Structural Reliability

Thuong Van DANG

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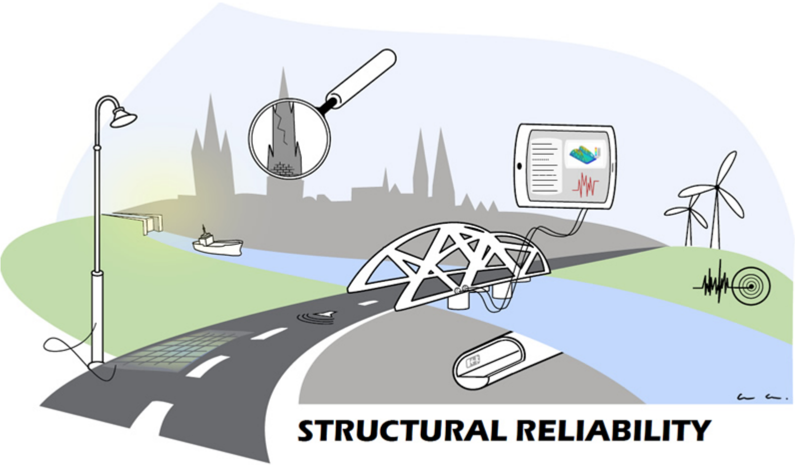

STRUCTURAL RELIABILITY

Introduction to Structural Reliability

Civil engineering facilities such as bridges, buildings, power plants, dams and offshore platforms are all intended to contribute to the benefit and quality of life. Therefore when it is important that the benefit of the facility can be identified considering all phases of the life of the facility, i.e. including design, manufacturing, construction, operation and eventually decommissioning.
For many years it has been assumed in design of structural systems that all loads and strengths are deterministic. The strength of an element was determined in such a way that it exceeded the load with a certain margin. The ratio between the strength and the load was denoted the safety factor. This number was considered as a measure of the reliability of the structure.

Introduction to Structural Reliability

In codes of practice for structural systems values for loads, strengths and safety factors are prescribed.

However, uncertainties in the loads, strengths and in the modeling of the systems require that methods based on probabilistic techniques in a number of situations have to be used.

A structure is usually required to have a satisfactory performance in the expected lifetime, i.e. it is required that it does not collapse or becomes unsafe and that it fulfills certain functional requirements.

Generally structural systems have a rather small probability that they do not function as intended.

Introduction to Structural Reliability

The study of structural reliability is concerned with the calculation and prediction of the probability of limit state violation for an engineered structural system at any stage during its life.

The objective of any structural design is to ensure safety and economy of the structure operating under a given environment.

Capacity (C) $>$ **Demand (D)**

So long this condition is satisfied, the safety of the structure is ensured for the intended purpose for which the structure is built. Besides this, designers also ensure that there is an optimal use of the materials which, in turn, ensures economy.

Concept of Limit State and Reliability

Limit state equation can be represented in the form:
g(X) = Capacity - Demand = C - D = 0

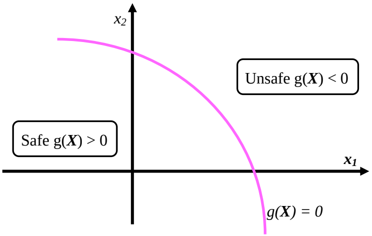


Figure: Limit State showing safe and unsafe regions

It is important to note that the limit state does not define a unique failure function, i.e. the limit state can be described by a number of equivalent failure functions. In structural reliability, the limit state function usually results from a mechanical analysis of the structure.

Concept of Limit State and Reliability

- **Reliability**
  The reliability of an engineering design is the probability that it response certain demands under conditions for a specified period of time. Reliability often mentions as the ability of a component or system to function at a specified moment or interval of time.
  There are some definitions of reliability in national and international documents. In ISO 2394, *reliability is the ability of a structure to comply with given requirements under specified conditions during the intended life for which it was designed.*
  While Eurocode provides a description: *reliability is the ability of a structure or a structural member to fulfill the specified requirements, including the design working life, for which it has been designed and it is usually expressed in probabilistic terms.*


Uncertainty
Reliability
Safety
Risk

Levels of Reliability Methods

Generally, methods to measure the reliability of a structure can be divided into four groups:

- **Level I methods:** The uncertain parameters are modeled by one characteristic value. For example in codes based on the partial safety factor concept (load and resistance factor formats).
- **Level II methods:** The uncertain parameters are modeled by the mean values and the standard deviations, and by the correlation coefficients between the stochastic variables. The stochastic variables are implicitly assumed to be normally distributed. The reliability index method is an example of a level II method.

