Smart Equity Investing:

Implementing Risk Optimization Techniques on Strategic Beta Portfolios

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Abstract

We examine the performance of risk-optimization techniques on equity style portfolios. To form these portfolios, also called Strategic Beta factors by practitioners and data providers, we group stocks based on size, value and momentum characteristics through either independent or dependent sorting. Overall, performing risk-oriented strategies on style portfolios constructed with a dependent sort deliver greater abnormal returns. On average, we observe these strategies to significantly outperform 42% of the risk-oriented ETFs listed on US exchanges, compared to 31% when the risk-oriented strategies are performed on portfolios formed with an independent sort. We attribute the outperformance yielded by dependent sorting to the fact that it provides a better stratification of the set of stocks' opportunity and diversification properties.

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For more than fifty years, passive investors have considered capitalization-weighted (CW) indices to be a suitable proxy for the tangency portfolios, namely the Maximum Sharpe Ratio (MSR) portfolio. Although CW indices provide a simple, cost-effective and intuitive manner to allocate to stocks, they are also exposed to certain inherent weakness, notably their embedded momentum bias (see, for instance, Hsu and Kalesnik (2014)) and their exposure to greater idiosyncratic risks through their larger allocation to certain stocks.

This evidence has incentivized the investors to seek alternative ways to construct equity portfolios. We observe a dual paradigm shift to so called "smart beta strategies" and "style investing" (also called "strategic beta factors"). On the one hand, smart beta strategies provide an alternative weighting scheme for stocks, i.e. alternative way to diversify risk. Although there is no consensus on whether smart beta strategies should be considered as passive or active management, we can all agree that they follow a systematic and rules-based process. Smart beta ranges from scientific diversification (such as the minimum variance portfolio or risk efficient indexing), risk-based heuristic methods (maximum diversification index, diversity-weighted index or risk parity indexing) to fundamental indexing (e.g., using dividend yield as a proxy for asset market value). Recent debates have emerged between those who believe the term “smart beta” is simply marketing hype (Malkiel (2014), Podkaminer (2015)) and those who believe there is true value to these strategies (Amenc, Goltz and Lodh (2016)). On the other hand, strategic beta investing looks to allocate more efficiently to “style” portfolios to capture systemic sources of market risk premiums. This technique has however existed for decades and firms such as Dimensional Fund Analysis have successfully marketed these strategies since the 1980s. But over the last few years, a number of market developments have led to a variety of new and innovative products being offered to investors by asset managers and banks. New investment vehicles such as ETFs, greater market liquidity, lower transaction costs, and increasingly sophisticated investors have all led to a proliferation of these new investment strategies. Yet, the border between smart and strategic beta is not always clear, leading to many sweeping generalities both for and against alternative portfolio construction techniques.

The recent literature categorizes these new investment schemes and analyze the potential performance of Smart Beta strategies (reviewed in Section I). These strategies have been implemented at the individual stock level as the equity building block to construct portfolios that aim to satisfy specific investor objectives or gain exposure to specific systematic risk factors (see for instance, Clarke, Silva and Thorley (2013), Arnott, Hsu, Kalesnik, and Tindall (2013)).

Although most of the research and product has focused on establishing stock level characteristics to form style portfolios and stock level optimization/weighting schemes, investors can also find benefits in performing strategic beta allocations at the portfolio level (Boudt and Benedict (2013)) or even at the asset class level (Ardia et al. (2016)). In fact, Froot and Teo (2008) observe that institutional investors tend to reallocate their funds across style groupings which suggest that our objective to perform Smart Beta strategies on investment style portfolios may be in line with this reallocation practice of institutional investors. In fact, recent studies have recognized the use of asset or factor portfolios as the new opportunity set (Izorek and Kowara (2013), Roncalli and Weisang (2016)). To the best of our knowledge, the value-added of working at the equity portfolio level (rather than asset classes or individual assets) when implementing risk-based optimizations and the importance of the sorting method used to construct those portfolios have not been deeply studied. Our paper addresses this gap. We demonstrate that there is a potential to performance improvement when performing strategic optimizations on "smart" characteristic-sorted equity portfolios and we then decompose this outperformance.

