 1_{[-ct,ct]}(x)\varphi(x)dx &= \frac{d}{dt}\left(\frac{1}{2c}\int_{-ct}^{+ct}\varphi(x)dx\right)\\ &= \frac{1}{2c}c\varphi(ct) - \frac{1}{2c}(-c)\varphi(-ct)\\ &= \int_{\mathbb{R}} \frac{1}{2}\big(\delta(x-ct)+\delta(x+ct)\big)\varphi(x)dx\end{aligned}$$

    for all smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with closed and bounded support.
  - Using $\Phi_t(x) = \frac{1}{2c} 1_{[-ct,ct]}(x)$ and $\Psi_t(x) = \frac{1}{2}\big(\delta(x-ct) + \delta(x+ct)\big)$, we obtain

$$\begin{aligned}\Psi_t \star g(x) + \Phi_t \star w(x) &= \int_{\mathbb{R}} \frac{1}{2}\big(\delta(x-z-ct)+\delta(x-z+ct)\big)g(z)dz + \int_{\mathbb{R}} \frac{1}{2c} 1_{[-ct,ct]}(x-z)w(z)dz\\ &= \frac{1}{2}\Big(g(x+ct)+g(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} w(z)dz.\end{aligned}$$

  - Setting $f(\cdot,t) = 0$ for $t < 0$, we obtain

$$\Phi \star w(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(x-z,t-s)f(z,s)dzds = \int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(z,s)dzds.$$

Superposition formulae in 1D ($m = 1$) (continued)

- One way of proving the superposition formulae involves the use of Fourier analysis:
  - For example, let us consider the initial-value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{in } \mathbb{R} \times]0, +\infty[, \\ u = g \quad \text{and} \quad \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

  - Taking the Fourier transform of these equations with respect to the x variable, we obtain

$$\begin{cases} \dfrac{\partial^2 \hat{u}}{\partial t^2} + c^2 |\xi|^2 \hat{u} = 0 & \text{in } \mathbb{R} \times]0, +\infty[, \\ \hat{u} = \hat{g} \quad \text{and} \quad \dfrac{\partial \hat{u}}{\partial t} = \hat{w} & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

  - The solution to this problem reads as

$$\hat{u}(\xi, t) = \hat{g}(\xi) \cos(c|\xi|t) + \hat{w}(\xi) \frac{\sin(c|\xi|t)}{c|\xi|}.$$

  - Because $\Phi_t(x) = \frac{1}{2c} 1_{[-ct,ct]}(x)$ and $\sin(c|\xi|t)/c|\xi|$, as well as $\Psi_t(x) = \frac{1}{2}\big(\delta(x - ct) + \delta(x + ct)\big)$ and $\cos(c|\xi|t)$, form Fourier transform pairs, it follows that

$$u(x, t) = \Psi_t \star g(x) + \Phi_t \star w(x).$$

  - Of course, to make this proof mathematically fully rigorous, the functions g and w must be sufficiently regular for the Fourier transforms and the convolutions to make sense.

Superposition formulae in 3D ($m = 3$)

- In 3D ($m = 3$), the aforementioned superposition formulae take the following form (proof omitted):
  - Initial-value problem:
    Given sufficiently regular functions g and w, the solution $u(\boldsymbol{x}, t) = \Psi_t \star g(\boldsymbol{x}) + \Phi_t \star w(\boldsymbol{x})$ to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = 0 & \text{in } \mathbb{R}^3 \times]0, +\infty[, \\ u = g \quad \text{and} \quad \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

    can be written equivalently in the form of the **Kirchhoff representation**

$$\boxed{u(\boldsymbol{x}, t) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi c^2 t} \int_{\|\boldsymbol{z}-\boldsymbol{x}\|=ct} g(\boldsymbol{z}) dS_{\boldsymbol{z}} \right) + \frac{1}{4\pi c^2 t} \int_{\|\boldsymbol{z}-\boldsymbol{x}\|=ct} w(\boldsymbol{z}) dS_{\boldsymbol{z}}.}$$

  - Inhomogeneous problem:
    Given a sufficiently regular function f, the solution $u(\boldsymbol{x}, t) = \Phi \star f(\boldsymbol{x}, t)$ to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = f & \text{in } \mathbb{R}^3 \times]0, +\infty[, \\ u = 0 \quad \text{and} \quad \dfrac{\partial u}{\partial t} = 0 & \text{on } \mathbb{R}^3 \times \{t = 0\}, \end{cases}$$

    can be written equivalently as follows:

