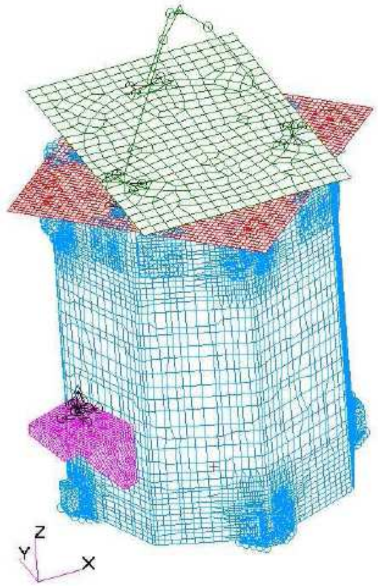
ICOSSAR 2017 International Conference on Structural Safety & Reliability

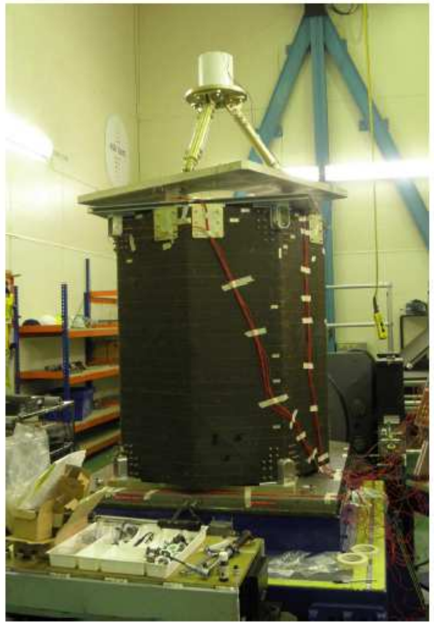
Guiding model improvement in dynamic substructuring: sensitivity analysis and nonparametric probabilistic modeling approach

Maarten Arnst

August 10, 2017

Motivation





Sensitivity analysis of parametric uncertainties and modeling errors in multiple components in the context of nonparametric probabilistic modeling.

Outline

- Motivation.
- Outline.
- Sensitivity analysis.
- Nonparametric probabilistic modeling.
- Illustration.
- Conclusion.

- **Characterization of uncertainty**:
  - Two statistically independent sources of uncertainty modeled as two statistically independent random variables X and Y with probability distributions P_X and P_Y:
    $$(X, Y) \sim P_X \otimes P_Y.$$

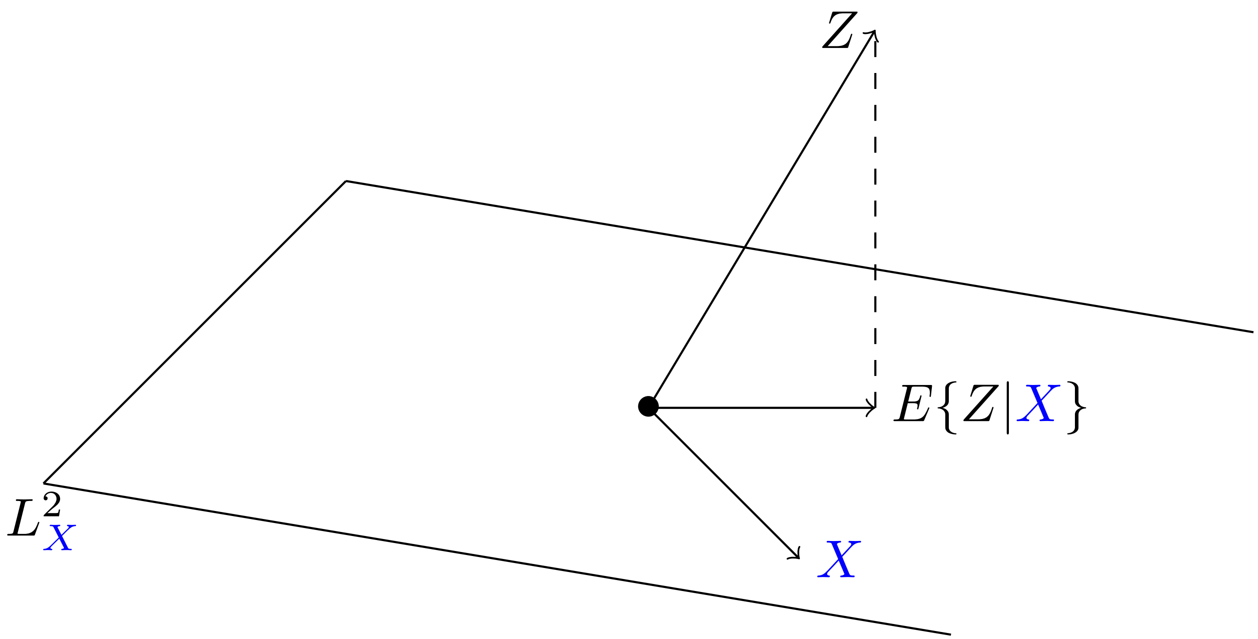
- **Propagation of uncertainty**:
  - We assume that the relationship between the sources of uncertainty and the predictions is represented by a nonlinear function g:

Sources of uncertainty	Problem	Prediction
(X, Y)	$Z = g(X, Y)$	Z

  - The probability distribution P_Z of the prediction is obtained as the image of the probability distribution $P_X \otimes P_Y$ of the sources of uncertainty under the function g:
    $$Z \sim P_Z = (P_X \otimes P_Y) \circ g^{-1}.$$

- **Sensitivity analysis**:
  - Is either X and Y most significant in inducing uncertainty in Z?