Our theoretical framework builds on the research of Barberis and Shleifer (2003), who demonstrate the natural tendency of investors to allocate funds according to asset categories, and of Berk (2000), who explains that forming groups of stocks into style indices circumvents the burden of estimating large covariance matrix of returns.

Our research contributions to the literature are twofold. First, we contribute to the literature about Smart Beta by reconstructing a proxy for tangent/well-diversified (US equity) market portfolios by applying risk-based strategies to characteristic-sorted equity portfolios (i.e., an opportunity set sorted by market capitalization, book-to-market ratio and momentum characteristics). This method ensures style neutrality of the investment solution and simplifies the allocation by reducing the errors in the covariance matrix of returns.

Second, we contribute to the literature regarding style investing (Strategic Beta) and provide guidelines for how these style indices (portfolios) should be constructed to improve the potential of any type of optimization strategy. To this end, we contrast the empirical results of an *independent* sort, as in Fama and French (1993), with those of a *dependent* sort (Lambert and Hübner (2013), Lambert, Fays and Hübner (2016)). The construction method used by Fama and French (1993) sets a standard but many of the methodological choices (e.g., breakpoints or asymetrical sort) that the authors use are not intended to produce portfolios with the highest Sharpe ratio for each level of fundamentals. By using the Fama-French methodology, Lambert et al. (2016) uncover that sorting stocks independently based on correlated variables (e.g., the negative correlation between firms’ market equity and book-to-market equity) might lead to very unequal numbers of securities in portfolios and hence to poor diversification in sorted portfolios. To control for the impact of correlated variables on the classifications assigned to firms' characteristics as well as ensuring a good balance between portfolios, the authors use a dependent sort. This simple but fundamental methodological change enables proper stratification of the US equity opportunity set. Other researchers have also used dependent sorting to group stocks into portfolios. Among others, Daniel, Grinblatt, Titman and Wermers (1997) perform a triple dependent sort on the size, value and momentum characteristics of a stock to construct benchmarks to test the performance of mutual funds, whereas Novy-Marx (2013) briefly review the positive effect of sorting stocks depending on their value and profitability characteristics, and Wahal and Yavuz (2013) apply a dependent sort as a robustness test to construct portfolios according to stocks' comovement and their past returns. These last authors also motivate the choice of applying a dependent rather than independent sort to control for correlation between the variables to sort.

Our research focuses on stock level US data, allowing us to construct and test strategies using a variety of protocols and thereby draw robust conclusions as to the benefits of investing in portfolios that do not simply rely on market capitalization as an input. We demonstrate that a dependent sorting methodology also helps to deliver a significantly higher Sharpe ratio for Strategic Beta strategies. We claim that performing asset allocation on well diversified portfolios is key to avoid exposure to the idiosyncratic risks as often pointed out by the literature for factor investing. Our stratificiation of the equity market allows us to achieve this goal.We decompose the source of the outperformance of Strategic Beta strategies according to four value drivers: the choices of stock classifications (dependent vs independent), the rebalancing frequency, the number of portfolios that stratify the US equity market, and the risk-oriented optimizations used to form a Strategic Beta strategy.

The rest of the paper is organized as follows. Section I presents a literature review regarding risk-based and heuristic assets allocation techniques. Section II describes the opportunity set, i.e. the data and methodology used to construct the characteristic-based portfolios. Section III review the procedure to estimate the covariance matrix implemented in our risk-based optimizations and the methodology used to account for transaction costs. Section IV reports the results of mean-variance spanning tests to evaluate the efficiency of the risk-based strategies across the diffent sorting methodologies. Section V presents implications of the sorting methodologies in term of portfolio diversification. Section VI concludes.