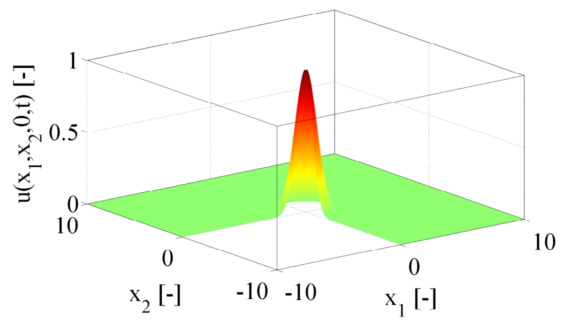
$$\boxed{u(\boldsymbol{x}, t) = \int_0^t \frac{1}{4\pi c^2 (t-s)} \int_{\|\boldsymbol{z}-\boldsymbol{x}\|=c(t-s)} f(\boldsymbol{z}, s) dS_{\boldsymbol{z}} ds.}$$

Superposition formulae in 3D ($m = 3$) (continued)

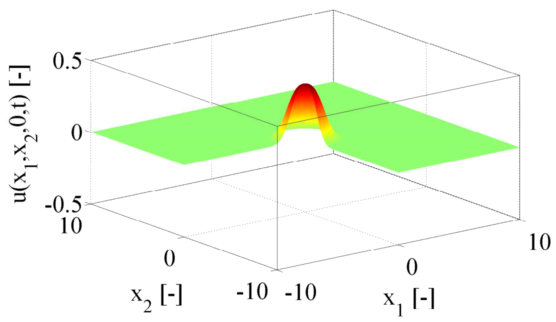
- As an example, let us consider the initial-value problem with initial data $g(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|^2/\rho^2)$ and $w(\boldsymbol{x}) = 0$. Then, leaving the details as an exercise, the Kirchhoff representation leads to

$$u(\boldsymbol{x}, t) = \exp\left(-\frac{c^2t^2 + \|\boldsymbol{x}\|^2}{\rho^2}\right)\left(\cosh\left(\frac{2ct\|\boldsymbol{x}\|}{\rho^2}\right) - \frac{ct}{\|\boldsymbol{x}\|}\sinh\left(\frac{2ct\|\boldsymbol{x}\|}{\rho^2}\right)\right).$$

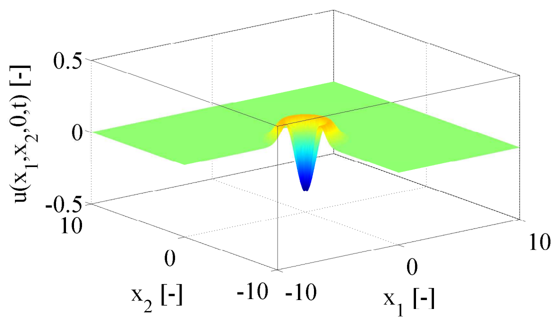
The wave generated by these initial conditions with $c = 1$ and $\rho = 1$:



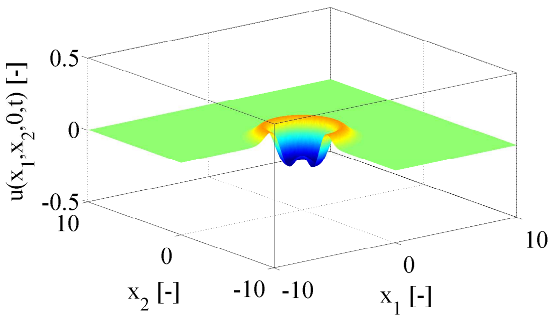
$t = 0.$



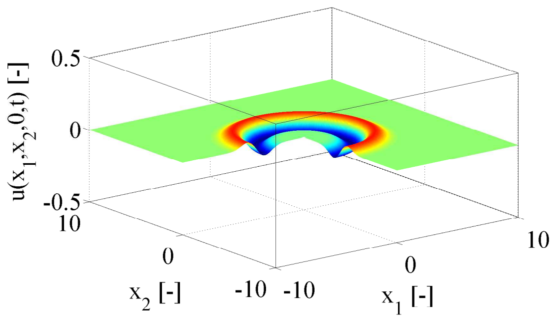
$t = 0.5.$



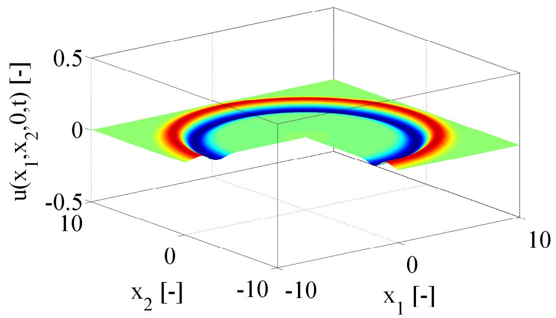
$t = 1.$



$t = 2.$



$t = 4.$



$t = 8.$

The solution $u(x_1, x_2, x_3, t)$ is plotted as a function of x_1 and x_2 for $x_3 = 0$ at different times t.

