

System-wise equivalent static loads for the design of flexible mechanisms

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Highlights

- System-wise equivalent static loads (ESL) are defined for mechanism design.
- Global displacements of the mechanism are naturally included in the optimization problem formulation.
- The generality of the resulting static response optimization problem is extended.
- System-wise ESLs are detailed for a particular Lie group nonlinear finite element formalism.
- Properties of the adopted formalism are exemplified for three mechanism designs.

Abstract

Structural optimization of flexible multibody systems (MBS) is a challenging design problem which has received increased attention in the last decade. The weakly coupled method is a powerful method to solve this design problem as it transforms the underlying dynamic response optimization problem into a static response optimization problem subject to multiple load cases. These load cases can be conveniently defined using the so-called Equivalent Static Load (ESL) method. With the usual ESL method, ESLs are defined at a component-level assuming that each component is isolated from the rest of the system. Thus, this method cannot account for design functions involving the global response of the system but it is restricted to optimization problems that are solely formulated in terms of local responses within the components. In order to get rid of this restriction, a system-wise extension of the ESL method is proposed. It is shown that the solutions of the equivalent static optimization problem and of the initial MBS optimization problem are equivalent and that the ESL formulation naturally leads to a weakly coupled solution algorithm. Standard benchmarks exemplify the proposed method.

Keywords: structural optimization, flexible multibody systems, weakly coupled method, equivalent Static Load (ESL), nonlinear finite element method, local frame formalism (SE(3))

1 Introduction

1.1 State of the art

The optimal design of mechanical system components usually relies on a component-based approach wherein standard methods of static response optimization are applied. In this approach, the optimized components are isolated from the system and subjected to boundary and loading conditions to predict deformations, stress distributions, etc. These boundary and loading conditions generally come from the designer experience, empirical relations or experimental efforts. Although the majority of loads are dynamic, structural optimization is then typically performed under (quasi-)static or frequency domain loadings, thereby avoiding the difficulties of evaluating the dynamic response and incorporating it in the optimization.

With the evolution of multibody system (MBS) analysis, the component-based approach has been extended to a system-based approach. The latter captures the system behavior by employing a MBS simulation that evaluates the response of the whole system. Bruns and Tortorelli [7] were amongst the first to propose such an approach by combining rigid MBS analysis and optimization techniques. To perform the optimization, they extracted some load cases evaluated during the MBS analysis. Later on, Oral and Kemal Ider [23] proposed a methodology to consider the coupled rigid-elastic motion of MBS system within the optimization loop and they incorporated time-dependent constraints in the optimization problem. This holistic approach is essential since the optimal design may be very sensitive to the support and loading conditions.

Adopting the system-based approach to conduct optimization of mechanical systems, two optimization methods can currently be identified: the weakly coupled and the fully coupled methods [40]. The fully coupled method solves a dynamic response optimization problem which incorporates the time response coming directly from the MBS simulation. Hence, this method naturally accounts for the system global response and dynamic effects. The exact sensitivities of the MBS time response can be evaluated and exploited for a fast convergence of the optimization algorithm in the vicinity of the optimal design. However, the formulation of the MBS optimization problem should be carefully addressed as it significantly impacts the optimal solution and also the convergence of the optimization process [37]. In particular, the treatment of time-dependent responses is rather complex and the influence of the changes of component inertial property on vibrations and the interactions between flexible components generally result in complex design problems. Nonetheless, even if the optimization process may be arduous, the fully coupled method is totally general to design optimal mechanisms and has been employed in numerous works, for instance [6, 14, 25, 35, 39].

Instead of solving a dynamic response optimization problem, the weakly coupled method reformulates the design problem in a two-step approach. Firstly, a MBS simulation predicts the dynamic loading applied to the system and secondly, a quasi-static approach is adopted to solve the design problem. In this second step, the Equivalent Static Load (ESL) method has become the reference method to define a series of static load cases mimicking the dynamic loading in a rational manner. Even though ESLs are design-dependent, the static response optimization process considers them as constant, i.e. design-independent. Consequently, cycles between MBS analysis and optimization process are required to factor in the effect of mass redistribution over the load cases. Compared to the fully coupled method, it is however not necessary to evaluate the sensitivity of the MBS dynamic response, which is a significant advantage. The weakly coupled method has been used in various works, for instance [16, 19, 21, 27, 32, 33, 41].

1.2 Motivations for the present work

The Equivalent Static Load method is the cornerstone of the weakly coupled method. Indeed, the ESL method defines equivalent static load cases rationally whereupon the dynamic response optimization problem can be transformed into an equivalent static response optimization problem. The latter is then efficiently solved using standard techniques of structural optimization.

The ESL method was first introduced by Choi and Park [8, 9] to perform structural optimization of a single structure (not mechanism) subject to dynamic loading. The main motivations of developing such a method were to avoid the expensive computation of state variable derivatives, the possibility of using efficient commercial solvers for structural optimization of the static response and to circumvent the difficult treatment of time-dependent constraints.

Kang *et al.* [21] extended the ESL method to optimize mechanical systems modeled using a floating frame of reference formulation. This MBS formalism is well-suited to derive ESLs as originally proposed by Choi and Park for a structure [10, 20]. Indeed, this MBS formalism describes the flexibility of each component in a body-attached frame. Thereby, ESLs for each body can be derived by applying the ESL method developed for classical structures. Such ESLs, defined as if the body was isolated from the rest of the system, are hereafter qualified as *component-wise ESLs*. The component-wise ESL derivation for other formalisms may raise further questions because of the necessity to select an appropriate reference frame for the static load definition. For instance, a classical nonlinear finite element method [13] describes the flexibility with respect to an inertial frame and not to body-attached frames. Consequently, the choice of the frame to define ESLs is ambiguous. Studies concerning ESL derivations for other formalisms [21, 36, 38, 32] also adopted the definition of ESLs with respect to body-attached frames. We thus propose a new definition applicable to mechanical systems independently of the adopted MBS formalism, which specifies that ESLs are derived for isolated components in body-attached frames, yielding,

Definition 1. When a dynamic load is applied to a mechanism, the **component-wise equivalent static load** for an isolated body is defined as the static load that produces the same relative displacement field, expressed in a body-attached frame, as the one created by the dynamic load at an arbitrary time.

This component-wise ESL definition leads to an efficient approach to optimize mechanical system components, provided that the initial optimization problem is only formulated in terms of local responses of each body with respect to their body-attached frame. This means that the cost and constraint functions should only depend on local behaviors and not on the full motion of the system. In this case, the initial MBS optimization problem can be transformed into a static response optimization problem based on component-wise ESLs. By construction, the solution of the initial and equivalent static response optimization problems are rigorously equivalent [40]. Under the assumption that the sensitivity of the ESLs can be neglected in the static response optimization process, the modified formulation can then be solved efficiently using a weakly coupled algorithm which involves iterations between the MBS analysis, the processing of ESLs and the solution of the static response optimization problem. We note that even if the coupling between components are disregarded in the resulting static response optimization problems, ESLs account for interactions within the system since they stem from the MBS analysis.

Formulating the optimization process only with respect to local displacements of components is restrictive, not always feasible and counter-intuitive. An engineering approach would rather formulate the optimization problem with respect to a combination of system response and of component responses when needed, such as to account for stress constraints. In addition to the previous limitation, the decoupling of components implies that kinematic joints are disregarded during the static analysis. As a consequence, rigid body modes must be prevented by enforcing boundary conditions to fix the body-attached frames. Consequently, the definition of body-attached frames, which bares some arbitrariness, may have an important influence on the optimal design.

In this research, a generalized ESL definition for the optimal design of mechanisms is proposed. The generalization consists in defining *system-wise ESLs* in contrast to *component-wise ESLs*. The proposed definition is stated as follows,

Definition 2. When a dynamic load is applied to a mechanism, the **system-wise equivalent static load** is the static load defined at the system level, that produces the same deformed configuration of the mechanism as the one created by the dynamic load at an arbitrary time.

As a result, the limitation of formulating an optimization problem only with respect to local responses is removed. Thus, any optimization problem whose formulation involves the global displacement field, reaction forces in the kinematic joints of the system, as well as local displacements can be treated. Moreover, the extension can be performed with a similar level of complexity as with the component-wise ESL method.

The layout of this paper is as follows. Section 2 introduces the equations of motion of flexible multi-body systems in a general setting as well as the formulation of structural optimization problems for flexible mechanisms. In Section 3, the definition of system-wise ESLs is proposed and discussed. The framework of the weakly coupled method to solve the optimization problem using system-wise ESLs is also presented. Following the general definition of system-wise ESLs, the derivation of system-wise ESLs for a particular nonlinear finite element formalism is proposed in Section 4. A local frame formalism, relying on the representation of the motion on the special Euclidean group (SE(3)) is adopted. It is demonstrated that the properties of this formalism lead to an efficient and convenient approach to evaluate system-wise ESLs. Classical benchmarks are proposed in Section 5 to illustrate the developed method. Section 6 concludes the paper.