1 Literature Review

The seminal work of Markowitz (1952) on Modern Portfolio Theory (MPT) has pioneered the industry of portfolio management regarding the construction of passive portfolios. Under several assumptions[1], the MPT describes how the optimal asset allocation can be reached by minimizing the risk-return tradeoff of a portfolio and being tangent to the efficient frontier. Popularized by the introduction of Capital Asset Pricing Model (CAPM) and the principle of market's prices efficiency (Sharpe (1964), Lintner (1965), Mossin (1966)), the "market" portfolio, which weighs assets relative to their market capitalization, is considered as the optimal mean-variance portfolio. However, a plethora of papers have recently fueled the debate on the sub-optimality of CW allocations when the assumptions of price efficiency is disregarded (see for

[1] The main assumptions refer to unlimited risk-free borrowing and short selling, homogenous preferences, expectations and horizons, no frictions (taxes, transaction costs) and non-tradable assets (social security claims, housing, human capital). Thus, under real-world conditions, the market portfolio may not be efficient according to Sharpe (1991) and Markowitz (2005).

instance Arnott, Hsu and Moore (2005, p. 85), Hsu (2006)). The recent literature has thus proposed non-capitilization-weighted strategies to circumvent the drawbacks of CW allocation schemes.

For instance, Amenc, Goltz, Lodh and Martellini (2014) indicate that traditional CW allocations suffer from poor diversification (mainly invested in large capitalization stocks) and from exposure to uncontrolled sources of risk. One simple way to ensure good diversification and low idiosyncratic risk is to equal weight all of the *N* constituents of the portfolio. An *Equal-Weighted* scheme, referred to as "1/N", is a heuristic[2] that approximates a mean-variance optimality only when the assets have the same expected return and covariances (Chaves et al. (2012)). This naïve weighting scheme has increased in popularity since DeMiguel et al. (2009) demonstrated that none of the "optimal" allocation schemes the authors put under review (Bayesian methods as well as the CW portfolio) significantly outperform out-of-sample the "1/N" portfolio in terms of the Sharpe ratio, and certainty equivalent value. The only advantage that the CW portfolio has is the zero turnover of its buy-and-hold policy, i.e. the investor does not need to trade any assets, compared to the 1/N policy. Moreover, Plyakha, Uppal and Vilkov (2015) decompose the sources of outperformance between CW and 1/N portfolios and suggest that the equal-weighted strategy produces additional returns from the rebalancing frequencies and the embedded reversal strategy it captures. For the simplicity of the strategy, DeMiguel et al. (2009) claim that the 1/N should be defined as a benchmark to evaluate alternative weighting schemes.

Another debated issue around the mean-variance optimality of the CW portfolio concerns the price as a measure of fair value, if one believes that the stock prices do not fully reflect firm

[2] A heuristic method is by definition a method that requires resources with lower complexity to obtain a solution that is sufficient but does not guarantee optimality.

fundamentals, then the CW portfolio is sub-optimal because it over- (under) weighs over- (under) priced stocks (Hsu (2006)). To integrate this matter, *Fundamental indexing* has led to the creation of characteristics-based indices that weight stocks according to their economic footprints (such as revenues, book values, and earnings). According to Arnott, Hsu and Moore (2005), this new heuristic scheme provides consistently superior mean-variance performance compared to traditional CW indices. Hsu and Kalesnik (2014) demonstrate that among four allocation strategies (i.e., fundamental weight, minimum variance, CW and 1/N), the traditional CW index is the only allocation scheme that produces a negative measure of "skill". In theory, skill in portfolio management is related to alpha, and by definition, broad indices should not produce any form of abnormal return. However CW portfolios exhibit (by construction) a drag in their expected returns because the strategy involves buying stocks when prices are high and selling stocks when prices are low. Overall, Graham (2011) and Perold (2007) conclude that if there is some evidence that CW indices can underperform fundamental indices in some time periods, there is no evidence that because of this return drag, they systematically underperform regardless of the period. In reality, Fundamental indexing is another method to implement style investing: it produces a significant bias toward distressed stocks (Jun and Malkiel (2007), Perold (2007)). This method therefore has exactly the same risk of concentration as traditional CW portfolios.