Superposition formulae in 3D ($m = 3$) (continued)

- A key consequence of the Kirchhoff formula is that in 3D ($m = 3$), the value taken by the solution to an initial-value problem involving the wave equation at a given point $\boldsymbol{x}$ and time t depends only on the values of the initial data on the sphere centered at $\boldsymbol{x}$ and of radius ct. In physics, this property is known as the **Huygens principle**.

  The Huygens principle ensures that if the initial data are concentrated near a given point, that is, if the initial data are nonvanishing only near this point, then the solution will remain concentrated near a progressively expanding sphere surrounding this point.

  For example, the Huygens principle ensures that any light that we now observe from a distant star was emitted at a past time proportional to the distance between this star and Earth.

  The Huygens principle does not hold in in 1D ($m = 1$) or 2D ($m = 2$), where concentrated initial data may spread out as time advances.

Superposition formulae in 3D ($m = 3$) (continued)

- As an alternative to obtaining it by making explicit the convolutions in the superposition formulae, the Kirchhoff representation can also be proven by the **method of spherical means**:
  - Given sufficiently regular initial data g and w (specifically, let g be in $C^3(\mathbb{R}^3)$ and w in $C^2(\mathbb{R}^3)$), let u be the solution to the initial-value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = 0 & \text{in } \mathbb{R}^3 \times]0, +\infty[, \\ u = g \quad \text{and} \quad \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R}^3 \times \{t = 0\}. \end{cases}$$

  - The first step is to define the so-called spherical mean $m_u(\boldsymbol{x}, r, t)$ as the mean of $u(\cdot, t)$ over the sphere centered at $\boldsymbol{x}$ of radius $r > 0$, that is,

$$m_u(\boldsymbol{x}, r, t) = \frac{1}{4\pi r^2} \int_{\|\boldsymbol{z}-\boldsymbol{x}\|=r} u(\boldsymbol{z}, t) dS_{\boldsymbol{z}} = \frac{1}{4\pi} \int_{\|\boldsymbol{y}\|=1} u(\boldsymbol{x} + r\boldsymbol{y}, t) dS_{\boldsymbol{y}}.$$

  - The next step is to show that for a fixed $\boldsymbol{x}$ in $\mathbb{R}^3$, the function $(r, t) \mapsto r m_u(\boldsymbol{x}, r, t)$ is the solution to an IBVP involving the 1D wave equation. Differentiating with respect to r, we obtain

$$\frac{\partial m_u}{\partial r}(\boldsymbol{x}, r, t) = \frac{1}{4\pi} \int_{\|\boldsymbol{y}\|=1} \boldsymbol{y} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}} u(\boldsymbol{x} + r\boldsymbol{y}, t) dS_{\boldsymbol{y}}.$$

Superposition formulae in 3D ($m = 3$) (continued)

- Because $\boldsymbol{y}$ is the unit outward normal, we can apply Stokes's theorem to obtain
$$\frac{\partial m_u}{\partial r}(\boldsymbol{x}, r, t) = \frac{1}{4\pi}\int_{\|\boldsymbol{y}\|\leq 1} \text{div}_{\boldsymbol{y}} \boldsymbol{\nabla}_{\boldsymbol{x}} u(\boldsymbol{x} + r\boldsymbol{y}, t) dV_{\boldsymbol{y}}.$$
- Changing variables, we obtain
$$\begin{aligned}\frac{\partial m_u}{\partial r}(\boldsymbol{x}, r, t) &= \frac{1}{4\pi}\int_{\|\boldsymbol{y}\|\leq 1} r\triangle_{\boldsymbol{x}} u(\boldsymbol{x} + r\boldsymbol{y}, t) dV_{\boldsymbol{y}} \\ &= \frac{1}{4\pi r^2}\int_{\|\boldsymbol{z}\|\leq r} \triangle_{\boldsymbol{x}} u(\boldsymbol{x} + \boldsymbol{z}, t) dV_{\boldsymbol{z}} \\ &= \frac{1}{4\pi r^2}\int_0^r \int_{\|\boldsymbol{y}\|=1} \triangle_{\boldsymbol{x}} u(\boldsymbol{x} + s\boldsymbol{y}, t) s^2 dS_{\boldsymbol{y}} ds.\end{aligned}$$
- Multiplying by r^2 and differentiating with respect to r, we obtain
$$\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right) m_u(\boldsymbol{x}, r, t) = \frac{1}{4\pi}\int_{\|\boldsymbol{y}\|=1} \triangle_{\boldsymbol{x}} u(\boldsymbol{x} + r\boldsymbol{y}, t) r^2 dS_{\boldsymbol{y}}.$$
- Manipulating the left-hand side, we obtain
$$\left(\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right) m_u(\boldsymbol{x}, r, t) = r^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) m_u(\boldsymbol{x}, r, t) = r\frac{\partial^2 r m_u}{\partial r^2}(\boldsymbol{x}, r, t).$$