Levels of Reliability Methods

- **Level III methods:** The uncertain quantities are modeled by their joint distribution functions. The probability of failure is estimated as a measure of the reliability.
- **Level IV methods:** In these methods, the consequences (cost) of failure are also taken into account and the risk (consequence multiplied by the probability of failure) is used as a measure of the reliability. In this way, different designs can be compared on an economic basis taking into account uncertainty, costs and benefits.

Review of Probability Theory

Events and basis probability rules

An event E is defined as a subset of the sample space (all possible outcomes of a random quantity) Ω . The failure event E of e.g. a structural element can be modeled by E = R$\leq$ S where R is the strength and S is the load.

The probability of failure is the probability:

P_f = P(E) = P(R$\leq$ S).

Axioms of probability:

- *Axiom 1:* for any event E :

$$0 \leq P(E) \leq 1 \tag{1}$$

- *Axiom 2:* for the sample space

$$P(\Omega) = 1 \tag{2}$$

- *Axiom 3:* for mutually exclusive events $E_1, E_2, ..., E_m$:

$$P\left(\bigcup_{i=1}^{m} E_i\right) = \sum_{i=1}^{m} P(E_i) \tag{3}$$

Review of Probability Theory

The conditional probability of an event E_1 given another event E_2 is defined by:

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} \tag{4}$$

if E_1 and E_2 are statistically independent:

$$P(E_1 \cap E_2) = P(E_1)P(E_2) \tag{5}$$

Bayes theorem:

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(A)} = \frac{P(A|E_i)P(E_i)}{\sum_{j=1}^{m} P(A|E_i)P(E_i)} \tag{6}$$

where A is an event.

Continuous stochastic variables

Consider a continuous stochastic variable X . The distribution function of X is denoted $F_X(x)$ and gives the probability $F_X(x) = P(X \leq x)$. The probability density function $f_X(x)$ is defined by:

$$f_X(x) = \frac{d}{dx} F_x(x) \tag{7}$$

$$\tag{8}$$

The expected value is defined by:

$$\mu = \int_{-\infty}^{\infty} (x f_X(x) dx \tag{9}$$

The variance σ^2 is defined by:

$$\sigma^2 = \int (x - \mu)^2 f_X(x) dx \tag{10}$$

where σ is the standard deviation.

Review of Probability Theory

- **Normal distribution**

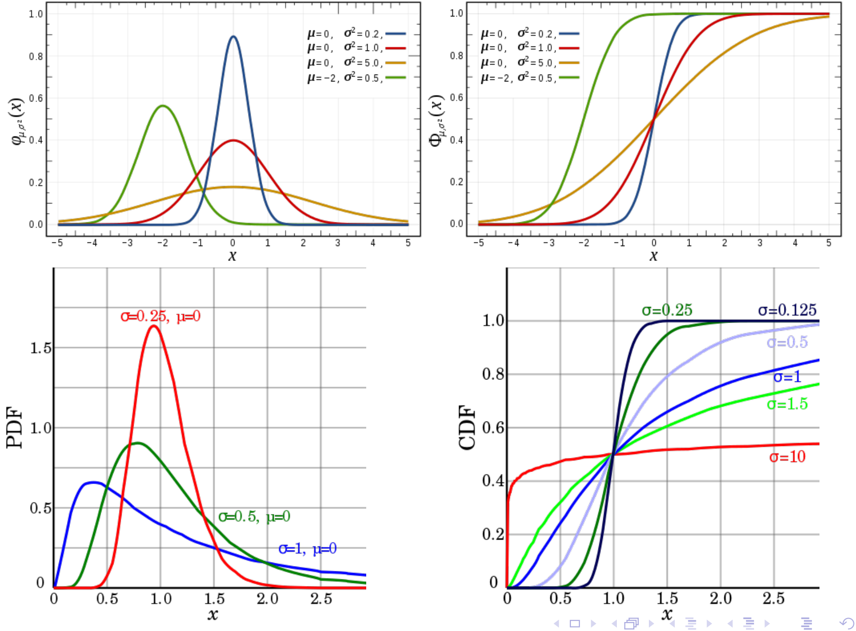
  The normal distribution often occurs in practical applications because the sum of a large number of statistically independent random variables converges to a normal (known as central limit theorem). The normal distribution can be used to represent material properties, fatigue uncertainty. It is also used in stress-strength interference models in reliability studies.