- **Least-squares-best approximation of function g with function of only one input**:
  - Assessment of the significance of the source of uncertainty X:
$$g_X^* = \arg\min_{f_X^*} \iint \left| g(x, y) - f_X^*(x) \right|^2 P_X(dx) P_Y(dy).$$
  - By means of the calculus of variations, it can be readily shown that the solution is given by
$$g_X^* = \int g(\cdot, y) P_Y(dy).$$
  - In the geometry of the space of $P_X \otimes P_Y$-square-integrable functions, g_X^* is the orthogonal projection of function g of x and y onto the subspace of functions of only x:



- **Expansion of function g in terms of main effects and interaction effects**:
  - Extension to assessment of significance of both sources of uncertainty X and Y:

$$g(x,y) = g_0 + \underbrace{g_X(x)}_{\text{main effect of } X} + \underbrace{g_Y(y)}_{\text{main effect of } Y} + \underbrace{g_{(X,Y)}(x,y)}_{\text{interaction effect of } X \text{ and } Y},$$

where

$$g_0 = \iint g(x,y) P_X(dx) P_Y(dy),$$

$$g_X(x) = g_X^*(x) - g_0 = \int g(x,y) P_Y(dy) - g_0,$$

$$g_Y(y) = g_Y^*(y) - g_0 = \int g(x,y) P_X(dx) - g_0.$$

  - Because they are obtained via orthogonal projection, the functions g_0, g_X, g_Y, and $g_{(X,Y)}$ are orthogonal functions.
  - The property that g_0, g_X, g_Y, and $g_{(X,Y)}$ are orthogonal provides a link with other expansions, such as the polynomial chaos expansion.

- **Sensitivity indices = mean-square values of main effects and interaction effects**:
  - Quantitative insight into the significance of X and Y in inducing uncertainty in Z:

$$\underbrace{\iint \left|g(x,y)-g_0\right|^2 P_X(dx)P_Y(dy)}_{=\sigma_Z^2}$$

$$= \underbrace{\int \left|g_X(x)\right|^2 P_X(dx)}_{=s_X} + \underbrace{\int \left|g_Y(y)\right|^2 P_Y(dy)}_{=s_Y} + \underbrace{\iint \left|g_{(X,Y)}(x,y)\right|^2 P_X(dx)P_Y(dy)}_{=s_{(X,Y)}} .$$

  - Because g_X, g_Y, and $g_{(X,Y)}$ are orthogonal, there are no double product terms.
  - Thus, the expansion of g (geometry) reflects a **partitioning of the variance** of Z into terms that are the variances of the main and interaction effects of X and Y (statistics), where:

    s_X = portion of the variance of Z that is explained as stemming from X,

    s_Y = portion of the variance of Z that is explained as stemming from Y.

- Computation by means of a **stochastic expansion method**:

$$s_X \approx \sum_{\alpha \neq 0} c^2_{(\alpha,0)}, \qquad s_Y \approx \sum_{\beta \neq 0} c^2_{(0,\beta)}, \qquad \text{with} \quad g(x,y) = \sum_{(\alpha,\beta)} c_{(\alpha,\beta)} \varphi_\alpha(x) \psi_\beta(y).$$

- Computation by means of **deterministic numerical integration**:

$$s_X \approx Q_X\left(|Q_Y g - Q_X Q_Y g|^2\right),$$
$$s_Y \approx Q_Y\left(|Q_X g - Q_X Q_Y g|^2\right).$$

- Computation by means of **Monte Carlo integration**:

$$s_X \approx \frac{1}{\nu} \sum_{\ell=1}^{\nu} \left(g(x_\ell, y_\ell) - \frac{1}{\nu} \sum_{k=1}^{\nu} g(x_k, y_k) \right) \left(g(x_\ell, \tilde{y}_\ell) - \frac{1}{\nu} \sum_{k=1}^{\nu} g(x_k, \tilde{y}_k) \right),$$
$$s_Y \approx \frac{1}{\nu} \sum_{\ell=1}^{\nu} \left(g(x_\ell, y_\ell) - \frac{1}{\nu} \sum_{k=1}^{\nu} g(x_k, y_k) \right) \left(g(\tilde{x}_\ell, y_\ell) - \frac{1}{\nu} \sum_{k=1}^{\nu} g(\tilde{x}_k, y_k) \right).$$

- References: [B. Sudret. Reliab. Eng. Syst. Safe., 2008], [Crestaux et al. Reliab. Eng. Syst. Safe., 2009], [I. Sobol. Math. Comput. Simulat., 2001], and [A. Owen. ACM T. Model. Comput. S., 2013].