Superposition formulae in 3D ($m = 3$) (continued)

- From $\frac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = 0$ and $r \frac{\partial^2 r m_u}{\partial r^2}(\boldsymbol{x}, r, t) = \frac{1}{4\pi} \int_{\|\boldsymbol{y}\|=1} \triangle_{\boldsymbol{x}} u(\boldsymbol{x} + r\boldsymbol{y}, t) r^2 dS_{\boldsymbol{y}}$, it follows that

$$\frac{\partial^2 r m_u}{\partial t^2}(\boldsymbol{x}, r, t) - c^2 \frac{\partial^2 r m_u}{\partial r^2}(\boldsymbol{x}, r, t) = 0 \qquad \text{in }]0, +\infty[\times]0, +\infty[.$$

- Because of the continuity of u, we have $\lim_{r\to 0} r m_u(\boldsymbol{x}, r, t) = 0$.

- As a conclusion, we find that $\tilde{m}_u(\boldsymbol{x}, r, t) = r m_u(\boldsymbol{x}, r, t)$ satisfies the IBVP

$$\begin{cases} \dfrac{\partial^2 \tilde{m}_u}{\partial t^2}(\boldsymbol{x}, r, t) - c^2 \dfrac{\partial^2 \tilde{m}_u}{\partial r^2}(\boldsymbol{x}, r, t) = 0 & \text{in }]0, +\infty[\times]0, +\infty[, \\ \lim\limits_{r\to 0} \tilde{m}_u(\boldsymbol{x}, r, t) = 0 & \text{on } \{r = 0\}\times]0, +\infty[, \\ \tilde{m}_u(\boldsymbol{x}, r, 0) = \tilde{m}_g(\boldsymbol{x}, r) = \dfrac{1}{4\pi r} \displaystyle\int_{\|\boldsymbol{z}-\boldsymbol{x}\|=r} g(\boldsymbol{z}) dS_{\boldsymbol{z}} & \text{on }]0, +\infty[\times\{t = 0\}, \\ \dfrac{\partial \tilde{m}_u}{\partial t}(\boldsymbol{x}, r, 0) = \tilde{m}_w(\boldsymbol{x}, r) = \dfrac{1}{4\pi r} \displaystyle\int_{\|\boldsymbol{z}-\boldsymbol{x}\|=r} w(\boldsymbol{z}) dS_{\boldsymbol{z}} & \text{on }]0, +\infty[\times\{t = 0\}. \end{cases}$$

Because this IBVP is defined on $]0, +\infty[\times]0, +\infty[$, specifically, because the spatial domain is limited to the positive real line $]0, +\infty[$, there is a boundary condition at $r = 0$.

Superposition formulae in 3D ($m = 3$) (continued)

- The subsequent step is to reformulate the IBVP mentioned previously (spatial domain limited to the positive real line) as an equivalent IVP (spatial domain encompasses the whole real line), thus facilitating the representation of the solution by means of the D'Alembert formula.
- This step is completed by the method of images, which means here that the initial data already specified on the positive real line are extended to the whole real line by odd reflection:

$$\tilde{m}_u(\boldsymbol{x}, r, 0) = \begin{cases} \tilde{m}_g(\boldsymbol{x}, r) & \text{if } r \geq 0, \\ -\tilde{m}_g(\boldsymbol{x}, -r) & \text{if } r \leq 0, \end{cases} \qquad \frac{\partial \tilde{m}_u}{\partial t}(\boldsymbol{x}, r, 0) = \begin{cases} \tilde{m}_w(\boldsymbol{x}, r) & \text{if } r \geq 0, \\ -\tilde{m}_w(\boldsymbol{x}, -r) & \text{if } r \leq 0, \end{cases}$$