  The cumulative distribution function of a normal random variable X with the mean μ_X and standard deviation σ_X is given by the exponential expression:

$$F_X(x) = \Phi(\frac{x-\mu}{\sigma}) = \int_{-\infty}^{x} \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\frac{x-\mu_X}{\sigma_X})^2\right]dx \tag{12}$$

  where $\Phi(u)$ is is the standardized distribution function for a Normal distributed stochastic variable with expected value μ = 0 and standard deviation σ = 1.

Review of Probability Theory



- **Lognormal distribution**
  The cumulative distribution function for a stochastic variable with expected value μ_X and standard deviation σ_X is denoted LN(μ_X, σ_X), and is defined by:

$$F_X(x) = \Phi(\frac{lnx - \mu_X}{\sigma_X}) = \int_{-\infty}^{lnx} \frac{1}{\sigma_X \sqrt{2\pi}} \exp\left[-\frac{1}{2}(\frac{lnx - \mu_X}{\sigma_X})^2\right] dx \tag{13}$$

  where:
  $\sigma_X = \sqrt{ln(\frac{\sigma}{\mu}^2 + 1)}$ and $\mu_X = ln\mu - \frac{1}{2}\sigma_X^2$ is the standard deviation and expected value for the Normal distributed stochastic variable: Y = lnX

- **Exponential distribution**
  The exponential distribution is most widely used distribution in reliability and risk assessment. It is the only distribution having constant hazard rate and is used to model useful life of many engineering systems. The exponential distribution is closely related to the Poisson distribution which is discrete. If the number of failure per unit time is Poisson distribution then the time between failures follows the exponential distribution. The cumulative distribution function is given by:

$$F(x, \lambda) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{14}$$

where $\lambda > 0$ is inverse scale

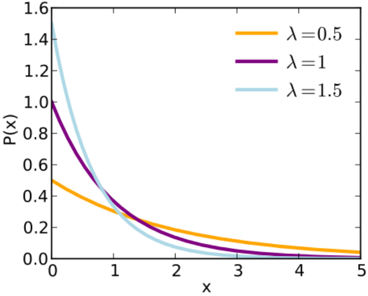
Review of Probability Theory

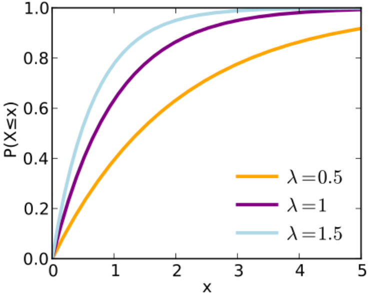
- **Weibull distribution**
  The Weibull distribution is a continuous probability distribution. It is named after Swedish mathematician Waloddi Weibull, who described it in detail in 1951, although it was first identified by Frchet (1927) and first applied by Rosin & Rammler (1933) to describe a particle size distribution.
  The cumulative distribution function is given by:

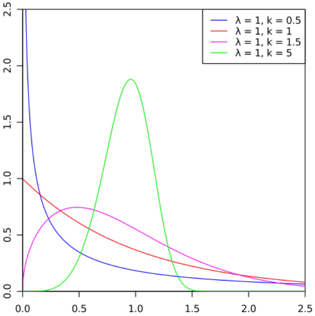
$$F(x, k, \lambda) = \begin{cases} 1 - e^{-(\frac{x}{\lambda})^k} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{15}$$

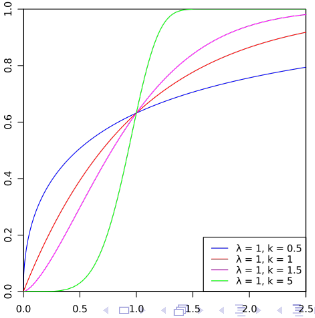
where $\lambda \epsilon (0, +\infty)$ and $k \epsilon (0, +\infty)$ are scale and shape parameter.