Nonparametric probabilistic modeling

- C. Soize. “Nonparametric model of random uncertainties for reduced matrix models in structural dynamics.” In: Probabilistic Engineering Mechanics 15, pp. 277–294, 2000.

  C. Soize and H. Chebli. “Random Uncertainties Model in Dynamic Substructuring Using a Nonparametric Probabilistic Model.” In: Journal of Engineering Mechanics 129, pp. 449–457, 2003.

- Baseline starting point = FE model of linear dynamical behavior of dissipative structure:

$$[M]\ddot{\boldsymbol{u}}(t) + [D]\dot{\boldsymbol{u}}(t) + [K]\boldsymbol{u}(t) = \boldsymbol{f}(t),$$

  where

  $\boldsymbol{u} = (u_1, \ldots, u_m)$ is the (generalized) displacement vector,

  $\boldsymbol{f}$ the (generalized) external forces vector,

  and $[M]$, $[D]$, and $[K]$ the mass, damping, and stiffness matrices.

Nonparametric probabilistic modeling

- **Step 1**: Associate with the deterministic model a **reduced-order model**:

$$
\begin{aligned}
&[M_n]\ddot{\boldsymbol{q}}(t) + [D_n]\dot{\boldsymbol{q}}(t) + [K_n]\boldsymbol{q}(t) = \boldsymbol{f}^n(t),\\
&\boldsymbol{u}^n(t) = [\Phi]\boldsymbol{q}(t),
\end{aligned}
$$

where

$[M_n]$, $[D_n]$, and $[K_n]$ are the reduced mass, damping, and stiffness matrices,

and $[\Phi]$ the matrix collecting in its columns the reduction basis $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, . . ., $\boldsymbol{\varphi}_n$.

Such a reduced-order probabilistic model can be obtained, for instance, by solving the eigenvalue problem associated with the mass and stiffness matrices of the deterministic model,

$$[K]\boldsymbol{\varphi}_j = \lambda_j [M]\boldsymbol{\varphi}_j;$$

in which case the reduced matrices of the reduced-order model are given by

$$[M_n]_{ij} = \delta_{ij}, \quad [D_n]_{ij} = \boldsymbol{\varphi}_i \cdot [D]\boldsymbol{\varphi}_j, \quad [K_n]_{ij} = \lambda_i \delta_{ij}.$$

- **Step 2**: represent the reduced matrices by using **random matrices**:

$$
\begin{aligned}
&[\boldsymbol{M}_n]\ddot{\boldsymbol{Q}}(t) + [\boldsymbol{D}_n]\dot{\boldsymbol{Q}}(t) + [\boldsymbol{K}_n]\boldsymbol{Q}(t) = \boldsymbol{f}^n(t),\\
&\boldsymbol{U}^n(t) = [\Phi]\boldsymbol{Q}(t),
\end{aligned}
$$

To accommodate in the reduced matrices a probabilistic representation of parametric uncertainties and modeling errors, the nonparametric probabilistic approach represents them as follows:

$$
\begin{aligned}
[\boldsymbol{M}_n] &= [L_M][\mathbf{Y}_M][L_M]^{\mathrm{T}},\\
[\boldsymbol{D}_n] &= [L_D][\mathbf{Y}_D][L_D]^{\mathrm{T}},\\
[\boldsymbol{K}_n] &= [L_K][\mathbf{Y}_K][L_K]^{\mathrm{T}},
\end{aligned}
$$

with $[L_M]$, $[L_D]$, and $[L_K]$ the Cholesky factors of $[M_n]$, $[D_n]$, and $[K_n]$.

Nonparametric probabilistic modeling

- To assign a suitable probability distribution to the random matrices $[\boldsymbol{Y}_M]$, $[\boldsymbol{Y}_D]$, and $[\boldsymbol{Y}_K]$, the nonparametric probabilistic approach uses the **maximum entropy principle**.

  The probability distribution thus obtained is such that the mean values of $[\boldsymbol{Y}_M]$, $[\boldsymbol{Y}_D]$, and $[\boldsymbol{Y}_K]$ are all equal to the identity matrix, that is,

$$E\{[\mathbf{Y}_M]\} = [I_n],$$
$$E\{[\mathbf{Y}_D]\} = [I_n],$$
$$E\{[\mathbf{Y}_K]\} = [I_n],$$

  and the amount of uncertainty expressed in $[\boldsymbol{Y}_M]$, $[\boldsymbol{Y}_D]$, and $[\boldsymbol{Y}_K]$ is tunable by free dispersion parameters δ_M, δ_D, and δ_K, respectively, defined by

$$\delta_M = \sqrt{E\{\|[\mathbf{Y}_M] - [I_n]\|_{\mathsf{F}}^2\}/\|[I_n]\|_{\mathsf{F}}^2},$$
$$\delta_D = \sqrt{E\{\|[\mathbf{Y}_D] - [I_n]\|_{\mathsf{F}}^2\}/\|[I_n]\|_{\mathsf{F}}^2},$$
$$\delta_K = \sqrt{E\{\|[\mathbf{Y}_K] - [I_n]\|_{\mathsf{F}}^2\}/\|[I_n]\|_{\mathsf{F}}^2}.$$

- **The dispersion parameters must be calibrated such that the uncertainty in $[\boldsymbol{Y}_M]$, $[\boldsymbol{Y}_D]$, and $[\boldsymbol{Y}_K]$ reflects the significance of the parametric uncertainties and modeling errors**.