Instead of looking at heuristic methods, academics and practitioners have explored risk-based optimization techniques which simplify the mean-variance estimation process by disregarding (or subistuting) the expected returns of an asset by its volatility (risk). In other words, the techniques assume that the expected return of an asset increases proportionally to its risks. Clarke, Silva and Thorley (2013), Amenc, Goltz and Martellini (2013), Frazzini and Pedersen (2014) have shown evidence that these techniques exploit a recently discovered market anomaly:

the low-beta anomaly. The low beta anomaly contradicts the MPT theory in the sense that stocks with high-volatility (high beta) should earn higher returns than low-volatility (low beta) stocks. However, the low beta anomaly shows the opposite is true on many international markets: low risk stocks outperforms high risk stocks (Baker, Bradley and Taliaferro (2014)). Exploiting this market anomaly may thus deliver higher Sharpe ratios than the traditional CW. There are among the risk-based optimizations three common techniques that disregard (or subistut) the expected return by the volatility of an asset and are shown to exploit the low-beta anomaly.

First, the *minimum variance* portfolio discards the estimation of the expected return and simply focuses on finding the portfolio with the lowest risk. One of its advantages lies in the simplicity of the parameter estimation. Indeed, the objective function of a minimum variance portfolio only requires the estimation of the assets covariance matrix to attribute weights to the portfolio constituents.

Second, the *maximum diversification* portfolio substitutes the expected returns from the Sharpe ratio by the volatility (risk) of the assets posing the assumptions that the expected return of an asset increases proportionally to its risk (Choueifaty and Coignard (2008)) - here, the standard deviation is a proxy for expected return. Under this hypothesis, the maximum diversification portfolio is the portfolio that is tangent to the efficient frontier (the MSR portfolio).

Third, the *risk parity* is the most widely adopted and touted risk-based portfolio allocation. Risk parity aims to equalize the marginal contribution of each asset to the global portfolio risk (Maillard, Roncalli and Teïletche (2010)). Asness, Frazzini and Pedersen (2012) provide a theoretical foundation for risk parity portfolios: in the presence of leverage-averse investors, safer assets should outperform riskier ones on a risk-adjusted basis. Risk parity overweighs safer assets to achieve an equal risk contribution between asset classes. For example,if we consider a stock/bond portfolio, risk parity will overweigh the allocation to bonds, because it it is the asset class with the lower volatility. Such a strategy would obviously benefit greatly from a decreasing interest rate environment, as has been the case for the last 30 years (Chaves, Hsu, Li and Shakernia (2011), Fisher, Maymin and Maymin (2015)). Nevertheless, traditional risk parity strategies do not come without risk, as risk parity implies not only a low concentration in asset holdings but also a low concentration in risk contributions (Steiner (2012)). It can therefore be a low diversified portfolio in the MPT sense.

Table 1 recalls the analytical forms of the heuristic and risk-based allocations that will serve as a practical base in our empirical analysis, namely minimum variance, risk parity, maximum diversification and equal weighting.

[Table 1 near here]

Overall, all these common risk-based optimization techniques aim at substituting the traditional CW allocation to find the optimal mean-variance portfolio. While claiming that one strategy is able to rule them all remains fairly optimistic, the objective of our paper is to demonstrate that the selection of the underlying assets is at least as much as important as the selection of the allocation technique to reach a mean-variance optimality.

2 Investment Opportunity Set

This section describes our opportunity set, i.e. the set of portfolios that constitute our test assets, by stratifying the US stocks universe in investment style portfolios. All allocation techniques will be performed on the two sorting methodologies. The first construction methodology is based on an *independent* sort of the stocks universe and has become a standard in the asset-pricing literature for constructing characteristic-sorted portfolios (Fama and French

(1993)). The databases used to form the construction of portfolios are based on the merge of the Center for Research in Security Prices (CRSP) and Compustat. CRSP database contains historical prices information, whereas Compustat provides accounting information on all stocks listed on the major US stock exchanges. The sample period ranges from July 1963[3] to December 2015 and comprises all stocks listed on NYSE, AMEX, and NASDAQ. For stocks listed on the NASDAQ, the data start in 1973. The analysis covers a total of 618 monthly observations. Following Fama and French (1993) to filter the database and construct cross-sectional portfolios, we keep stocks with a CRSP share[4] code (SHRCD) of 10 or 11 at the beginning of month *t*, an exchange code (EXCHCD) of 1, 2 or 3, available shares (SHROUT) and price (PRC) data at the beginning of month *t*, available return (RET) data for month t, at least 2 years of listing on Compustat to avoid survival bias Fama and French (1993) and a positive book-equity value at the end of December of year *y*-1.