2 Mechanism design problem formulation

2.1 Flexible multibody system dynamics

Considering that spatial semi-discretization of the flexible bodies has been conducted, the configuration of a flexible mechanism can be described by a time-dependent configuration variable $q(t)$. The equations of motion of flexible multibody systems can be established by applying fundamental principles of mechanics

and can be stated in a general way as [1, 13, 26]

$$\dot{q}(t) = Z(q(t), \mathbf{v}(t)), \quad (1)$$

$$\mathbf{M}(q(t)) \dot{\mathbf{v}}(t) + \mathbf{B}^T(q(t), t) \boldsymbol{\lambda}(t) = \mathbf{g}(q(t), \mathbf{v}(t), t), \quad (2)$$

$$\Phi(q(t)) - \Psi(t) = \mathbf{0}, \quad (3)$$

where Z expresses the velocity compatibility, $\mathbf{v}(t)$ is the velocity vector, $\mathbf{M}$ is the mass matrix and $\mathbf{g}(q, \mathbf{v}, t) = \mathbf{g}_{ext}(q, t) - \mathbf{g}_{gyr}(q, \mathbf{v}) - \mathbf{g}_{int}(q)$. $\mathbf{g}_{ext}$, $\mathbf{g}_{gyr}$ and $\mathbf{g}_{int}$ are respectively the external, gyroscopic and internal force vectors. Equation (3) represents the kinematic constraints wherein the function Φ stands for the kinematics of the constraints and the function $\Psi(t)$ for the time evolution of the driven motion in the case of a rheonomous system. If the system is scleronomous, kinematic constraints do not depend explicitly on time ($\Psi(t) = \mathbf{0}$) and we have $\Phi(q(t)) = \mathbf{0}$. The matrix $\mathbf{B}$ represents the kinematic constraint gradients and $\boldsymbol{\lambda}(t)$ is the Lagrange multiplier vector associated with the enforcement of constraints.

The variable $q(t)$ can be either a vector evolving on a linear k -dimensional space $\mathbb{R}$, as in classical finite element formulations for instance, or on a more general quantity, such as a element of a k -dimensional manifold G with a Lie group structure. The latter is adopted in Section 4. The velocity compatibility equation (1) can take several expressions depending on the adopted definition for the variables q and $\mathbf{v}$. The simplest case boils down to $\dot{q}(t) = \mathbf{v}(t)$.

Since kinematic constraints are represented by algebraic equations, the equations of motion (1)-(3) form a mixed set of second order Differential and Algebraic Equations (DAEs). Several solution strategies exist to perform time integration.

2.2 Optimization problem formulation - Fully coupled method

Mechanism design problems can be formulated as mathematical optimization problems that are subsequently solved using numerical techniques. Compared to standard structural optimization problems, the dynamic analysis brings the time parameter t in the optimization problem formulation, leading to a dynamic response optimization problem.

The design problem is typically formulated as the minimization of an objective function $f_0(\mathbf{p})$ subject to n_c constraints $f_j(\mathbf{p}) \leq \bar{f}_j$, where the n_v independent design variables p_i are gathered in the vector $\mathbf{p} \in \mathbb{R}^{n_v}$. These constraints ensure the integrity of the structural design and its manufacturability [4, 15, 22]. Side-constraints $\underline{p}_i \leq p_i \leq \bar{p}_i$ restrict the design space and usually reflect technological considerations.

Adopting a *nested* approach [15] and incorporating explicitly the equations of motion (1)-(3), the design problem reads

$$\begin{aligned} & \underset{\mathbf{p}}{\text{minimize}} && f_0(\mathbf{p}, s(\cdot)) \\ & \text{subject to} && \text{EOM}(\mathbf{p}, s(t), t) = \mathbf{0}, && t \in [t_0, t_f], \\ & && q(t_0) = q^0(\mathbf{p}), \\ & && \mathbf{v}(t_0) = \mathbf{v}^0(\mathbf{p}), \\ & && f_j(\mathbf{p}, s(t), t) \leq \bar{f}_j, && j = 1, \dots, n_{ct}, \\ & && f_j(\mathbf{p}, s(\cdot)) \leq \bar{f}_j, && j = n_{ct} + 1, \dots, n_c, \\ & && \underline{p}_i \leq p_i \leq \bar{p}_i, && i = 1, \dots, n_v, \end{aligned} \quad (4)$$

where $q(t_0) = q^0(\mathbf{p})$ and $\mathbf{v}(t_0) = \mathbf{v}^0(\mathbf{p})$ are the initial conditions of the equations of motion represented by $\text{EOM}(\mathbf{p}, s(t), t) = \mathbf{0}$. In the equations of motion, the operators $\mathbf{M}(\mathbf{p}, q)$, $\mathbf{g}(\mathbf{p}, q, \mathbf{v}, t)$, $\Phi(\mathbf{p}, q)$ and $\mathbf{B}(\mathbf{p}, q, t)$ are design-dependent. The variable $s(t)$, defined as

$$s(t) = [q(t), \mathbf{v}(t), \dot{\mathbf{v}}(t), \boldsymbol{\lambda}(t)], \quad t \in [t_0, t_f], \quad (5)$$

is a set of “dependent” variables since the MBS response can be evaluated for a given value of $\mathbf{p}$ by time integration of the equation of motion with the initial conditions q^0 and $\mathbf{v}^0$. As the equations of motion and the initial conditions depend on the design variables, the variable $s(t)$ thus implicitly depends on the design variables $\mathbf{p}$. A distinction is made between n_{ct} time-dependent constraints that are imposed at every time instant and n_{cg} global constraints involving aggregated quantities over the whole time interval. We thus have $n_{ct} + n_{cg} = n_c$ constraints. Cost and constraint functions can be, for instance,

the mass of the system, stress measures at particular material points for specific time instants or a mean displacement measure aggregated over a certain period of time.

The optimization problem formulation (4) is totally general and the design problem can be solved by various optimization algorithms. The fully coupled method tackles directly the dynamic response optimization problem (4).

3 The weakly coupled method

3.1 Component-wise equivalent static load

Employing the component-wise ESL method, only a simplified version of the original design problem (4) can be addressed [40]. Indeed, only the local displacements $\mathbf{q}_e = [\mathbf{q}_e^{1T} \dots \mathbf{q}_e^{n_b T}]^T$ of the system components, where n_b corresponds to the number of bodies, are available to formulate the design functions. The local displacements $\mathbf{q}_e$ are extracted from the system configuration q based on a formula that depends on the MBS formalism adopted to describe the system dynamics. The simplified design problem reads

$$\begin{aligned}
& \underset{\mathbf{p}}{\text{minimize}} && f_0(\mathbf{p}, \mathbf{q}_e(\cdot)) \\
& \text{subject to} && \text{EOM}(\mathbf{p}, q(t), \mathbf{v}(t), \dot{\mathbf{v}}(t), \boldsymbol{\lambda}(t), t) = \mathbf{0}, && t \in [t_0, t_f], \\
& && q(t_0) = q^0(\mathbf{p}), \\
& && \mathbf{v}(t_0) = \mathbf{v}^0(\mathbf{p}), \\
& && f_j(\mathbf{p}, \mathbf{q}_e(t), t) \leq \bar{f}_j, && j = 1, \dots, n_{ct}, \\
& && f_j(\mathbf{p}, \mathbf{q}_e(\cdot)) \leq \bar{f}_j, && j = n_{ct} + 1, \dots, n_c, \\
& && \underline{p}_i \leq p_i \leq \bar{p}_i, && i = 1, \dots, n_v.
\end{aligned} \tag{6}$$

To transform the dynamic response optimization problem (6), the weakly coupled method relies on the definition of a well-posed static problem for each body. The static equilibrium equation (SEE) for body b is expressed as

$$\text{SEE}^b(\mathbf{p}, \mathbf{q}_e^b, \mathbf{g}_{eq}^b) := \mathbf{g}_{int}^b(\mathbf{p}, \mathbf{q}_e^b) - \mathbf{g}_{eq}^b = \mathbf{0}, \tag{7}$$

where $\mathbf{g}_{int}^b$ and $\mathbf{g}_{eq}^b$ represent respectively the inner force vector and the equivalent static load vector acting on body b . It is important to note that as components are isolated from the system, boundary conditions must be enforced in this equivalent static problem to fix rigid body modes and to obtain a well-posed problem. Subsequently, the simplified design problem (6) is reformulated as

$$\begin{aligned}
& \underset{\mathbf{p}}{\text{minimize}} && f_0(\mathbf{p}, \hat{\mathbf{q}}_e(\cdot)) \\
& \text{subject to} && \text{EOM}(\mathbf{p}, q(t), \mathbf{v}(t), \dot{\mathbf{v}}(t), \boldsymbol{\lambda}(t), t) = \mathbf{0}, && t \in [t_0, t_f], \\
& && q(t_0) = q^0(\mathbf{p}), \\
& && \mathbf{v}(t_0) = \mathbf{v}^0(\mathbf{p}), \\
& && \text{SEE}^b(\mathbf{p}, \hat{\mathbf{q}}_e^b(t), \mathbf{g}_{eq}^b(t)) = \mathbf{0}, && b = 1, \dots, n_b, \\
& && \text{SEE}^b(\mathbf{p}, \mathbf{q}_e^b(t), \mathbf{g}_{eq}^b(t)) = \mathbf{0}, && b = 1, \dots, n_b, \\
& && f_j(\mathbf{p}, \hat{\mathbf{q}}_e(t), t) \leq \bar{f}_j, && j = 1, \dots, n_{ct}, \\
& && f_j(\mathbf{p}, \hat{\mathbf{q}}_e(\cdot)) \leq \bar{f}_j, && j = n_{ct} + 1, \dots, n_c, \\
& && \underline{p}_i \leq p_i \leq \bar{p}_i, && i = 1, \dots, n_v.
\end{aligned} \tag{8}$$

The optimization problem formulations (6) and (8) are formally equivalent. Indeed, since the static equilibrium equation is well-posed, it admits a unique solution. As a result, the solution of the optimization problem (8) is such that $\hat{\mathbf{q}}_e(t) = \mathbf{q}_e(t)$. The interpretation of the load $\mathbf{g}_{eq}^b(t)$ as the equivalent static load for body b is justified here since it gives a static displacement field equivalent to the one obtained from the dynamic analysis for the considered time instant. Also, the optimization problem formulation (8) highlights that ESLs are design-dependent. As it will be discussed in Section 3.3, this problem can be solved efficiently using an iterative scheme wherein the sensitivity of the ESLs is neglected during the static response optimization [17].