Review of Probability Theory









Review of Probability Theory

Conditional distributions

The conditional distribution function for X_1 given $X_2$is defined by:

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} \tag{16}$$

X_1 and X_2 are statistically independent if $f_{X_1|X_2}(x_1|x_2) = f_{X_1}(x_1)$ implying that:

$$f_{X_1|X_2}(x_1|x_2) = f_{X_1}(x_1)f_{X_2}(x_2) \tag{17}$$

Covariance and correlation

The covariance between X_1 and X_2 is defined by:

$$Cov[X_1, X_2] = E[(X_1 - \mu_1)(X_2 - \mu_2)] \tag{18}$$

It is seen that

$$Cov[X_1, X_1] = Var[X_1] = \sigma_1^2 \tag{19}$$

Review of Probability Theory

Covariance and correlation

The correlation coefficient between X_1 and X_2 is defined by:

$$\rho_{X_1,X_2} = \frac{Cov[X_1, X_2]}{\sigma_1, \sigma_2} \tag{20}$$

If $\rho_{X_1,X_2} = 0$ then X_1 and X_2 is uncorrelated, but not necessarily statistically independent.

For a stochastic vector $X = (X_1, X_2, ..., X_n)$ the covariance-matrix is defined by:

$$C = \begin{pmatrix} Var[X_1, X_1] & Cov[X_1, X_2] & \cdots & Cov[X_1, X_n] \\ Cov[X_1, X_2] & Var[X_2, X_2] & \cdots & Cov[X_2, X_n] \\ \vdots & \vdots & \ddots & \vdots \\ Cov[X_1, X_n] & Cov[X_2, X_n] & \cdots & Var[X_n, X_n] \end{pmatrix} \tag{21}$$

First Order Second Moment Method (FOSM)

The First Order Second Moment FOSM method is based on a first order Taylors approximation of the performance function. It uses only second moment statistics (means and covariances) of the random variables. Consider a case where C and D are normally distributed variables and they are statistically independent. μ_C and μ_D are mean and σ_C and σ_D are standard deviations of C and D respectively.

Then mean of g(X) is:

$$\mu_g = \mu_C - \mu_D \tag{22}$$

and standard deviation of g(X) is:

$$\sigma_g = \sqrt{\mu_C^2 + \mu_D^2} \tag{23}$$

First Order Second Moment Method (FOSM)

So that failure probability is:

$$p_f = P(g(X) < 0) = \Phi\left(\frac{0-(\mu_C-\mu_D)}{\sqrt{\mu_C^2+\mu_D^2}}\right) = \Phi\left(-\frac{\mu_g}{\sigma_g}\right) \quad (24)$$

where β is measure of the reliability for the limit state g(X). It was proposed by Cornell and hence, is termed as Cornell's Reliability Index:

$$\beta = \frac{\mu_g}{\sigma_g} \quad (25)$$

Consider a case of generalized performance function of many random variables: $X_1, X_2, ..., X_n$

First Order Second Moment Method (FOSM)

Expanding this performance function about the mean gives:

$$LSF = g(X_1, X_2, ..., X_n) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}(X_i - \mu_{X_i}) + ... \tag{26}$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g}{\partial X_i \partial X_j}(X_i - \mu_{X_i})(X_j - \mu_{X_j}) + ... \tag{27}$$

where the derivatives are evaluated at the mean values.
Considering first two terms in the Taylor's series expression and taking expectation on both sides one can prove that :

$$\mu_g = E[g(X)] = g(\mu_1, \mu_2, ..., \mu_n) \tag{28}$$

$$Var(g) = \sigma_g^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial g}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right) Cov(X_i, X_j) \tag{29}$$

where $Cov(X_i, X_j)$ is the covariance of X_i and X_j.

First Order Reliability Method (FORM)

- **Reliability index**
  In 1974 Hasofer & Lind proposed a definition of the reliability index which is invariant with respect to the mathematical formulation of the safety margin.