We defined the book value of equity as the Compustat book value of stockholders' equity (SEQ) plus the balance-sheet deferred taxes and investment tax credit (TXDITC). If available, we decrease this amount by the book value of preferred stock (PSTK). If the book value of

[3] Compustat and CRSP information are available from January 1950 and January 1926, respectively. However, after we correct for survival and backfill biases, our sample starts in July 1953 (Fama and French (1993) rebalancing date). Moreover, 60 daily observations are required to estimate the covariance matrix. We decided to start the sample at the same date as in Fama and French (1993): July 1963.

[4] In his paper, Hasbrouck (2009, p. 1455) explains this restriction as "*restricted to ordinary common shares (CRSP share code 10 or 11) that had a valid price for the last trading day of the year, and had no changes of listing venue or large splits within the last 3 months of the year*".

stockholders' equity (SEQ) plus the balance-sheet deferred taxes and investment tax credit (TXDITC) is not available, we use the firm's total assets (AT) minus its total liabilities (LT).

Book-to-market equity (B/M) is the ratio of the book value of equity for the fiscal year ending in calendar year *y*-1 divided by market equity. Market equity is defined as the price (PRC) of the stock times the number of shares outstanding (SHROUT) at the end of June *y* to construct the size characteristic and at the end of December of year *y*-1 to construct the B/M ratio.

We also include the extension of the Fama and French's three-factor model by Carhart (1997) with a momentum factor (i.e., a *t*-2 until *t*-12 cumulative prior return) to add an additional dimension to our investment style portfolios. The momentum reflects the return differential between the highest and lowest prior-return portfolios.

In the original Fama-French approach, portfolios are constructed using a 2x3 independent sorting procedure: two-way sorting (small and big) on market capitalization and three-way sorting (low, medium, high) on the book-to-market equity ratio. These style classifications are defined according to NYSE[5] stocks exchange only and then applied to the whole sample (AMEX, NASDAQ and NYSE). Six portfolios are constructed at the intersection of the 2x3 classifications and rebalanced on a yearly basis at the end of June.

The second sorting methodology apply a *dependent* sort following Lambert et al. (2016) who use a simple but fundamental change to the independent sorting methodology to form characteristic-sorted portfolios. The authors consider the whole sample rather than only the NYSE as breakpoints and motivate their choice to avoid the remark from Daniel et al. (1997, p. 1057):

[5] The NYSE is represented by stocks that account for the largest capitalization in the CRSP database. The exchange codes 1, 2 and 3 are for the NYSE, NASDAQ and AMEX, respectively.

"*size breakpoints are designed so that there will be an equal number of NYSE firms in each of the [five] portfolios*". Lambert et al. (2016) uncover that these NYSE breakpoints create an imbalance in the (total) number of stocks between small- and large-cap portfolios such that independent sorting leads to a higher number of stocks in small-value portfolios. Using independent sorting on negatively (or positively) correlated variables can induce, by design, a strong tilt toward the extreme categories of inverse ranks (low-high and high-low). From January 1963 to December 2014, the market equity and book-to-market equity of a firm were on average negatively correlated (-5%). In Figure 1, we illustrate the implications of the choice of sorting methodology to stratify the US equity universe into (2x3) characteristic-sorted portfolios. The independent sorting methodology results in a large part of the universe falling into the small-value (28.1%) category, whereas dependent sorting delivers a well-balanced repartition of stocks in all portfolios (around 16%).