3.2 System-wise equivalent static load

The design problem (6) only involves the local displacements $\mathbf{q}_e$ of the individual bodies. Consequently, cost and constraint functions have to be formulated with respect to these local variables, which reduces the generality of the design problem compared to the original design problem (4). To overcome this limitation, the design problem must be formulated with respect to the global displacement $\mathbf{q}$ as

$$\begin{aligned}
& \underset{\mathbf{p}}{\text{minimize}} && f_0(\mathbf{p}, \mathbf{q}(\cdot)) \\
& \text{subject to} && \text{EOM}(\mathbf{p}, q(t), \mathbf{v}(t), \dot{\mathbf{v}}(t), \boldsymbol{\lambda}(t), t) = \mathbf{0}, && t \in [t_0, t_f], \\
& && q(t_0) = q^0(\mathbf{p}), \\
& && \mathbf{v}(t_0) = \mathbf{v}^0(\mathbf{p}), \\
& && f_j(\mathbf{p}, \mathbf{q}(t), t) \leq \bar{f}_j, && j = 1, \dots, n_{ct}, \\
& && f_j(\mathbf{p}, \mathbf{q}(\cdot)) \leq \bar{f}_j, && j = n_{ct} + 1, \dots, n_c, \\
& && p_i \leq \bar{p}_i \leq \bar{p}_i, && i = 1, \dots, n_v.
\end{aligned} \tag{9}$$

To transform the dynamic response optimization problem (9) into an equivalent static response optimization problem, we propose to supersede the static problem (7) defined for each body by a static problem defined system-wise as

$$\text{SEE}(\mathbf{p}, q, \boldsymbol{\lambda}_{eq}, \mathbf{g}_{eq}, \boldsymbol{\Psi}_{eq}, t) := \begin{aligned} & \mathbf{g}_{int}(\mathbf{p}, q) + \mathbf{B}_{eq}^T(\mathbf{p}, q, t) \boldsymbol{\lambda}_{eq} - \mathbf{g}_{eq} = \mathbf{0}, \\ & \boldsymbol{\Phi}_{eq}(\mathbf{p}, q) - \boldsymbol{\Psi}_{eq}(t) = \mathbf{0}, \end{aligned} \tag{10}$$

where the term $\mathbf{B}_{eq}^T \boldsymbol{\lambda}_{eq}$ explicitly accounts for the reaction forces of kinematic joints and from which the equivalent static load vector $\mathbf{g}_{eq}$ can be evaluated. In absence of rigid modes in the MBS, we simply have $\boldsymbol{\Phi}_{eq} = \boldsymbol{\Phi}$, $\mathbf{B}_{eq}^T = \mathbf{B}^T$ and $\boldsymbol{\lambda}_{eq} = \boldsymbol{\lambda}$. However, in presence of rigid modes, the solution of this static problem is not unique. In order to obtain a well-posed static problem, we propose to introduce additional kinematic constraints $\boldsymbol{\Phi}_r(q) - \boldsymbol{\Psi}_r(t) = \mathbf{0}$ which have to be selected in order to prevent rigid body modes. Consequently, we have

$$\boldsymbol{\Phi}_{eq} = \begin{bmatrix} \boldsymbol{\Phi} \\ \boldsymbol{\Phi}_r \end{bmatrix}, \quad \boldsymbol{\Psi}_{eq} = \begin{bmatrix} \boldsymbol{\Psi} \\ \boldsymbol{\Psi}_r \end{bmatrix}, \quad \mathbf{B}_{eq}^T = [\mathbf{B}^T, \mathbf{B}_r^T] \quad \text{and} \quad \boldsymbol{\lambda}_{eq} = \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\lambda}_r \end{bmatrix}. \tag{11}$$

Solving (10) gives the system configuration whereupon the optimization problem can be formulated at the system level and the formulation is no longer restricted to local displacements. By construction of the additional kinematic constraints, for given $\mathbf{p}$ and $\mathbf{g}_{eq}$, the SEE possesses a unique solution. The equivalent static load is thus represented by the variables $\mathbf{g}_{eq}$ and $\boldsymbol{\Psi}_{eq}$. In comparison to the component-wise ESL method, the major difference is that ESLs are now defined at the system level. Hence, to account for the connectivity between components, kinematic constraints are incorporated in the equivalent static problem.

The general design problem (4) is then reformulated as

$$\begin{aligned}
& \underset{\mathbf{p}}{\text{minimize}} && f_0(\mathbf{p}, \hat{q}(\cdot), \hat{\boldsymbol{\lambda}}(\cdot)) \\
& \text{subject to} && \text{EOM}(\mathbf{p}, q(t), \mathbf{v}(t), \dot{\mathbf{v}}(t), \boldsymbol{\lambda}(t), t) = \mathbf{0}, && t \in [t_0, t_f], \\
& && q(t_0) = q^0(\mathbf{p}), \\
& && \mathbf{v}(t_0) = \mathbf{v}^0(\mathbf{p}), \\
& && \text{SEE}(\mathbf{p}, \hat{q}(t), \hat{\boldsymbol{\lambda}}_{eq}(t), \mathbf{g}_{eq}(t), \boldsymbol{\Psi}_{eq}(t), t) = \mathbf{0}, \\
& && \text{SEE}(\mathbf{p}, q(t), \boldsymbol{\lambda}_{eq}(t), \mathbf{g}_{eq}(t), \boldsymbol{\Psi}_{eq}(t), t) = \mathbf{0}, \\
& && f_j(\mathbf{p}, \hat{q}(t), \hat{\boldsymbol{\lambda}}(t), t) \leq \bar{f}_j, && j = 1, \dots, n_{ct}, \\
& && f_j(\mathbf{p}, \hat{q}(\cdot), \hat{\boldsymbol{\lambda}}(\cdot)) \leq \bar{f}_j, && j = n_{ct} + 1, \dots, n_c, \\
& && p_i \leq \bar{p}_i \leq \bar{p}_i, && i = 1, \dots, n_v.
\end{aligned} \tag{12}$$

Since the system-level static equilibrium equation (10) is well-posed, it admits a unique solution. As a result, the solution of the optimization problem (12) is such that $\hat{q}(t) = q(t)$ and $\hat{\boldsymbol{\lambda}}_{eq}(t) = \boldsymbol{\lambda}_{eq}(t)$. This proves the formal equivalence between the design problems (9) and (12). It is important to note that system-wise ESLs are also design-dependent.

3.3 The optimization framework

Although ESLs are design-dependent, the weakly coupled method assumes that ESLs are fixed during the static response optimization process. Ergo, to account for the effects of design modifications over the ESLs, cycles are needed between MBS analysis and static response optimization process. The purpose of performing the static response optimization process with fixed ESLs is that for systems with moderate flexibility, less MBS analyses are usually required to achieve convergence. Hence, the overall computation time can be reduced since dynamic analyses are more time consuming than static analyses.

The iterative process of the weakly coupled method can deal with system-wise ESLs without difficulty. Consequently, the standard framework is adopted to solve (12) and consists of the following steps:

1. Initialization: initialize the design variables $\mathbf{p}$ and set the cycle counter it to 1.
2. MBS analysis: perform the forward time integration of the equations of motion,

$$\begin{aligned} \text{EOM}(\mathbf{p}, q(t), \mathbf{v}(t), \dot{\mathbf{v}}(t), \boldsymbol{\lambda}(t), t) &= \mathbf{0}, \quad t \in [t_0, t_f], \\ q(t_0) &= q^0(\mathbf{p}), \\ \mathbf{v}(t_0) &= \mathbf{v}^0(\mathbf{p}). \end{aligned} \quad (13)$$

3. ESL process: evaluate the system-wise ESLs $\mathbf{g}_{eq}$ and $\boldsymbol{\Psi}_{eq}$ based on the response $q(t)$ obtained in step 2,

$$\text{SEE}(\mathbf{p}, q, \boldsymbol{\lambda}_{eq}, \mathbf{g}_{eq}, \boldsymbol{\Psi}_{eq}, t) = \mathbf{0}. \quad (14)$$

4. Static response optimization: solve the equivalent static response optimization problem assuming that ESLs are design-independent:

$$\begin{aligned} &\underset{\mathbf{p}}{\text{minimize}} && f_0(\mathbf{p}, \hat{q}(\cdot), \hat{\boldsymbol{\lambda}}(\cdot)) \\ &\text{subject to} && \text{SEE}(\mathbf{p}, \hat{q}(t), \hat{\boldsymbol{\lambda}}_{eq}(t), \mathbf{g}_{eq}(t), \boldsymbol{\Psi}_{eq}(t), t) = \mathbf{0}, \quad t \in [t_0, t_f], \\ & && f_j(\mathbf{p}, \hat{q}(t), \hat{\boldsymbol{\lambda}}(t), t) \leq \bar{f}_j, \quad j = 1, \dots, n_{ct}, \\ & && f_j(\mathbf{p}, \hat{q}(\cdot), \hat{\boldsymbol{\lambda}}(\cdot)) \leq \bar{f}_j, \quad j = n_{ct} + 1, \dots, n_c, \\ & && \underline{p}_i \leq p_i \leq \bar{p}_i, \quad i = 1, \dots, n_v. \end{aligned} \quad (15)$$

The iterations to solve this optimization problem are denoted as inner iterations.

5. Convergence check:

$$\text{If } \frac{\|p_{i,it} - p_{i,it-1}\|}{\|p_{i,it-1}\|} \leq \varepsilon, \quad \text{for } i = 1, \dots, n_v, \quad (16)$$

stop, otherwise set $it = it + 1$ and go to step 2.

The steps 2 to 5 compose one cycle of the iterative process. A stopping criterion based on a relative change of the design variable values is used to terminate the cycles. This criterion is also adopted to stop the static response optimization problem. Nonetheless other choices are possible. While the values of ε are the same for the two processes, the value of the stopping criterion may be slightly relaxed for the first static response optimization problems since ESLs may strongly vary from one cycle to the other so that a tight convergence of the inner iterations is not necessary. Figure 1 illustrates the flowchart of the weakly coupled method.