Nonparametric probabilistic modeling

- Extension to structures with multiple components = **dynamic substructuring approach**.

- **Step 1**: Associate with the deterministic model a **reduced-order model**:

$$
\begin{bmatrix} [\mathcal{M}_{\mathsf{i}}^1] & [0] & [M_{\mathsf{c}}^1] \\ [0] & [\mathcal{M}_{\mathsf{i}}^2] & [M_{\mathsf{c}}^2] \\ [M_{\mathsf{c}}^1]^{\mathrm{T}} & [M_{\mathsf{c}}^2]^{\mathrm{T}} & [M_{\Sigma}^1]+[M_{\Sigma}^2] \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}}^1(t) \\ \ddot{\boldsymbol{q}}^2(t) \\ \ddot{\boldsymbol{u}}_{\Sigma}(t) \end{bmatrix} + \begin{bmatrix} [\mathcal{D}_{\mathsf{i}}^1] & [0] & [D_{\mathsf{c}}^1] \\ [0] & [\mathcal{D}_{\mathsf{i}}^2] & [D_{\mathsf{c}}^2] \\ [D_{\mathsf{c}}^1]^{\mathrm{T}} & [D_{\mathsf{c}}^2]^{\mathrm{T}} & [D_{\Sigma}^1]+[D_{\Sigma}^2] \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{q}}^1(t) \\ \dot{\boldsymbol{q}}^2(t) \\ \dot{\boldsymbol{u}}_{\Sigma}(t) \end{bmatrix}
$$

$$
+ \begin{bmatrix} [\mathcal{K}_{\mathsf{i}}^1] & [0] & [K_{\mathsf{c}}^1] \\ [0] & [\mathcal{K}_{\mathsf{i}}^2] & [K_{\mathsf{c}}^2] \\ [K_{\mathsf{c}}^1]^{\mathrm{T}} & [K_{\mathsf{c}}^2]^{\mathrm{T}} & [K_{\Sigma}^1]+[K_{\Sigma}^2] \end{bmatrix} \begin{bmatrix} \boldsymbol{q}^1(t) \\ \boldsymbol{q}^2(t) \\ \boldsymbol{u}_{\Sigma}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^1(t) \\ \boldsymbol{f}^2(t) \\ \boldsymbol{f}_{\Sigma}(t) \end{bmatrix},
$$

$$
\begin{bmatrix} \boldsymbol{u}^1(t) \\ \boldsymbol{u}_{\Sigma}(t) \end{bmatrix} = \begin{bmatrix} [\Phi^1] & -[K_{\mathsf{i}}^1]^{-1}[K_{\mathsf{c}}^1] \\ [0] & [I] \end{bmatrix} \begin{bmatrix} \boldsymbol{q}^1(t) \\ \boldsymbol{u}_{\Sigma}(t) \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{u}^2(t) \\ \boldsymbol{u}_{\Sigma}(t) \end{bmatrix} = \begin{bmatrix} [\Phi^2] & -[K_{\mathsf{i}}^2]^{-1}[K_{\mathsf{c}}^2] \\ [0] & [I] \end{bmatrix} \begin{bmatrix} \boldsymbol{q}^2(t) \\ \boldsymbol{u}_{\Sigma}(t) \end{bmatrix}.
$$

Nonparametric probabilistic modeling

- **Step 2**: Represent the reduced matrices by using **random matrices**:

$$\begin{bmatrix} [\boldsymbol{\mathcal{M}}_{\mathsf{i}}^1] & [0] & [\boldsymbol{M}_{\mathsf{c}}^1] \\ [0] & [\boldsymbol{\mathcal{M}}_{\mathsf{i}}^2] & [\boldsymbol{M}_{\mathsf{c}}^2] \\ [\boldsymbol{M}_{\mathsf{c}}^1]^{\mathrm{T}} & [\boldsymbol{M}_{\mathsf{c}}^2]^{\mathrm{T}} & [\boldsymbol{M}_{\Sigma}^1] + [\boldsymbol{M}_{\Sigma}^2] \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{Q}}^1(t) \\ \ddot{\boldsymbol{Q}}^2(t) \\ \ddot{\boldsymbol{Q}}_{\Sigma}(t) \end{bmatrix} + \begin{bmatrix} [\boldsymbol{\mathcal{D}}_{\mathsf{i}}^1] & [0] & [\boldsymbol{D}_{\mathsf{c}}^1] \\ [0] & [\boldsymbol{\mathcal{D}}_{\mathsf{i}}^2] & [\boldsymbol{D}_{\mathsf{c}}^2] \\ [\boldsymbol{D}_{\mathsf{c}}^1]^{\mathrm{T}} & [\boldsymbol{D}_{\mathsf{c}}^2]^{\mathrm{T}} & [\boldsymbol{D}_{\Sigma}^1] + [\boldsymbol{D}_{\Sigma}^2] \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{Q}}^1(t) \\ \dot{\boldsymbol{Q}}^2(t) \\ \dot{\boldsymbol{Q}}_{\Sigma}(t) \end{bmatrix}$$