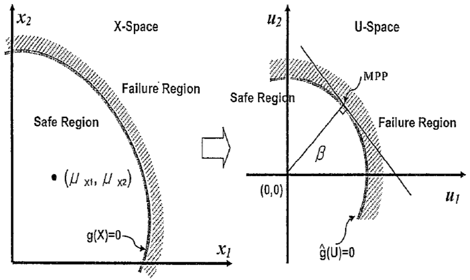


Figure: An example transformation to standard normal space

Reliability index β is defined as the smallest distance from the origin O(0,0) in the U-space to the failure surface g(U) = 0.

First Order Reliability Method (FORM)

The reliability index is thus determined by the optimization problem:

$$\beta = \underbrace{min}_{g(U)=0} \sqrt{\sum_{i=1}^{n} U_i^2} \tag{30}$$

Reliability index is also known as an equivalent value to the probability of failure, formally defined as a negative value of a standardized normal variable corresponding to the probability of failure P_f:

$$\beta = -\Phi_U^{-1}(P_f) \tag{31}$$

where $-\Phi_U^{-1}(P_f)$ denotes the inverse standardized normal distribution function

For independent variables of any distribution, the principle of the transformation into standard normal space consists of writing the equality of the distribution functions:

$$\Phi(u) = F_X(x) \Rightarrow u = \Phi(u)^{-1}(F_X(x)) \tag{32}$$

First Order Reliability Method (FORM)

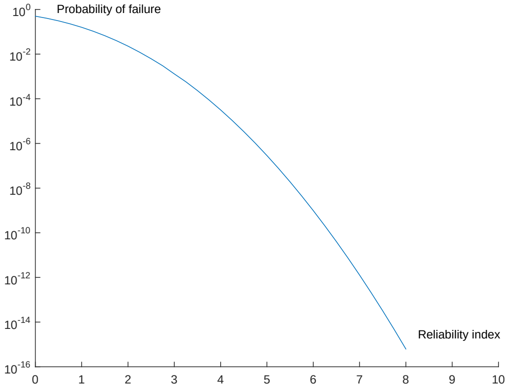
- **Probability of failure**
  The probability of failure is an important term in the theory of structural reliability. It is assumed that X is the vector of random variables that influence a system's load (L) and resistance (R), the limit state function is formulated in terms of these basic variables and given as:

$$g(X) = g(X_1, X_2, ..., X_n) = L - R \tag{33}$$

  In the general sense, the probability of failure based on the given limit state for a time-invariant reliability problem is:

$$P_f = P[g(X) \leq 0] = \int_{g(X) \leq 0} f_x(x) dx \tag{34}$$

  Where $f_x(x)$ and P_f are the joint probability density function of X.



Pf	beta	Pf	beta
0.5	0.00	1.07E-05	4.25
0.4013	0.25	3.40E-06	4.50
0.3085	0.50	1.02E-06	4.75
0.2266	0.75	2.87E-07	5.00
0.1587	1.00	7.61E-08	5.25
0.1056	1.25	1.90E-08	5.50
0.0668	1.50	4.46E-09	5.75
0.0401	1.75	9.87E-10	6.00
0.0228	2.00	2.05E-10	6.25
0.0122	2.25	4.02E-11	6.50
0.0062	2.50	7.39E-12	6.75
0.003	2.75	1.28E-12	7.00
0.0013	3.00	2.08E-13	7.25
5.77E-04	3.25	3.19E-14	7.50
2.33E-04	3.50	4.59E-15	7.75
8.84E-05	3.75	6.22E-16	8.00
3.17E-05	4.00	7.92E-17	8.25

First Order Reliability Method (FORM)

The FORM (First Order Reliability Method) is one of the basic and very efficient reliability methods. The FORM method is used as a fundamental procedure by a number of software products for the reliability analysis of structures and systems. It is also mentioned in EN 1990 [1] that the design values are based on the FORM reliability method.

The procedure of applying the algorithm by Rackwitz and Fiessler algorithm for reliability calculation can be listed as follows:

- *Step 1*: Write the limit state function g(X)$=$ 0 in terms of the basic variables. Transform the limit state function g(X) into standard normal space g(U).
- *Step 2*: Assume initial value of design point U_0 (mean value).
- *Step 3*: Calculate gradient vector

$$\frac{\partial g}{\partial u} \tag{35}$$

First Order Reliability Method (FORM)