To solve the equivalent static response optimization problem, gradient-based methods can be employed. These methods are well-suited to solve static response optimization problems as demonstrated by many examples, see for instance [11, 24, 28]. In this study, the *Method of Moving Asymptotes* (MMA) [34] is adopted.

4 Derivation of system-wise ESLs adapted to a local frame formalism

This section is dedicated to exemplify the derivation of system-wise ESLs for a particular MBS formalism. Although the component-wise and system-wise ESL definitions introduced respectively in (7) and (10)

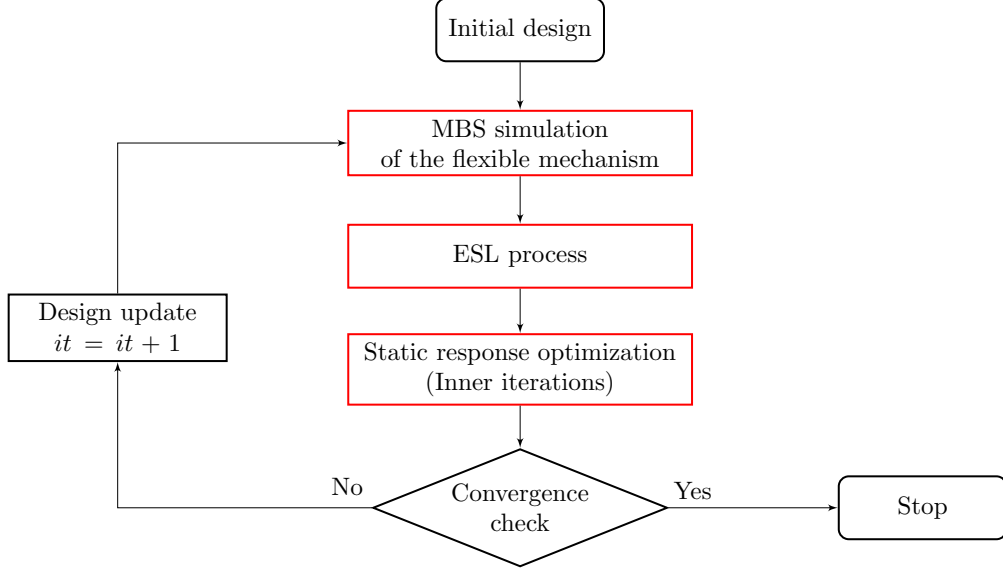


Figure 1: Flowchart of the weakly coupled method.

are general, their derivations depend on the MBS formalism adopted to describe the system dynamics. For component-wise ESLs, the derivation adapted to standard MBS formalisms are reviewed in [40].

For the novel system-wise ESL definition, we propose to derive them for a particular Lie group nonlinear finite element formalism which has interesting properties to derive ESLs and solve efficiently the equivalent static problem. Component-wise ESLs for this formalism have been derived in [38]. Hereafter, the equations of motion in a Lie group setting are first presented and system-wise ESLs are subsequently derived.

4.1 Equations of motion in a Lie group setting

Nonlinear finite element formulations are totally general to analyze complex flexible MBS and are well-suited to perform advanced optimization processes, like topology optimization [2], which renders this formalism appealing. However, a classical nonlinear finite element formalism deals with a non-trivial issue regarding the treatment of rotations [13, 1]. While continuing with a nonlinear finite element approach, it has been proposed to describe the dynamics of flexible MBS on a k -dimensional manifold G with a Lie group structure [5, 30, 31]. This mathematical framework offers a more natural answer to the problem of rotation parameterization. From a mathematical viewpoint, a Lie group G is a differentiable manifold for which the product (or composition) and inversion operations are smooth maps and on which differential geometry can be employed to perform operations. Interested readers may refer to [18, 3] for a more detailed description.

In the present approach, the configuration space G of a flexible MBS is described as a set of frame transformations. A frame is attached at each node and is represented by 4×4 matrix $\mathbf{H}$ defined as

$$\mathbf{H} = \begin{bmatrix} \mathbf{R} & \mathbf{x} \\ \mathbf{0}_{3 \times 1} & 1 \end{bmatrix}, \quad (17)$$

where $\mathbf{R} \in SO(3)$ is a rotation matrix and $\mathbf{x} \in \mathbb{R}^3$ is a position vector. Such frames are actually used to describe the absolute position and orientation of each node of the rigid and flexible bodies as well as to describe the relative transformations occurring in kinematic joints, leading to a mixed (absolute and relative) variable formulation.

Likewise, for each node, a velocity vector $\mathbf{v} = [\mathbf{u}^T \boldsymbol{\omega}^T]^T$ can be defined as

$$\dot{\mathbf{H}} = \mathbf{H} \tilde{\mathbf{v}}, \quad (18)$$

where $\tilde{\bullet}$ is a linear operator such that

$$\tilde{\mathbf{v}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & u_1 \\ \omega_3 & 0 & -\omega_1 & u_2 \\ -\omega_2 & \omega_1 & 0 & u_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (19)$$

It turns out that $\mathbf{u}$ is the 3-dimensional linear velocity of the node and $\boldsymbol{\omega}$ the 3-dimensional angular velocity of the nodal frame. Hence, the velocity vector $\mathbf{v}$ represents the absolute velocity of the node expressed in a nodal local frame whereupon the interpretation of the velocity vector is similar to that of a classical nonlinear finite element approach.

For a MBS including several interconnected bodies, the state of the system is represented by a block diagonal matrix

$$\mathbf{H} = \text{diag}(\mathbf{H}^1, \dots, \mathbf{H}^{n_n}), \quad (20)$$

and the nodal velocities are collected in global vector defined as

$$\mathbf{v} = [\mathbf{v}^{1T}, \dots, \mathbf{v}^{n_n T}]^T, \quad (21)$$

where n_n is the number of nodes.

The Lie group formalism is at the core of the development of the so-called local frame formulation for MBS, which significantly attenuates the influence of geometric nonlinearities on the equations of motion. Moreover, the Lie group formalism offers a convenient framework to derive the equations of motion without introducing a global parameterization of the matrix $\mathbf{H}$. The equations of motion are actually derived in terms of quasi-coordinates. Using classical principles of mechanics, the equations of motion can be expressed as a set of index-3 differential and algebraic equations, that reads

$$\dot{\mathbf{H}} = \mathbf{H}\tilde{\mathbf{v}}, \quad (22)$$

$$\mathbf{M}(\mathbf{H})\dot{\mathbf{v}} - \mathbf{g}_{gyr}(\mathbf{H}, \mathbf{v}) + \mathbf{g}_{int}(\mathbf{H}) + \mathbf{B}^T(\mathbf{H})\boldsymbol{\lambda} - \mathbf{g}_{ext}(\mathbf{H}, t) = \mathbf{0}_{k \times 1}, \quad (23)$$

$$\Phi(\mathbf{H}) - \Psi(t) = \mathbf{0}_{m \times 1}. \quad (24)$$

These equations of motion represent the dynamics of a general class of conservative flexible multibody systems but the formulation can be easily extended to account for non-conservative forces. As for the velocities, the different forces appearing in the equations of motion are also represented in the local frames. Ergo, they are less sensitive to geometric nonlinearities. The equations of motion (22)-(24) can be integrated using the Lie group version of the generalized- α scheme [5], which preserves the Lie group structure, i.e. the time integration can be performed without introducing a parameterization of rotations. This time integrator scheme exhibits second-order accuracy for index-3 differential-algebraic problems and numerical damping can be introduced to lessen high frequency content.

The linearized form of the equations of motion is useful to derive ESLs. Consequently, considering infinitesimal variations about a given state $(\mathbf{H}^*, \mathbf{v}^*, \dot{\mathbf{v}}^*, \boldsymbol{\lambda}^*)$ and defining the vector $\mathbf{r}$ as the residual of (23), the linearized form is expressed as

$$\delta(\mathbf{r}) = \mathbf{K}_t \delta \mathbf{h} + \mathbf{C}_t \delta(\mathbf{v}) + \mathbf{M} \delta(\dot{\mathbf{v}}) + \mathbf{B}^T \delta \boldsymbol{\lambda}, \quad (25)$$

$$\delta(\Phi) = \mathbf{B} \delta \mathbf{h}, \quad (26)$$

where the tangent damping and stiffness matrices are defined from $\partial \mathbf{r}(\mathbf{H}, \mathbf{v}, \dot{\mathbf{v}}, \boldsymbol{\lambda}) / \partial \mathbf{v} = \mathbf{C}_t$ and from the directional derivative $D \mathbf{r}(\mathbf{H}, \mathbf{v}, \dot{\mathbf{v}}, \boldsymbol{\lambda}) \cdot \tilde{\delta \mathbf{h}} = \mathbf{K}_t \delta \mathbf{h}$, respectively. The local frame virtual motion $\delta \mathbf{h}$ is defined as $\delta(\mathbf{H}) = \mathbf{H} \tilde{\delta \mathbf{h}}$.

4.2 System-wise ESL derivation for a local frame formalism

Based on the general definition of the equivalent static equilibrium problem (10) and the equations of motion (22-24), it is straightforward to define the equivalent static equilibrium problem for this local frame formalism, that reads

$$\text{SEE}(\mathbf{p}, \hat{\mathbf{H}}, \hat{\boldsymbol{\lambda}}_{eq}, \mathbf{g}_{eq}, \Psi_{eq}, t) := \begin{aligned} &\mathbf{g}_{int}(\mathbf{p}, \hat{\mathbf{H}}) + \mathbf{B}_{eq}^T(\mathbf{p}, \hat{\mathbf{H}}, t) \boldsymbol{\lambda}_{eq} - \mathbf{g}_{eq} = \mathbf{0} \\ &\Phi_{eq}(\mathbf{p}, \hat{\mathbf{H}}) - \Psi_{eq}(t) = \mathbf{0} \end{aligned} \quad (27)$$

wherein the kinematic constraint equations and boundary conditions are included. It is interesting to point out that the evaluation of system-wise ESLs is simply a byproduct of the MBS analysis and does

not require additional processing since the terms stepping in (27) have already been computed during the primal analysis. Solving this static equilibrium problem gives the statically deformed configuration of the mechanism $\hat{\mathbf{H}}$ and $\hat{\boldsymbol{\lambda}}$ resulting from applying the system-wise ESL $\mathbf{g}_{eq}$ and $\boldsymbol{\Psi}_{eq}$. By definition of the system-wise ESL, the statically deformed configuration of the mechanism $\hat{\mathbf{H}}$ is equivalent to the dynamically deformed configuration of the mechanism $\mathbf{H}$ for the considered time instant.