$$+ \begin{bmatrix} [\boldsymbol{\mathcal{K}}_{\mathsf{i}}^1] & [0] & [\boldsymbol{K}_{\mathsf{c}}^1] \\ [0] & [\boldsymbol{\mathcal{K}}_{\mathsf{i}}^2] & [\boldsymbol{K}_{\mathsf{c}}^2] \\ [\boldsymbol{K}_{\mathsf{c}}^1]^{\mathrm{T}} & [\boldsymbol{K}_{\mathsf{c}}^2]^{\mathrm{T}} & [\boldsymbol{K}_{\Sigma}^1] + [\boldsymbol{K}_{\Sigma}^2] \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}^1(t) \\ \boldsymbol{Q}^2(t) \\ \boldsymbol{Q}_{\Sigma}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}^1(t) \\ \boldsymbol{f}^2(t) \\ \boldsymbol{f}_{\Sigma}(t) \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{U}^1(t) \\ \boldsymbol{U}_{\Sigma}(t) \end{bmatrix} = \begin{bmatrix} [\Phi^1] & -[K_{\mathsf{i}}^1]^{-1}[K_{\mathsf{c}}^1] \\ [0] & [I] \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}^1(t) \\ \boldsymbol{U}_{\Sigma}(t) \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{U}^2(t) \\ \boldsymbol{U}_{\Sigma}(t) \end{bmatrix} = \begin{bmatrix} [\Phi^2] & -[K_{\mathsf{i}}^2]^{-1}[K_{\mathsf{c}}^2] \\ [0] & [I] \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}^2(t) \\ \boldsymbol{U}_{\Sigma}(t) \end{bmatrix}.$$

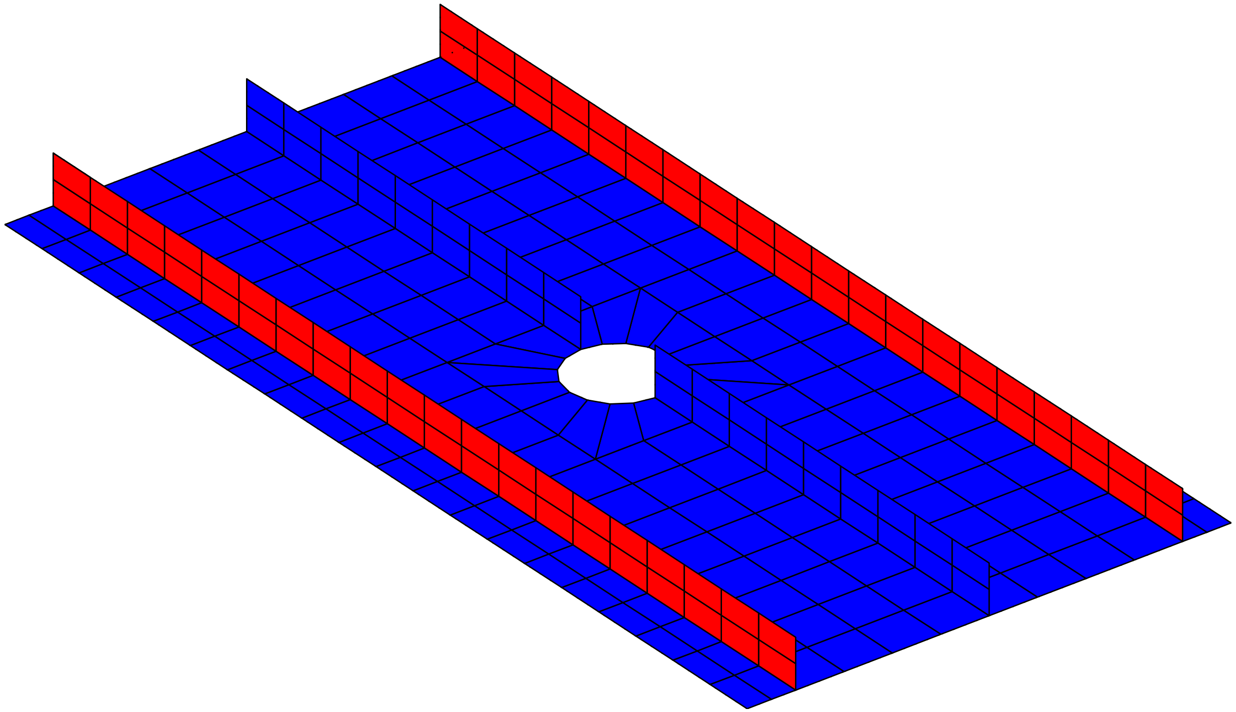
$$[\boldsymbol{M}^1] = \begin{bmatrix} [\boldsymbol{\mathcal{M}}_{\mathsf{i}}^1] & [\boldsymbol{M}_{\mathsf{c}}^1] \\ [\boldsymbol{M}_{\mathsf{c}}^1]^{\mathrm{T}} & [\boldsymbol{M}_{\Sigma}^1] \end{bmatrix}, \quad [\boldsymbol{D}^1] = \begin{bmatrix} [\boldsymbol{\mathcal{D}}_{\mathsf{i}}^1] & [\boldsymbol{D}_{\mathsf{c}}^1] \\ [\boldsymbol{D}_{\mathsf{c}}^1]^{\mathrm{T}} & [\boldsymbol{D}_{\Sigma}^1] \end{bmatrix}, \quad [\boldsymbol{K}^1] = \begin{bmatrix} [\boldsymbol{\mathcal{K}}_{\mathsf{i}}^1] & [\boldsymbol{K}_{\mathsf{c}}^1] \\ [\boldsymbol{K}_{\mathsf{c}}^1]^{\mathrm{T}} & [\boldsymbol{K}_{\Sigma}^1] \end{bmatrix}$$