thus leading to the following IVP on $\mathbb{R} \times]0, +\infty[$:

$$\begin{cases} \dfrac{\partial^2 \tilde{m}_u}{\partial t^2}(\boldsymbol{x}, r, t) - c^2 \dfrac{\partial^2 \tilde{m}_u}{\partial r^2}(\boldsymbol{x}, r, t) = 0 & \text{in } \mathbb{R} \times]0, +\infty[, \\ \tilde{m}_u(\boldsymbol{x}, r, 0) = \begin{cases} \tilde{m}_g(\boldsymbol{x}, r) & \text{if } r \geq 0, \\ -\tilde{m}_g(\boldsymbol{x}, -r) & \text{if } r \leq 0, \end{cases} & \text{on } \mathbb{R} \times \{t = 0\}, \\ \dfrac{\partial \tilde{m}_u}{\partial t}(\boldsymbol{x}, r, 0) = \begin{cases} \tilde{m}_w(\boldsymbol{x}, r) & \text{if } r \geq 0, \\ -\tilde{m}_w(\boldsymbol{x}, -r) & \text{if } r \leq 0, \end{cases} & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

The odd reflection of the initial data ensures that $\tilde{m}_u(\boldsymbol{x}, r, t) = 0$ on $\{r = 0\} \times]0, +\infty[$, thus ensuring that the restriction to $]0, +\infty[\times]0, +\infty[$ of the solution to the IVP coincides with the solution to the IBVP mentioned previously.

Superposition formulae in 3D ($m = 3$) (continued)

- Thus, for $0 \leq r \leq ct$, we have the D'Alembert representation
$$\tilde{m}_u(\boldsymbol{x}, r, t) = \frac{1}{2}\Big(\tilde{m}_g(\boldsymbol{x}, r + ct) - \tilde{m}_g(\boldsymbol{x}, ct - r)\Big) + \frac{1}{2c}\int_{ct-r}^{ct+r} \tilde{m}_w(\boldsymbol{x}, s)ds.$$

- The last step is to recover the solution therefrom by using de l'Hôpital's formula as follows:
$$u(\boldsymbol{x}, t) = \lim_{r\to 0} m_u(\boldsymbol{x}, r, t) = \lim_{r\to 0} \frac{r m_u(\boldsymbol{x}, r, t)}{r} = \lim_{r\to 0} \frac{\partial \tilde{m}_u}{\partial r}(\boldsymbol{x}, r, t).$$
Denoting by $\frac{\partial \tilde{m}_g}{\partial r}(x, ct)$ the partial derivative of $(\boldsymbol{x}, r) \mapsto \tilde{m}_g(\boldsymbol{x}, r)$ with respect to r evaluated at $(\boldsymbol{x}, ct)$, we obtain
$$u(\boldsymbol{x}, t) = \frac{\partial \tilde{m}_g}{\partial r}(\boldsymbol{x}, ct) + \frac{1}{c}\tilde{m}_w(\boldsymbol{x}, ct).$$

- By the chain rule, the partial derivative of $(\boldsymbol{x}, r) \mapsto \tilde{m}_g(\boldsymbol{x}, r) = \frac{1}{4\pi r}\int_{\|\boldsymbol{z}-\boldsymbol{x}\|=r} g(\boldsymbol{z})dS_{\boldsymbol{z}}$ with respect to r evaluated at $(\boldsymbol{x}, ct)$ is equal to that of $(\boldsymbol{x}, t) \mapsto \frac{1}{c}\frac{1}{4\pi ct}\int_{\|\boldsymbol{z}-\boldsymbol{x}\|=ct} g(\boldsymbol{z})dS_{\boldsymbol{z}}$ with respect to t evaluated at $(\boldsymbol{x}, t)$, thus finally providing Kirchhoff's representation
$$u(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{\|\boldsymbol{z}-\boldsymbol{x}\|=ct} g(\boldsymbol{z})dS_{\boldsymbol{z}}\right) + \frac{1}{4\pi c^2 t}\int_{\|\boldsymbol{z}-\boldsymbol{x}\|=ct} w(\boldsymbol{z})dS_{\boldsymbol{z}}.$$

Superposition formulae in 2D ($m = 2$)