4.3 Linearizing the nonlinear equivalent static problem

While the equivalent static problem (27) is totally general and can account for nonlinearities such as large deformations, solving this system of equations for all time steps at each iteration of the static optimization process may be expensive. Therefore, instead of solving this nonlinear system of equations, it is appealing to linearize it about a reference configuration of the mechanism $\mathbf{H}^*$. If the relative transformation between the statically deformed configuration $\hat{\mathbf{H}}$ and the reference configuration $\mathbf{H}^*$ is sufficiently small, it can be linearized and represented by a vector $\Delta\mathbf{h}$ such that

$$\hat{\mathbf{H}} \simeq \mathbf{H}^*(\mathbf{I} + \widetilde{\Delta\mathbf{h}}), \quad (28)$$

where $\mathbf{I}$ is the identity matrix.

Linearizing the static equilibrium equation (27) around the reference configuration $\mathbf{H}^*$ yields

$$\begin{aligned} \mathbf{g}_{int}(\mathbf{p}, \hat{\mathbf{H}}) + \mathbf{B}_{eq}^T(\mathbf{p}, \hat{\mathbf{H}}) \boldsymbol{\lambda}_{eq} - \mathbf{g}_{eq} &\simeq \mathbf{g}_{int}(\mathbf{p}, \mathbf{H}^*) + \mathbf{B}_{eq}^T(\mathbf{p}, \mathbf{H}^*) \boldsymbol{\lambda}_{eq}^* - \mathbf{g}_{eq} \\ &\quad + \mathbf{K}_t(\mathbf{p}, \mathbf{H}^*) \Delta\mathbf{h} + \mathbf{B}_{eq}^T(\mathbf{p}, \mathbf{H}^*) \Delta\hat{\boldsymbol{\lambda}}_{eq}, \\ \boldsymbol{\Phi}_{eq}(\mathbf{p}, \hat{\mathbf{H}}) - \boldsymbol{\Psi}_{eq} &\simeq \boldsymbol{\Phi}_{eq}(\mathbf{p}, \mathbf{H}^*) - \boldsymbol{\Psi}_{eq} + \mathbf{B}_{eq}(\mathbf{p}, \mathbf{H}^*) \Delta\mathbf{h}. \end{aligned} \quad (29)$$

Adopting a matrix formulation, the linearized system of equations (29) for the time step t_n takes the form

$$\begin{bmatrix} \mathbf{K}_t & \mathbf{B}_{n,eq}^T \\ \mathbf{B}_{n,eq} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{h} \\ \Delta\hat{\boldsymbol{\lambda}}_{eq} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{n,eq} - \mathbf{g}_{n,int}(\mathbf{p}, \mathbf{H}_n^*) - \mathbf{B}_{n,eq}^T(\mathbf{p}, \mathbf{H}_n^*) \boldsymbol{\lambda}_{n,eq}^* \\ -\boldsymbol{\Phi}_{eq}(\mathbf{p}, \mathbf{H}_n^*) + \boldsymbol{\Psi}_{n,eq} \end{bmatrix}. \quad (30)$$

Several variables of the right-hand-side depend on the design variables $\mathbf{p}$. The right-hand-side must thus be re-evaluated at each iteration of the static optimization and their sensitivity should also be computed in order to ensure the convergence of the static optimization problem when gradient-based algorithms are used.

However, in many practical situations and provided a suitable choice of the reference configuration $\mathbf{H}^*$, the right-hand-side of (30) may take a much simpler expression. In the examples below, the elastic deformations remain small so that $\mathbf{H}^*$ can be defined as the undeformed configuration which is such that the internal forces are zero ($\mathbf{g}_{int}(\mathbf{p}, \mathbf{H}_n^*) = \mathbf{0}$) and the kinematic constraints are satisfied ($\boldsymbol{\Phi}_{eq}(\mathbf{p}, \mathbf{H}^*) - \boldsymbol{\Psi}_{n,eq} = \mathbf{0}$). This reference configuration can thus be computed by solving (27) with $\mathbf{g}_{eq} = \mathbf{0}$ and the associated multipliers are zero ($\boldsymbol{\lambda}_{n,eq}^* = \mathbf{0}$). Indeed, since there is no load in the system, there is no reaction force. Moreover, because of the unicity of the SEE, its solution for $\mathbf{g}_{eq} = \mathbf{0}$ is necessarily $\mathbf{H}^*$ with $\boldsymbol{\lambda}_{eq}^* = \mathbf{0}$. In addition, in these examples, the kinematic constraints do not depend on $\mathbf{p}$ and the undeformed configuration $\mathbf{H}^*$ is design independent, which means that the identity $\mathbf{g}_{int}(\mathbf{p}, \mathbf{H}_n^*) = \mathbf{0}$ holds in the whole design space, for any value of $\mathbf{p}$. Consequently, the system of equations (30) can be simplified as

$$\begin{bmatrix} \mathbf{K}_t & \mathbf{B}_{n,eq}^T \\ \mathbf{B}_{n,eq} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{h} \\ \Delta\hat{\boldsymbol{\lambda}}_{eq} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_{n,eq} \\ \mathbf{0} \end{bmatrix}. \quad (31)$$

The system-wise ESL $\mathbf{g}_{eq}$ defined by (27) is then interpreted as the static loading which deforms the mechanism from the reference and undeformed configuration $\mathbf{H}^*$ to a statically deformed configuration $\hat{\mathbf{H}}$ that is equivalent to the dynamically deformed configuration $\mathbf{H}$.

Solving the system of equations (31) gives the two vectors $\Delta\mathbf{h}$ and $\Delta\hat{\boldsymbol{\lambda}}_{eq}$ defined in the nodal local frame for the time step t_n . The vector $\Delta\mathbf{h}$ can be interpreted as the increment of displacements with respect to the undeformed configuration and the vector $\Delta\hat{\boldsymbol{\lambda}}_{eq}$ as the increment of reaction forces in the kinematic joints. The statically deformed configuration of the mechanism $\hat{\mathbf{H}}$ can be recovered by employing (28).

The system of equations (31) can be efficiently solved with the adopted local frame formalism. Indeed, the tangent stiffness matrix $\mathbf{K}_t$ is evaluated for the undeformed configuration and this matrix is constant in the nodal local frame, that is the tangent stiffness matrix $\mathbf{K}_t$ is independent of the system configuration. As a result, only the gradient matrices $\mathbf{B}_{n,eq}$ of the kinematic constraints vary with the system configuration. Nonetheless, in the examples below, these matrices are design-independent. They can thus be computed before the optimization process and stored since their size is generally small, which avoids recomputing them at each iteration of the static response optimization.

5 Numerical Applications

The potential of the developed method is hereafter exemplified by solving three benchmarks.

5.1 Optimization of a 2-dof robot

The first example concerns the mass minimization of a 2-dof planar robot (Fig. 2) which is a standard benchmark for mechanism design, see for instance [23]. The goal of the design problem is to minimize the weight of the robot $W(\mathbf{p})$ subject to constraints on the tip deflection and on stresses of structural members.

This application is particularly interesting because the deflection constraint involves the flexibility of the whole system. Hence, the classical component-wise ESL method can only be applied if small deformations are assumed. Indeed, this assumption enables expressing the tip deflection as a linear contribution of the flexibility of each arm whereupon explicit constraints can be defined for each optimized components [36]. Adopting system-wise ESLs, the tip deflection constraint can be directly incorporated in the design problem since system-wise ESLs produce deformed configurations of the whole mechanism.

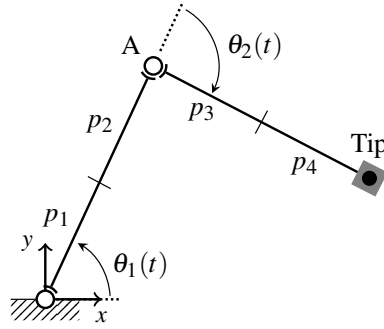


Figure 2: Kinematic model of the 2-dof robot.

The robot arms have a length of 600 mm and each arm is modeled by two equal length beam elements with a hollow cross section. The adopted beam element model is described in [31]. The robot is made of aluminum with a Young's modulus of $E = 72$ GPa, Poisson's ratio of $\nu = 0.3$ and mass density of $\rho = 2700$ kg/m³. Each revolute joint is driven by an ideal motor imposing a smooth joint trajectory $\theta_1(t)$ and $\theta_2(t)$ so that the robot tip deploys along a straight line. The initial joint angles are $\theta_1(t_i) = 120^\circ$ and $\theta_2(t_i) = 150^\circ$. The trajectory equation of the ideal tip displacement reads

$$\Delta x_{tip}(t) = \Delta y_{tip}(t) = \frac{0.5}{T} \left(t - \frac{T}{2\pi} \sin \frac{2\pi t}{T} \right), \quad (32)$$

where the period of the deployment motion T is set to 0.5 s. The prescribed joint motion is computed from a rigid kinematic model in order to satisfy (32). As the joint motions are imposed, the MBS has no rigid-body mode and there is no need to introduce additional kinematic constraints.