$$[\boldsymbol{M}^1] = [L_M^1][\mathbf{Y}_M^1][L_M^1]^{\mathrm{T}}, \quad [\boldsymbol{D}^1] = [L_D^1][\mathbf{Y}_D^1][L_D^1]^{\mathrm{T}}, \quad [\boldsymbol{K}^1] = [L_K^1][\mathbf{Y}_K^1][L_K^1]^{\mathrm{T}},$$

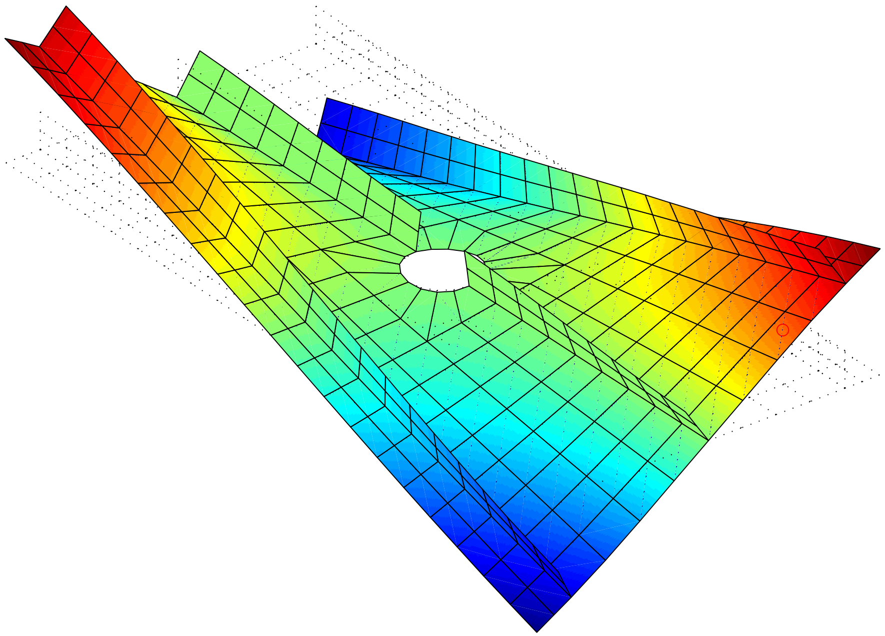
$$[\boldsymbol{M}^2] = \begin{bmatrix} [\boldsymbol{\mathcal{M}}_{\mathsf{i}}^2] & [\boldsymbol{M}_{\mathsf{c}}^2] \\ [\boldsymbol{M}_{\mathsf{c}}^2]^{\mathrm{T}} & [\boldsymbol{M}_{\Sigma}^2] \end{bmatrix}, \quad [\boldsymbol{D}^2] = \begin{bmatrix} [\boldsymbol{\mathcal{D}}_{\mathsf{i}}^2] & [\boldsymbol{D}_{\mathsf{c}}^2] \\ [\boldsymbol{D}_{\mathsf{c}}^2]^{\mathrm{T}} & [\boldsymbol{D}_{\Sigma}^2] \end{bmatrix}, \quad [\boldsymbol{K}^2] = \begin{bmatrix} [\boldsymbol{\mathcal{K}}_{\mathsf{i}}^2] & [\boldsymbol{K}_{\mathsf{c}}^2] \\ [\boldsymbol{K}_{\mathsf{c}}^2]^{\mathrm{T}} & [\boldsymbol{K}_{\Sigma}^2] \end{bmatrix}$$

$$[\boldsymbol{M}^2] = [L_M^2][\mathbf{Y}_M^2][L_M^2]^{\mathrm{T}}, \quad [\boldsymbol{D}^2] = [L_D^2][\mathbf{Y}_D^2][L_D^2]^{\mathrm{T}}, \quad [\boldsymbol{K}^2] = [L_K^2][\mathbf{Y}_K^2][L_K^2]^{\mathrm{T}}.$$