- In 2D ($m = 2$), the aforementioned superposition formulae take the following form (proof omitted):
  - Initial-value problem:
    Given sufficiently regular functions g and w, the solution $u(\boldsymbol{x}, t) = \Psi_t \star g(\boldsymbol{x}) + \Phi_t \star w(\boldsymbol{x})$ to
    $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = 0 & \text{in } \mathbb{R}^2 \times]0, +\infty[, \\ u = g \quad \text{and} \quad \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R}^2 \times \{t = 0\}, \end{cases}$$
    can be written equivalently in the form of the **Poisson representation**
    $$u(\boldsymbol{x}, t) = \frac{\partial}{\partial t} \left(\frac{1}{2\pi c} \int_{\|\boldsymbol{z}-\boldsymbol{x}\| \leq ct} g(\boldsymbol{z}) \sqrt{\frac{1}{c^2 t^2 - \|\boldsymbol{z}-\boldsymbol{x}\|^2}} dV_{\boldsymbol{z}} \right) + \frac{1}{2\pi c} \int_{\|\boldsymbol{z}-\boldsymbol{x}\| \leq ct} w(\boldsymbol{z}) \sqrt{\frac{1}{c^2 t^2 - \|\boldsymbol{z}-\boldsymbol{x}\|^2}} dV_{\boldsymbol{z}}.$$
  - Inhomogeneous problem:
    Given a sufficiently regular function f, the solution $u(\boldsymbol{x}, t) = \Phi \star f(\boldsymbol{x}, t)$ to
    $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = f & \text{in } \mathbb{R}^2 \times]0, +\infty[, \\ u = 0 \quad \text{and} \quad \dfrac{\partial u}{\partial t} = 0 & \text{on } \mathbb{R}^2 \times \{t = 0\}, \end{cases}$$
    can be written equivalently as follows:
    $$u(\boldsymbol{x}, t) = \int_0^t \frac{1}{2\pi c} \int_{\|\boldsymbol{z}-\boldsymbol{x}\| \leq c(t-s)} f(\boldsymbol{z}, s) \sqrt{\frac{1}{c^2 t^2 - \|\boldsymbol{z}-\boldsymbol{x}\|^2}} dV_{\boldsymbol{z}} ds.$$

Superposition formulae in 2D ($m = 2$) (continued)

- As an alternative to obtaining it by making explicit the convolutions in the superposition formulae, the Poisson representation can also be proven by the **method of descent**:
  - Given sufficiently regular initial data g and w (specifically, let g be in $C^3(\mathbb{R}^2)$ and w in $C^2(\mathbb{R}^2)$), let u be the solution to the initial-value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = 0 & \text{in } \mathbb{R}^2 \times]0, +\infty[, \\ u = g \quad \text{and} \quad \dfrac{\partial u}{\partial t} = w & \text{on } \mathbb{R}^2 \times \{t = 0\}. \end{cases}$$

  - The method of descent involves regarding this initial-value problem in 2D ($m = 2$) as a specialization of an initial-value problem in 3D ($m = 3$) in which the initial data do not depend on the third spatial variable. This allows the representation of the solution in 2D to be obtained from that in 3D in which the initial data do not depend on the third spatial variable.

    Specifically, with $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{z} = (z_1, z_2)$, the Kirchhoff representation leads to

$$u(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{\|(\boldsymbol{z},z_3)-(\boldsymbol{x},x_3)\|=ct} g(\boldsymbol{z}) dS_{(\boldsymbol{z},z_3)}\right) + \frac{1}{4\pi c^2 t}\int_{\|(\boldsymbol{z},z_3)-(\boldsymbol{x},x_3)\|=ct} w(\boldsymbol{z}) dS_{(\boldsymbol{z},z_3)}.$$

Superposition formulae in 2D ($m = 2$) (continued)

- To evaluate the integrals over the sphere, we parameterize the upper and lower hemispheres as

$$(z_1, z_2) \mapsto \phi(z_1, z_2) = x_3 \pm \sqrt{c^2t^2 - (z_1 - x_1)^2 - (z_2 - x_2)^2}, \qquad \|\boldsymbol{z} - \boldsymbol{x}\| \leq ct.$$