The actuator of the second robot arm is located at joint A and has a mass of 2 kg. The combined mass of the end-effector plus the payload is 1 kg. The gravity field is considered in the opposite y -direction. The generalized- α scheme with a time step $h = 5 \times 10^{-4}$ s and a spectral radius of $\rho_\infty = 0.8$ is employed to perform the MBS simulation of 0.66 s. The design variables p_i are the outer diameters of the hollow beam elements whose wall thickness is set to $0.1 \times p_i$. Initial values of the design variables are 50 mm. Thus, in this example, the design variables affect the internal and inertia forces but not the kinematic constraints.

The constraint enforcing a limit on the maximal deflection of the robot has the form

$$\sqrt{\Delta x_{tip}^2(t_j) + \Delta y_{tip}^2(t_j)}, \quad (33)$$

where Δx_{tip} and Δy_{tip} are the tip deflections in the inertial frame.

The second constraint limits the axial stresses at each time step t_j for each element i at the extremity of both sides of the neutral fiber. The stress constraint vector reads

$$\boldsymbol{\sigma}_{j,i} = [\sigma_{j,i}^N + \sigma_{j,i}^M \quad \sigma_{j,i}^N - \sigma_{j,i}^M]^T, \quad (34)$$

Table 1: Numerical results of the 2-dof robot design problem

	p_1 , mm	p_2 , mm	p_3 , mm	p_4 , mm	W , N	MBS analyses	Inner iterations
System-wise ESL method	56.17	44.54	35.02	23.60	15.55	6	11
Fully coupled method	55.66	45.32	34.93	23.47	15.56	9	/
Ref. [23]	54.27	44.15	37.55	26.32	15.72	38	/

where the superscripts $(\bullet)^N$ and $(\bullet)^M$ stand for the normal and bending stresses, respectively. The resulting design problem is formulated as

$$\begin{aligned}
& \underset{\mathbf{p}}{\text{minimize}} && W(\mathbf{p}) \\
& \text{subject to} && \sqrt{\Delta x_{tip}^2(t_j) + \Delta y_{tip}^2(t_j)} \leq 0.001 \text{ mm}, \\
& && [-75 \ -75]^T \leq \boldsymbol{\sigma}_{j,i} \leq [75 \ 75]^T \text{ MPa}, \quad j = 1, \dots, n_c, \\
& && 0.02 \text{ m} \leq p_i \leq 0.06 \text{ m}, \quad i = 1, \dots, 4,
\end{aligned} \tag{35}$$

where n_c equals the number of time steps.

The optimization problem starts with a feasible design, i.e. all design requirements are satisfied. The initial design has a weight of $W = 22.467$ N. The optimization process is deemed converged when the stopping criteria (16) is achieved with $\varepsilon = 10^{-3}$ and that the relative constraint violation is less than 10^{-6} . The same design problem is solved using the fully coupled method and serves as reference to compare and validate the proposed approach.

Both optimization methods exhibit a smooth convergence toward the optimized design (Fig. 3(a)). The deflection constraint mainly governs the optimized design since stresses are far from the admissible stress value of 75 MPa (Fig. 4). Figure 3(b) illustrates that the deflections of the optimized designs obtained either by the developed method or by the fully coupled method are similar and that the constraints are active at two time steps. Markers are plotted every 0.125 s for the sake of readability.

The numerical results are gathered in Table 1 where a weight reduction of 31% can be observed. The results are in excellent agreement with the results of the fully coupled method. The fully coupled method requires three more iterations to achieve convergence but does not have inner iterations at each cycle. Good agreement is also obtained with [23] in which a fully coupled method is also adopted but with a different MBS formalism.

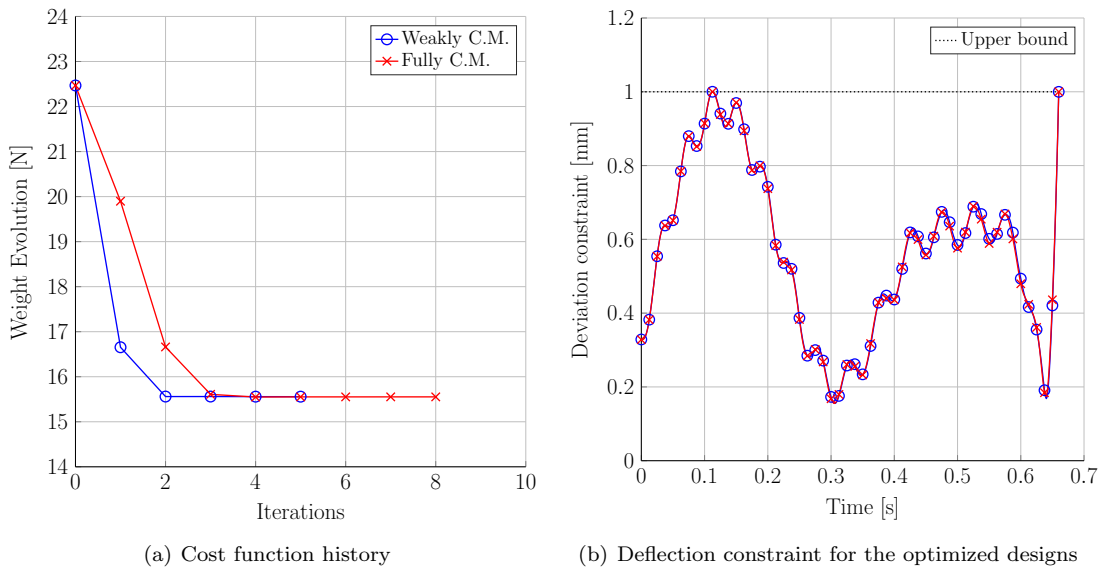


Figure 3: Optimization results of the 2-dof robot.

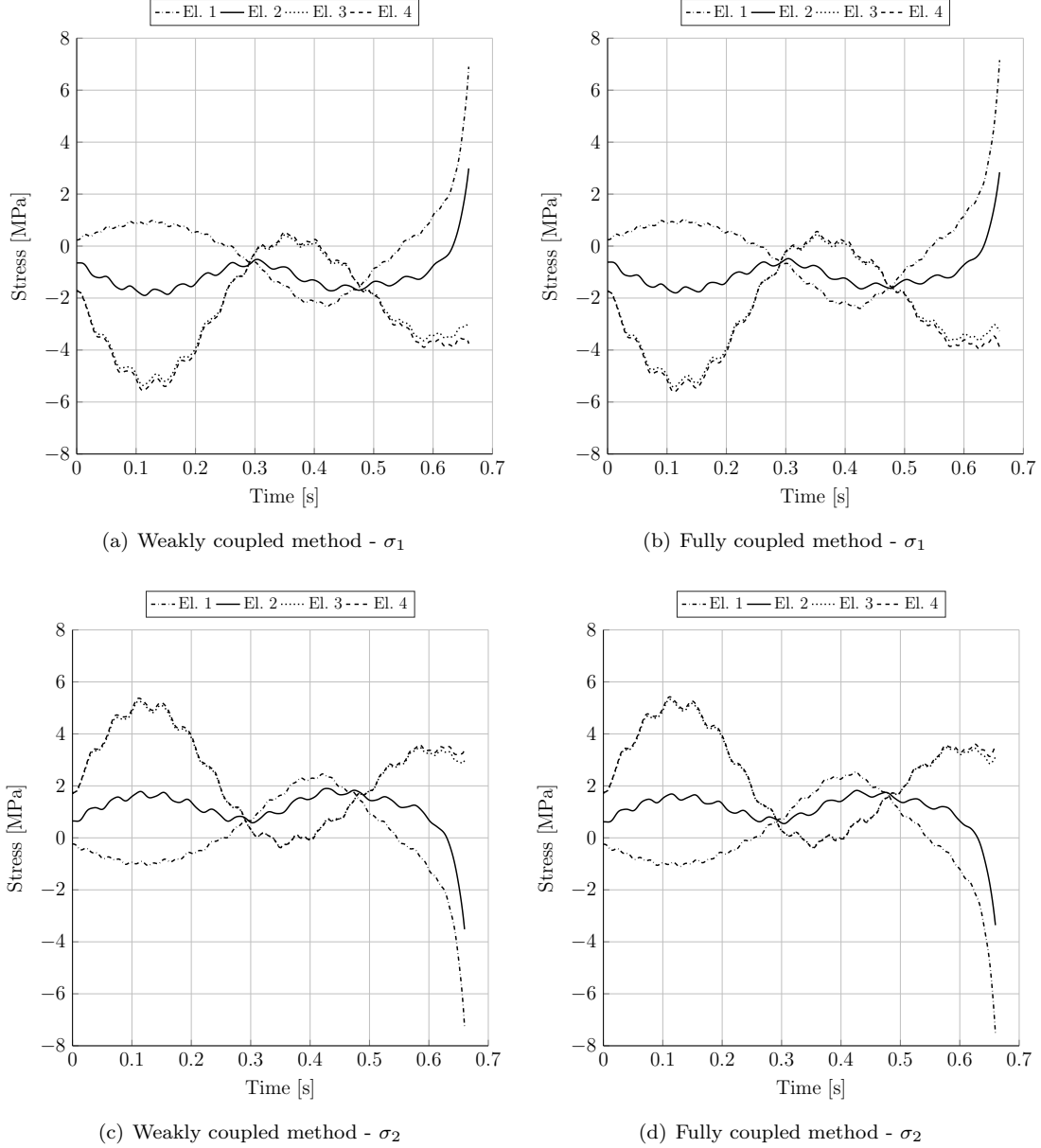


Figure 4: Stress constraints for the optimized design of the 2-dof robot. Elements are numbered from the ground to the tip.