- *Step 4*: Calculate an improved guess of the Reliability Index β

$$\alpha = -\frac{\nabla g(u)}{|\nabla g(u)|} \tag{36}$$

and then iterated point u^{i+1} is obtained:

$$u^{i+1} = (u\alpha^T)\alpha + \frac{g(u)}{|\nabla g(u)|}\alpha \tag{37}$$

- *Step 5*: Calculate the corresponding reliability index:

$$\beta^{i+1} = \sqrt{(u^{i+1})^T u^{i+1}} \tag{38}$$

- *Step 6*: If convergence in β: $|\beta^{i+1} - \beta^i| \leq 10^{-3}$, then stop, else i=i+1 and go to step 2

First Order Reliability Method (FORM)

Example: Suppose that the performance function of a problem is defined by:

$$g(X) = X1X2\text{-}1500$$

where X1 follows a normal distribution with mean $\mu_{X1} = 33.2$ and standard deviation $\sigma_{X1} = 2.1$. X2 follows a normal distribution with mean $\mu_{X2} = 50$ and standard deviation $\sigma_{X2} = 2.6$.
Using FORM, find reliability index and probability of failure?

Second Order Reliability Method (SORM)

In reality, the limit states are highly non-linear in standard normal space and hence a first order approximation may contribute significant error in reliability index evaluation. Thus, a better approximation by second order terms is required for a highly non-linear limit state.

The procedure to apply the algorithm of Breitung

- *Step 1*: Initial orthogonal matrix T0 is evaluated from the direction cosines, evaluated as explained in FORM under Rackwitz and Fiessler algorithm:

$$T0 = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ \alpha_1 & \alpha_2 & ... & \alpha_n \end{pmatrix} \tag{39}$$

  where $\alpha_1, \alpha_2, ..., \alpha_n$ are the direction cosines of the unit gradient vector at the Most Probable failure Point (MPP).

Second Order Reliability Method (SORM)

- *Step 2*: Consider a matrix T0 $= [t_{01}, t_{02}, ..., t_{0n}]^t$ is modified using Gram-Schmidt orthogonal procedure as:

$$t_k = t_{0k} - \sum_{i=k+1}^{n} \frac{t_i t_{0k}^t}{t_i t_i^t} t_i \tag{40}$$

  where t_k is row vectors of modified orthogonal matrix T $= [t_1, t_2, ..., t_n]^t$ and k ranges from n, n-1, n-2,...,2,1. The rotation matrix in produced by normalizing these vectors to unit.
- *Step 3*: An orthogonal transformation of random variables X into Y is evaluated using orthogonal matrix T (also known as rotation matrix) Y = TX. Again using orthogonal matrix T, another matrix A is evaluated as:

$$A = [a_{ij}] = \frac{(THT^t)_{ij}}{|G^*|} \qquad i, j = 1, 2, ..., n-1 \tag{41}$$

  where $|G^*|$ is the length of the gradient vector and H represents a double derivative matrix of the limit state in standard normal space at the design point.

$$H = \begin{pmatrix} \frac{\partial^2 g}{\partial u_1^2} & \frac{\partial^2 g}{\partial u_1 \partial u_2} & \cdots & \cdots \\ \frac{\partial^2 g}{\partial u_2 \partial u_1} & \frac{\partial^2 g}{\partial u_2^2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \frac{\partial^2 g}{\partial u_n^2} \end{pmatrix} \tag{42}$$

- *Step 4*: the last row and last column in the A matrix and last row in the Y matrix are eliminated to consider a factor that last variable y_n coincides with β computed in FORM.

$$y_n = \beta + \frac{1}{2} y^t A y \tag{43}$$

Now, the size of coefficient matrix A is reduced to (n-1)x(n-1) and main curvatures κ_i are given by computing eigen values of matrix A.

Second Order Reliability Method (SORM)

- *Step 5*: compute the failure probability P_f using Breitung equation:

$$P_f = \phi(-\beta) \prod_{i=1}^{n-1} (1 + \beta\kappa_i)^{-1/2} \tag{44}$$

where κ_i is the main curvatures of the limit state surface at design point.