The formula for the integration of a function over a parameterized surface then indicates that

$$dS_{(\boldsymbol{z}, z_3)} = \sqrt{1 + \|\boldsymbol{\nabla}_{\boldsymbol{z}}\phi(\boldsymbol{z})\|^2} dV_{\boldsymbol{z}} = \sqrt{1 + \frac{\|\boldsymbol{z} - \boldsymbol{x}\|^2}{c^2t^2 - \|\boldsymbol{z} - \boldsymbol{x}\|^2}} dV_{\boldsymbol{z}} = ct \sqrt{\frac{1}{c^2t^2 - \|\boldsymbol{z} - \boldsymbol{x}\|^2}} dV_{\boldsymbol{z}},$$

thus leading to, with the factor 2 appearing because we integrate over 2 hemispheres,

$$u(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\left(\frac{2}{4\pi c}\int_{\|\boldsymbol{z}-\boldsymbol{x}\| \leq ct} g(\boldsymbol{z}) \sqrt{\frac{1}{c^2t^2 - \|\boldsymbol{z} - \boldsymbol{x}\|^2}} dV_{\boldsymbol{z}}\right) + \frac{2}{4\pi c}\int_{\|\boldsymbol{z}-\boldsymbol{x}\| \leq ct} w(\boldsymbol{z}) \sqrt{\frac{1}{c^2t^2 - \|\boldsymbol{z} - \boldsymbol{x}\|^2}} dV_{\boldsymbol{z}},$$

that is, the Poisson representation, as asserted.

- We note that the Huygens principle is false in 2D ($m = 2$). In fact, in 2D, initially concentrated initial data leave a slowly decaying remnant that never entirely disappears.

Wave equation on a portion of space

Notion of initial-boundary-value problem

- If one wishes to solve the wave equation in an open bounded subset Ω of $\mathbb{R}^m$, it is appropriate to specify not only an initial value $u(\boldsymbol{x},0)$ on Ω but also a boundary condition, for example,
  - a Dirichlet boundary condition: $u = g$,
  - a Neumann boundary condition: $\boldsymbol{\nabla}_{\boldsymbol{x}} u \cdot \boldsymbol{n} = g$,

  the interpretation of which depends on the application.

Example of initial-boundary-value problem

- As an example, let us look more closely at the initial-boundary-value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \triangle_{\boldsymbol{x}} u = 0 & \text{in } \Omega \times]0,+\infty[, \\ u(\boldsymbol{x},0) = g(\boldsymbol{x}) \quad \text{and} \quad \dfrac{\partial u}{\partial t}(\boldsymbol{x},0) = w(\boldsymbol{x}) & \text{on } \Omega \times \{t=0\} \quad \text{(initial condition)}, \\ u(\boldsymbol{x},t) = 0 & \text{on } \partial\Omega \times]0,+\infty[\quad \text{(boundary condition)}. \end{cases}$$

- This problem can be solved by the method of separation of variables.

Example of initial-boundary-value problem (continued)

- **Eigenproblem**: With reference to the boundary condition $u(\boldsymbol{x},t)=0$ on $\partial\Omega\times]0,+\infty[$, we begin by solving the following eigenproblem:

$$\begin{cases} -\triangle_{\boldsymbol{x}}\varphi_k(\boldsymbol{x}) = \lambda_k\varphi_k(\boldsymbol{x}) & \text{in } \Omega, \\ \varphi_k(\boldsymbol{x}) = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us assume that we can find a sequence of eigenvalues λ_k with corresponding eigenfunctions φ_k, which form an orthonormal basis for $L^2(\Omega)$.

- **Function series**: Given the eigenfunctions $\{\varphi_k\}_{k=1}^{+\infty}$, we seek a solution of the following form:

$$u(\boldsymbol{x},t) = \sum_{k=1}^{+\infty} b_k(t)\varphi_k(\boldsymbol{x});$$

thus, we seek a solution of the form of a series of products of functions of fewer independent variables, namely, a solution of the form of a series of products of functions b_k of only t (yet to be determined) and the eigenfunctions φ_k of only $\boldsymbol{x}$ (already known).

Example of initial-boundary-value problem (continued)

- **System of uncoupled equations ("diagonalization")**: Inserting this function series into the IBVP and assuming the validity of term-by-term differentiation, we obtain

$$\begin{cases} \displaystyle\sum_{k=1}^{+\infty}\left(\frac{d^2 b_k}{dt^2}(t)\varphi_k(\boldsymbol{x}) - b_k(t)c^2\triangle_{\boldsymbol{x}}\varphi_k(\boldsymbol{x})\right) = 0 & \text{in } \Omega\times]0,+\infty[, \\ \displaystyle\sum_{k=1}^{+\infty} b_k(0)\varphi_k(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \text{and} \quad \sum_{k=1}^{+\infty}\frac{db_k}{dt}(0)\varphi_k(\boldsymbol{x}) = w(\boldsymbol{x}) & \text{on } \Omega\times\{t=0\}, \\ \displaystyle\sum_{k=1}^{+\infty} b_k(t)\varphi_k(\boldsymbol{x}) = 0 & \text{on } \partial\Omega\times]0,+\infty[. \end{cases}$$