5.2 Optimization of a 4-bar mechanism with deviation constraints

The second benchmark considers the mass minimization of a 4-bar mechanism inspired from [21, 12, 29]. The 4-bar mechanism consists of three flexible links connected to each other and to the ground by revolute joints (Fig. 5). The optimization problem is to find the mobile link diameters which minimize the total mass of the system $m(\mathbf{p})$ while satisfying a deviation constraint on point A . Mathematically, the design problem is stated as

$$\begin{aligned}
 & \underset{\mathbf{p}}{\text{minimize}} && m(\mathbf{p}) \\
 & \text{subject to} && \sqrt{\Delta x_A^2(t_j) + \Delta y_A^2(t_j)} \leq 0.001 \text{ m}, && j = 1, \dots, 501, \\
 & && 0.015 \text{ m} \leq p_i \leq 0.5 \text{ m}, && i = 1, \dots, 3,
 \end{aligned} \tag{36}$$

where Δx_A and Δy_A correspond to the deflections of point A computed in the inertial frame. The optimization problem differs from [12, 21, 29] wherein authors consider stress constraints instead of deviation constraints.

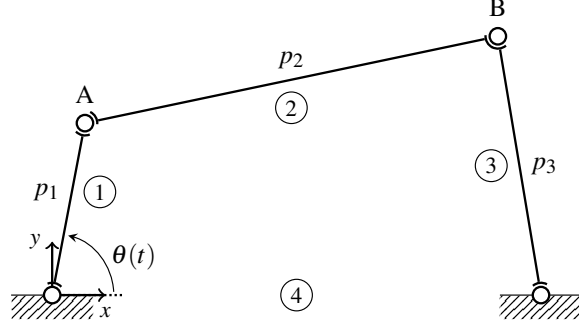


Figure 5: Kinematic model of the 4-bar mechanism.

Table 2: Numerical results of the 4-bar mechanism design problem with deviation constraints

	p_1 , mm	p_2 , mm	p_3 , mm	m , kg	MBS analyses	Inner iterations
System-wise ESL method	27.482	15.000	15.000	1.315	6	19
Fully coupled method	27.486	15.000	15.000	1.315	6	/

Solving this design problem with a component-wise ESL method is not feasible because the equivalent static problem cannot verify the loop closure kinematic [36]. In other words, component-wise ESLs do not permit reconstruction of the deformed configuration of the mechanism. The 4-bar mechanism design problem illustrates that system-wise ESLs extend the generality of optimization problem formulation. System-wise ESLs generate deformed configurations of the mechanism whereupon the optimization problem can naturally account for constraints involving the response of a single or several components.

The 4-bar mechanism is made of aluminum with a Young's modulus of $E = 68.95$ GPa, Poisson's ratio of $\nu = 0.3$ and mass density of $\rho = 2757$ kg/m³. The lengths of the links are $l_1 = 304.8$ mm, $l_2 = l_4 = 914.4$ mm and $l_3 = 762$ mm. Each link is modeled by six beam finite elements [31] with a constant solid circular cross section. The mechanism moves in the horizontal plane so that gravity is neglected. The input crank is driven by an ideal motor imposing the rotation $\theta(t)$ with a linear acceleration from 0 to 10π s⁻¹ in 0.3 s and then maintaining a constant angular velocity until 0.5 s. As this joint motion is imposed, the MBS has no rigid-body mode and there is no need to introduce additional kinematic constraints. The simulation is performed using the generalized- α scheme with a time step $h = 10^{-3}$ s and a spectral radius of $\rho_\infty = 0$. This spectral radius value filters high frequency loadings and smoothes the optimization problem process.

The design variables p_i are the link diameters with an initial value set to 250 mm. These values give a feasible initial design with a mass of 268 kg. As in the previous example, the design variables affect the internal and inertia forces but not the kinematic constraints. The optimization processes are deemed converged when the relative variations of the design variable values are less than 10^{-3} with a relative constraint violation less than 10^{-6} . These tolerance values ensure a tight convergence such that the comparison is performed on converged solutions. Again, the fully coupled method serves as a reference for comparison.

As for the previous example, both optimizations converge towards the same optimized design. Both methods achieve convergence in 6 iterations but it should be kept in mind that the weakly coupled method requires inner iterations (Fig. 6(a)). Numerical results are given in Table 2. Figure 6(b) illustrates that the deflection constraints for the two optimized designs match perfectly. Markers are plotted every 0.2 s for the sake of readability.

5.3 Optimization of a 4-bar mechanism with stress constraints

The third example also deals with the 4-bar mechanism illustrated in Figure 5. However, in contrast to the previous example, this design problem considers stress constraints instead of deviation constraints as initially proposed in [21, 12, 29]. Such a formulation only accounts for individual component responses and therefore, the previous problem related to the loop closure kinematic is not raised. Consequently, the

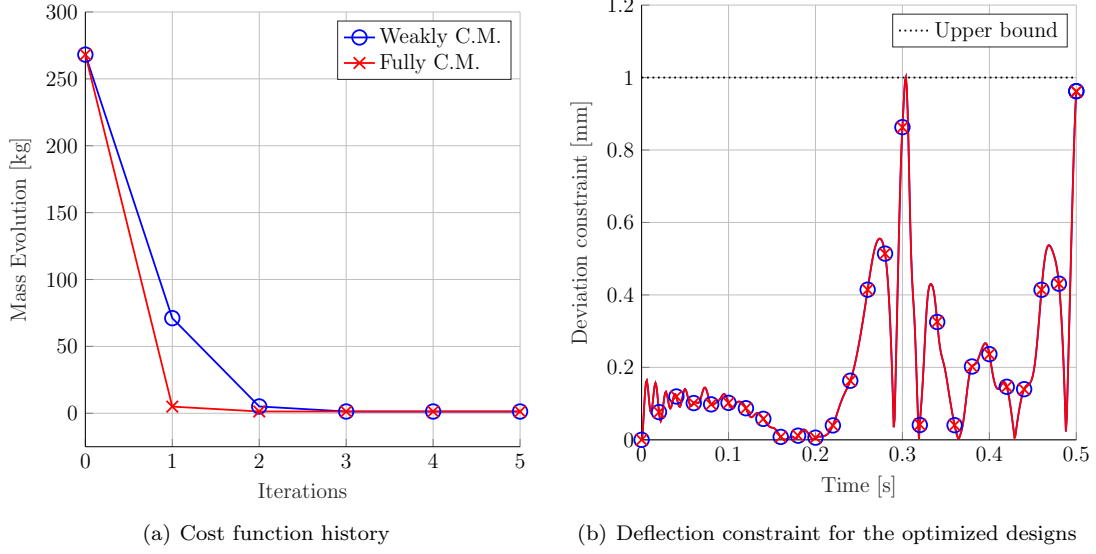


Figure 6: Optimization results of the 4-bar mechanism.

design problem can be solved using either the component-wise or the system-wise ESL method. Hence, both methods are hereafter employed to solve the design problem and to analyze their respective behavior.

Settings of the design problem are the same as for the previous example, except that the gravity field is considered in the opposite y -direction, that the input crank is driven by an ideal motor imposing a constant angular velocity $\omega = 10\pi \text{ s}^{-1}$ and that initial values of design variables are set to 356.8 mm. These modifications are considered to match the initial design problem.

The optimization problem is to find the mobile link diameters p_i which minimize the total mass of the system $m(\mathbf{p})$ subjected to bending stress constraints $\sigma_{j,e}$ at each time step t_j for each beam element e . The MBS analysis computes the bending moment $M_{j,e}$ for each element as a function of time whereupon the bending stresses are calculated from

$$\sigma_{j,e} = \frac{4\sqrt{\pi}}{A_e^{3/2}} M_{j,e}, \quad (37)$$

where A_e is the cross sectional area of element e . Mathematically, the optimization problem is stated as

$$\begin{aligned} & \underset{\mathbf{p}}{\text{minimize}} && m(\mathbf{p}) \\ & \text{subject to} && \sigma_{j,e} \leq 27.58 \text{ MPa}, && j = 301, \dots, 500, \\ & && 0.015 \text{ m} \leq p_i \leq 0.5 \text{ m}, && i = 1, \dots, 3, \end{aligned} \quad (38)$$

where e numbers the beam finite element and goes from 1 to 18. The optimization considers the time interval between 0.3 and 0.5 s which exactly covers one period of steady state motion, after which the transient response has died away.

For the component-wise ESL method, fixed-free beam conditions are employed to fix the singularity of the stiffness matrix during the static response optimization [21, 38]. This choice is important as it affects the optimization process. For the system-wise ESL method, as the input crank motion is imposed, the MBS has no rigid-body mode and additional kinematic constraints are not needed.

The initial design has a mass $m = 546 \text{ kg}$ and bending stress constraints are satisfied. The optimization process is deemed converged when the relative variations of the design variable values are less than 2.5×10^{-2} with a relative constraint violation less than 2.5×10^{-3} .

Both methods converge towards a similar optimized design in 10 iterations (Fig. 7(a)). Figures 7(b)-7(d) show stresses for the optimized design, wherein it can be observed that stress constraints are active for each link at one time step. Numerical results are gathered in Table 3 and are compared with the results given in [21, 12]. The overall trend is similar but those results require some comments. First, regarding the number of iterations, the stopping criteria adopted in this paper and the ones used in [21, 12] are different. For this particular problem, ESLs are very sensitive with respect to the design variables. Considering a tight stopping criteria may prevent the design process to convergence as it may face some

oscillating responses. Second, the MBS formalism adopted in this paper differs from the ones used the other references and therefore, the accuracy of the analysis varies. Performing a MBS analysis of the mechanism with the optimal values given in [21, 12] shows that stress constraints are largely violated. This reveals that the stress amplitudes predicted in our simulation are generally higher, which explains our more conservative and heavier design. Third, algorithms employed to solve the design process are also different. Last, this optimization problem has multiple local optima [12].

The detailed convergence history of the component-wise and system-wise ESL methods are given in A. It can be observed that the first iterations are exactly the same. The hypothesis of both methods are respected and the fixation of the rigid body in the component-wise ESL method is consistent. Small differences appear after several iterations but they are not significant.