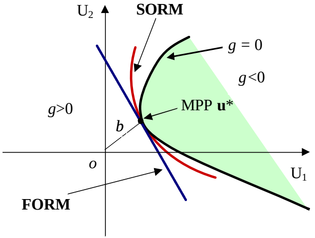


Figure: Comparison of FORM and SORM

Monte Carlo Method

Monte Carlo simulation is a statistical analysis tool and widely used in both non-engineering fields and engineering fields. Monte Carlo is also suitable for solving complex engineering problems because it can deal with a large number of random variables, various distribution types, and highly non-linear engineering models. Different from a physical experiment, Monte Carlo simulation performs random sampling and conducts a large number of experiments on the computer. Three steps are required in the simulation process:

- *Step 1*: Sampling on input random variables (Generating samples of random variables).
  The purpose of sampling on the input random variables $\mathbf{X} = (X_1, X_2, ..., X_n)$ is to generate samples that represent distributions of the input variable from their cdfs $F_{x_i}(x_i)(i = 1, 2, ..., n)$.

Monte Carlo Method

The samples of the random variables will then be used as inputs to the simulation experiments.

- *Step 2*: Numerical experimentation
  Suppose that N samples of each random variable are generated, then all the samples of random variables constitute N sets of inputs, $\mathbf{x_i} = (x_{i1}, x_{i2}, ..., x_{iN}), i = 1, 2, ..., N$ to the model Y=g(X). Solving the problem N times yields N sample points of the output Y:

$$y_i = g(x_i), i = 1, 2, ..., N \tag{45}$$

- *Step 3*: Extraction of probabilistic information of output variables
  After N samples of output Y have been obtained, statistical analysis can be carried out to estimate the characteristics of the output Y, such as the mean, variance, reliability, the probability of failure, pdf and cdf.

Monte Carlo Method

- The mean:

$$\mu = \overline{Y} = \frac{1}{N} \sum_{i=1}^{N} y_i \tag{46}$$

- The variance:

$$\sigma_Y^2 = \frac{1}{N-1} \sum_{i=1}^{N} (y_i - \overline{Y})^2 \tag{47}$$

- The probability of failure:

$$P_f = \frac{1}{N} \sum_{i=1}^{N} I[g(x_i)] = \frac{N_f}{N} \tag{48}$$

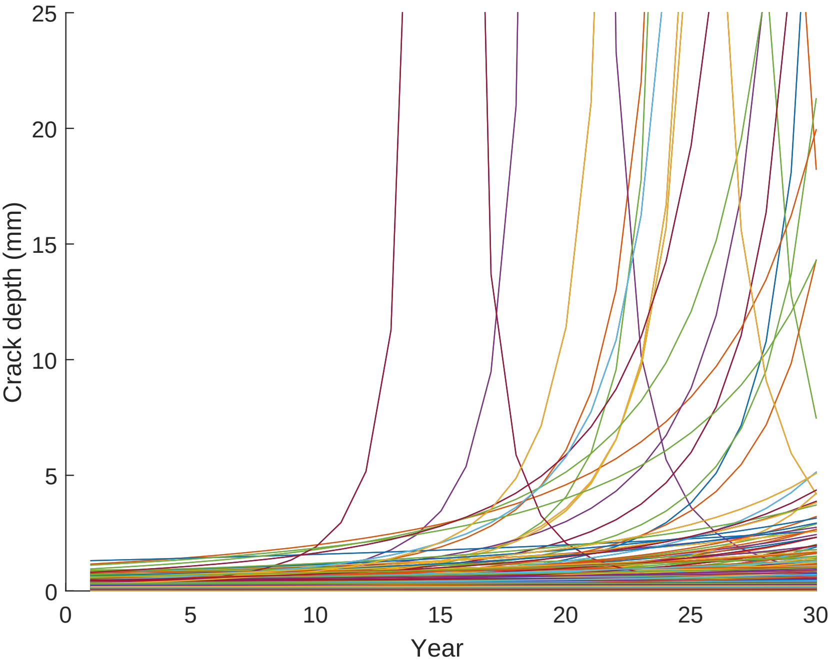
where N_f is the number of samples that have the performance function less than or equal to zero

Monte Carlo Method

- *Step 4*: Error analysis
  The commonly used confidence level is 95% under which the error is approximately given by:

  $$\varepsilon\% \approx 200\sqrt{\frac{(1-P_f}{NP_f}} \tag{49}$$

  where P_f is the true value of the probability of failure and N is required sample number to obtain certain P_f with a predefined error.


Crack depth (mm)
25
20
15
10
5
0
0
5
10
15
20
25
30
Year