  - If g and w are square-integrable, we have $g(\boldsymbol{x}) = \sum_{k=1}^{+\infty} g_k\varphi_k(\boldsymbol{x})$ and $w(\boldsymbol{x}) = \sum_{k=1}^{+\infty} w_k\varphi_k(\boldsymbol{x})$ with $g_k = \int_\Omega g(\boldsymbol{x})\varphi_k(\boldsymbol{x})d\boldsymbol{x}$ and $w_k = \int_\Omega w(\boldsymbol{x})\varphi_k(\boldsymbol{x})d\boldsymbol{x}$ for $k = 1, 2, 3, \ldots$; thus,

$$\begin{cases} \displaystyle\sum_{k=1}^{+\infty}\left(\frac{d^2 b_k}{dt^2}(t)\varphi_k(\boldsymbol{x}) + b_k(t)c^2\lambda_k\varphi_k(\boldsymbol{x})\right) = 0 & \text{in } \Omega\times]0,+\infty[, \\ \displaystyle\sum_{k=1}^{+\infty} b_k(0)\varphi_k(\boldsymbol{x}) = \sum_{k=1}^{+\infty} g_k\varphi_k(\boldsymbol{x}) \quad \text{and} \quad \sum_{k=1}^{+\infty}\frac{db_k}{dt}(0)\varphi_k(\boldsymbol{x}) = \sum_{k=1}^{+\infty} w_k\varphi_k(\boldsymbol{x}) & \text{on } \Omega\times\{t=0\}, \\ \displaystyle\sum_{k=1}^{+\infty} b_k(t)\varphi_k(\boldsymbol{x}) = 0 & \text{on } \partial\Omega\times]0,+\infty[. \end{cases}$$

Example of initial-boundary-value problem (continued)

- Clearly, the previous equations hold if the following system of uncoupled IVPs holds:

$$\left\{\begin{array}{lll} \dfrac{d^2 b_k}{dt^2}(t) + c^2 \lambda_k b_k(t) = 0 & \text{in }]0,+\infty[\\ b_k(0) = g_k \quad \text{and} \quad \dfrac{db_k}{dt}(0) = w_k & \text{at } t = 0 \end{array}\right. , \qquad \text{where } k = 1, 2, 3, \ldots.$$

It can be easily verified that the solution to this system of uncoupled IVPs is given by

$$b_k(t) = g_k \cos(c\sqrt{\lambda_k} t) + w_k \frac{1}{c\sqrt{\lambda_k}} \sin(c\sqrt{\lambda_k} t), \quad \text{where } k = 1, 2, 3, \ldots.$$

- As a conclusion, we obtain the following representation of the solution to the aforementioned IBVP:

$$u(\boldsymbol{x}, t) = \sum_{k=1}^{+\infty} \left(g_k \cos(c\sqrt{\lambda_k} t) + w_k \frac{1}{c\sqrt{\lambda_k}} \sin(c\sqrt{\lambda_k} t) \right) \varphi_k(\boldsymbol{x}).$$

- The convergence and term-by-term differentiation still require justification, but we omit these details.
- We find the solution as a superposition of eigenfunctions φ_k, also called **normal modes**, which oscillate in time at corresponding frequencies $c\sqrt{\lambda_k}$, also called **frequencies of vibration**.
- Because eigenfunctions φ_k associated with larger eigenvalues λ_k typically oscillate more rapidly in space, we can conclude from this representation that **components of the solution that oscillate more rapidly in space also oscillate more rapidly in time**.

Summary and conclusion

- The wave equation is the mathematical prototype of models of wave propagation phenomena.
- A fundamental solution for the wave equation is a solution that solves the wave equation for a Dirac impulse centered at the origin on the right-hand side. It can be shown that the so called retarded Green's function is a fundamental solution for the wave equation.
- A fundamental solution allows superposition formulae to be established for determining solutions to initial-value and inhomogeneous problems with general right-hand sides.
- The notion of fundamental solution allows interesting properties to be deduced:
  - Finite propagation speed.
  - No smoothening.
  - Finite domain of dependence.
- Certain initial-boundary-value problems involving the wave equation on a bounded portion of space can be solved by separation of variables. This leads to a representation of the solution as a superposition of normal modes that oscillate in time at corresponding frequencies of vibration.

References

Suggested reading material

- P. Olver. Introduction to Partial Differential Equations. Springer, 2014. Sections 2.4, 4.2, 9.5, 11.6, 12.5, and 12.6.

Additional references also consulted to prepare this lecture

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