Table 3: Numerical results of the 4-bar mechanism design problem with stress constraints

Compared items	System-wise ESL	Component-wise ESL	Ref. [21]	Ref. [12]
p_1 , mm	40.36	40.43	33.5	38.5
p_2 , mm	28.69	28.55	23.2	25.2
p_3 , mm	23.00	23.40	16.9	20.0
m , kg	3.578	3.597	2.28	2.89
MBS analyses	10	10	16	15
Inner iterations	23	23	22	/

6 Concluding remarks

The weakly coupled method is a powerful tool to perform structural optimization of mechanical systems within a flexible MBS approach. It relies on the equivalent static load (ESL) method that generates a series of static loads mimicking the dynamic loading whereupon the original dynamic response optimization problem is transformed into an equivalent static response optimization problem, the latter being easier to solve. However, the classical component-wise ESL method restrains the generality of the optimization problem formulation because it assumes that components are isolated from the system.

In this study, a generalized ESL definition at the system-level has been proposed. It is demonstrated that system-wise ESLs lead to a static response optimization problem that is equivalent to the original dynamic response optimization problem. Indeed, system-wise ESLs generate the deformed configuration of the entire mechanism and thus, any computed response can be incorporated in the optimization problem. The assumption of optimizing isolated components is in fact removed. Moreover, it turns out that evaluating system-wise ESLs is simply a by-product of the MBS analysis. Compared to the component-wise ESL method, the equivalent static problem is now solved system-wise. Hence, only one system-wise equivalent static problem is considered instead of as many component-wise equivalent static problems as the number of optimized bodies.

The proposed system-wise ESL definition is totally general. However, as for component-wise ESLs, their evaluation depends on the adopted MBS formalism. In this paper, system-wise ESLs have been derived for a local frame formalism, resulting from a nonlinear finite element approach. As highlighted, this formalism possesses interesting properties to efficiently derive ESLs and solve the equivalent static problem. A remarkable result is that the tangent stiffness matrix used in the static response optimization is independent of the system configuration, i.e. this matrix does not need to be stored or recomputed for each time step at each iteration of the static response optimization. Benchmarks have been solved to illustrate the novel approach and to compare the proposed method with the standard component-wise ESL method.

This paper demonstrated that the weakly coupled method combined with system-wise ESLs is equivalent to the fully coupled method from a formulation point of view. A meticulous comparison of both methods should now be conducted to identify their robustness and weaknesses as well as a deep investigation of their respective efficiency. Also, more general examples with free rigid-body modes should be studied to further explore the versatility of the proposed methodology.

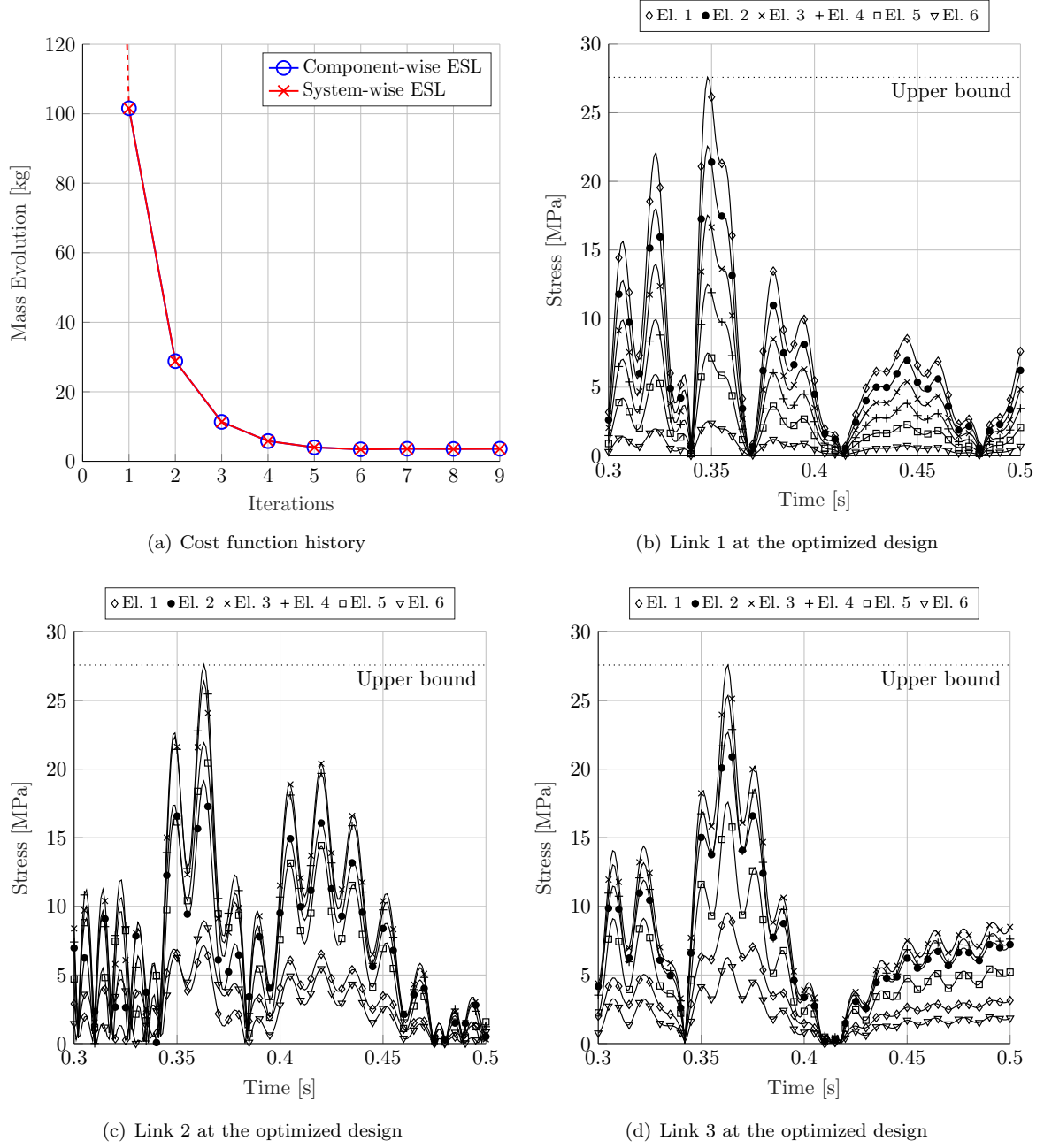


Figure 7: Optimization results of the 4-bar mechanism with stress constraints. Elements are numbered from the input crank and going clockwise with a counter reset at each new component. For the sake of readability, markers are plotted every 0.005 s and stress values are only plotted for the system-wise ESL method as the optimized designs are very similar.

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A Detailed results of the 4-bar mechanism design problem with stress constraints

Table 4: Convergence history of the design process - Component-wise ESL method.

iter	p_1	p_2	p_3	$m(\mathbf{p})$	link 1: $\max_{j,e} \sigma_{j,e}$	link 2: $\max_{j,e} \sigma_{j,e}$	link 3: $\max_{j,e} \sigma_{j,e}$
0	3.568e-01	3.568e-01	3.568e-01	5.461e+02	6.456e+00	1.796e+00	1.406e+00
1	2.199e-01	1.435e-01	1.323e-01	1.016e+02	2.758e+01	2.758e+01	2.758e+01
2	1.169e-01	7.821e-02	6.828e-02	2.882e+01	2.758e+01	2.758e+01	2.762e+01
3	7.641e-02	5.056e-02	3.831e-02	1.134e+01	2.839e+01	3.025e+01	4.184e+01
4	5.143e-02	3.688e-02	2.890e-02	5.817e+00	3.617e+01	3.250e+01	3.125e+01
5	4.500e-02	2.950e-02	2.388e-02	4.001e+00	2.974e+01	3.332e+01	3.193e+01
6	3.870e-02	2.842e-02	2.274e-02	3.441e+00	3.032e+01	2.779e+01	2.793e+01
7	4.121e-02	2.806e-02	2.348e-02	3.590e+00	2.828e+01	2.761e+01	2.775e+01
8	4.017e-02	2.829e-02	2.332e-02	3.547e+00	2.768e+01	2.759e+01	2.759e+01
9	4.043e-02	2.856e-02	2.340e-02	3.597e+00	2.759e+01	2.759e+01	2.758e+01

Table 5: Convergence history of the design process - System-wise ESL method.

iter	p_1	p_2	p_3	$m(\mathbf{p})$	link 1: $\max_{j,e} \sigma_{j,e}$	link 2: $\max_{j,e} \sigma_{j,e}$	link 3: $\max_{j,e} \sigma_{j,e}$
0	3.568e-01	3.568e-01	3.568e-01	5.461e+02	6.456e+00	1.796e+00	1.406e+00
1	2.199e-01	1.435e-01	1.323e-01	1.016e+02	2.758e+01	2.758e+01	2.758e+01
2	1.169e-01	7.822e-02	6.826e-02	2.882e+01	2.758e+01	2.758e+01	2.764e+01
3	7.641e-02	5.062e-02	3.820e-02	1.134e+01	2.839e+01	3.020e+01	4.210e+01
4	5.145e-02	3.698e-02	2.875e-02	5.819e+00	3.615e+01	3.242e+01	3.134e+01
5	4.505e-02	2.965e-02	2.371e-02	4.008e+00	2.971e+01	3.320e+01	3.202e+01
6	3.795e-02	2.859e-02	2.269e-02	3.418e+00	3.106e+01	2.778e+01	2.787e+01
7	4.138e-02	2.797e-02	2.360e-02	3.598e+00	2.903e+01	2.765e+01	2.784e+01
8	4.016e-02	2.828e-02	2.306e-02	3.526e+00	2.771e+01	2.760e+01	2.766e+01
9	4.036e-02	2.869e-02	2.300e-02	3.578e+00	2.758e+01	2.761e+01	2.758e+01

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