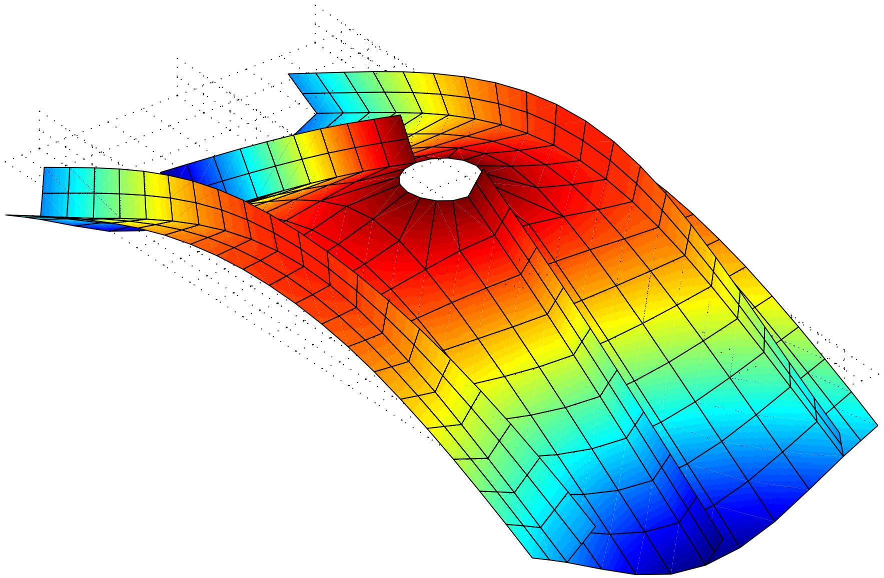
Stiffened panel with a hole.



First few dynamical eigenmodes.



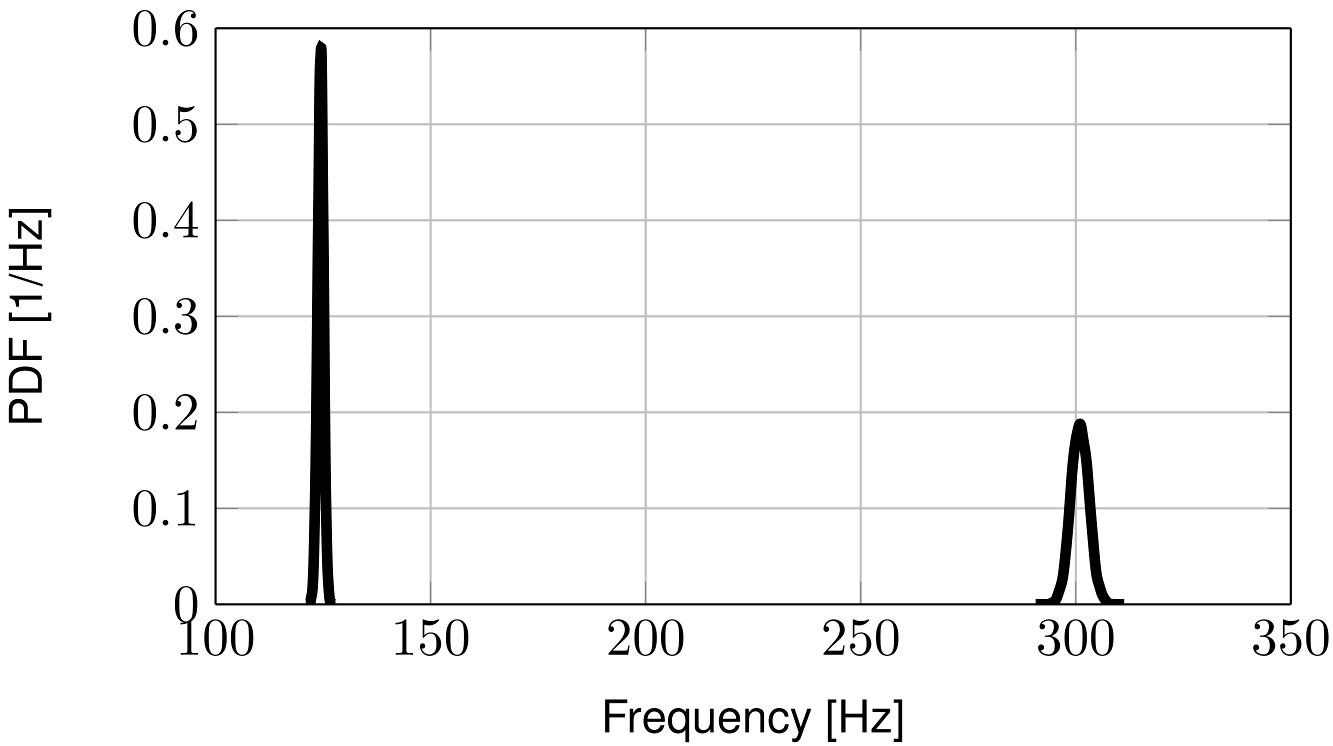
Mode 1 at 124.88 Hz.



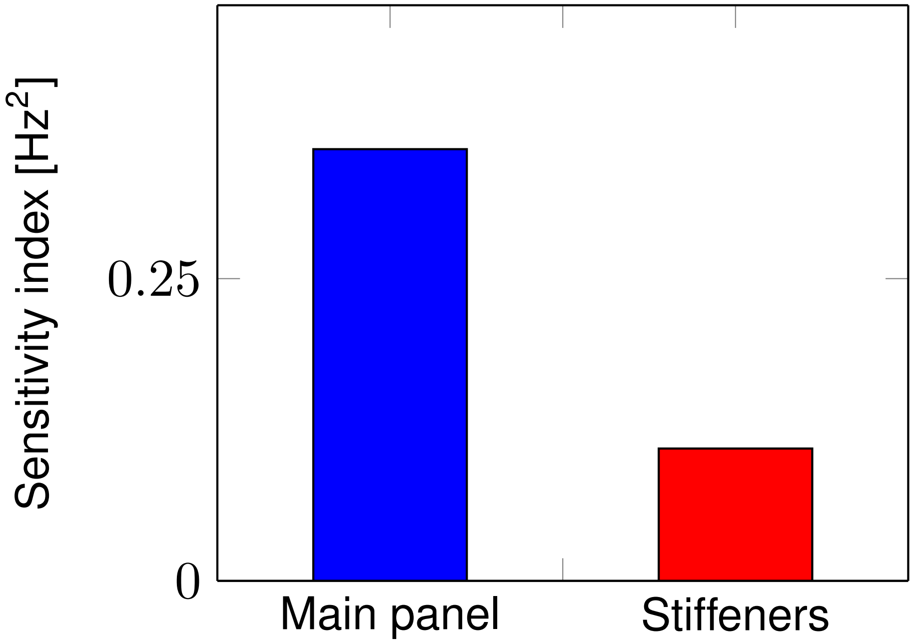
Mode 2 at 302.82 Hz.

Illustration

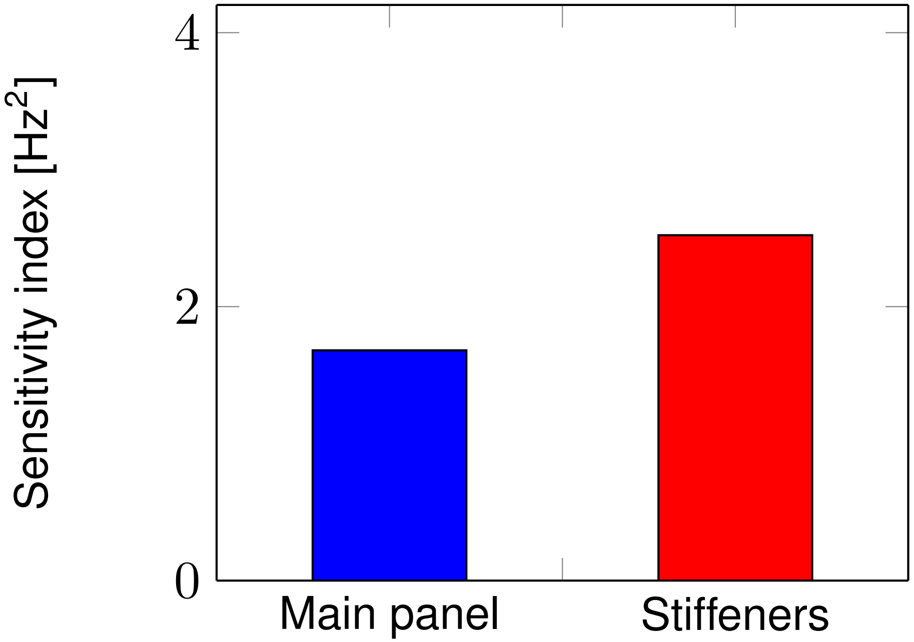
After a component mode synthesis, we used the nonparametric probabilistic approach to introduce uncertainties in the submodels of the main panel and the stiffeners.



PDFs of the first and second eigenfrequencies.



First eigenfrequency.



Second eigenfrequency.

Conclusion and acknowledgement

- Global sensitivity analysis methods can help ascertain which sources of uncertainty are most significant in inducing uncertainty in predictions.

- Although most applications in the literature involve scalar-valued sources of uncertainty, the concepts and methods of global sensitivity analysis are valid and useful more broadly for stochastic process, random fields, random matrices, and other sources of uncertainty.

- When combined with sub structuring approaches, nonparametric probabilistic modeling approaches allow to separately represent parametric uncertainties and modeling errors in separate structural components.

- We discussed global sensitivity analysis of such nonparametric probabilistic models and demonstrated its application in an illustration from structural dynamics.

Conclusion and acknowledgement

- This presentation can be downloaded from our institutional repository:

  http://orbi.ulg.ac.be.

- Other references:

  - M. Arnst and J.-P. Ponthot. An overview of nonintrusive characterization, propagation, and sensitivity analysis of uncertainties in computational mechanics. International Journal for Uncertainty Quantification, 4:387–421, 2014.

  - M. Arnst and K. Goyal. Sensitivity analysis of parametric uncertainties and modeling errors in computational-mechanics models by using a generalized probabilistic modeling approach. Reliability Engineering and System Safety, 167:394–405, 2017.

- Support of the University of Liège through a starting grant is gratefully acknowledged.