[Figure 1 near here]

Each of these methods for constructing investment styles portfolios will serve as testing assets in this paper. In Figure 2, we present the relation between the risk and return of the portfolios sorted with an *independent* and *dependent* methodology in red and blue, respectively. We illustrate the construction of portfolios by splitting the US stocks universe into six (2x3), nine (3x3) or twenty-seven (3x3x3) groups. The 3x3x3 splits is constructed on the size, value and momentum characteristics of a firm. For illustration purposes, portfolios displayed in Figure 2 are rebalanced and reallocated annually.

[Figure 2 near here]

3 Implementation of Strategic Beta Strategies

Our research focuses on stock level US retrieved from CRSP and Compustat databases, allowing us to construct and test strategies using a variety of protocols and thereby draw robust conclusions as to the benefits of investing in portfolios that do not simply rely on market capitalization as an input. More precisely, in addition to the choice of the two sorting methodologies to construct portfolios, we control for three other parameters when constructing our style indices.

First, the number of characteristic-based portfolios is set to either six (2x3), nine (3x3), or twenty-seven (3x3x3) and constitute the underlying securities of the final "market" portfolio of our studies. For a large number of securities in a portfolio, the estimation of the covariance matrix requires sophisticated techniques because an estimation solely based on a sample period can be fraught with considerable errors (see, e.g., Ledoit and Wolf (2004)). Referring to the covariance matrix, Berk (2000, p. 420) states that *"[g]iven a typical sample of 2000 stocks, this matrix has more than 2 million elements. With only 70 years of data, there is an obvious specification problem"*. He thus suggests grouping stocks to reduce this specification problem. However, this approach does not entirely resolve the issue because the sample covariance matrix has to estimate n(n-1)/2 pair-wise correlations. Increasing the number of groups of stocks can lead to strong sample dependency and consequently noisy estimates. Given that we form 6, 9, and 27 investment style portfolios and use 60 daily returns to estimate the covariance matrix. In the most extreme case (27 portfolios), we are left with 0.17 data points per parameter, which might present a potential issue if we only consider the sample covariance matrix in our optimizations. This problem is also referred to as sampling error. We use, in our applications, the *shrinkage* methodology from

Ledoit and Wolf (2004) to estimate the covariance with lower sampling errors. Further details on the shrinkage method can be found in the Appendix 1.

Second, the allocation scheme intra portfolio is either capitalization-weighted or equal-weighted. We do this because each allocation scheme has been set as standard throughout the years. For instance, the portfolios found on Ken French's website are either CW or 1/N.

And third, we control for different rebalancing frequency, i.e. monthly, quarterly, semi-annually, and annually. In their paper on the taxonomy of market equity anomalies, Novy-Marx and Velikov (2016) explain that the recent popularity of equal-weighted portfolios might be misleading after considering transaction costs because a naïve diversification allocation (1/N) places more weights on small-cap stocks, i.e. the most illiquid and expensive stocks to trade. Moreover, a higher rebalancing frequency may cannibalize a large part of the performance of a strategy on net returns (after transactions costs). To consider transaction costs, Plyakha et al. (2015) implement a decreasing function of transaction costs from 1% in 1978 to 0.5% in 1993 for their S&P500 sample. However, in our paper, we trade stocks on NYSE-NASDAQ-AMEX exchanges and consequently have to differentiate transactions costs for small and large-cap stocks. We thus follow an approach similar to that of Novy-Marx and Velikov (2016) and use the individual stocks estimates from the Gibbs sampling developed in Hasbrouck (2009). Further details on this method can be found in the Appendix 2.

In Figure 3, we show the annual box-and-whisker plot for the CRSP/Gibbs estimates of transaction costs (variable c from equation (5)) from 1963 to 2015.

[Figure 3 near here]

Novy-Marx and Velikov (2016) uncover a minor drawback to Hasbrouck's estimation technique, which requires relatively long series of daily prices to perform the estimation (250 days). This results in a number of missing observations for which Novy-Marx and Velikov (2016) perform a non-parametric matching method and attribute equivalent transaction costs to the stock with a missing value according to its closest match in size and idiosyncratic volatility. Since these missing observations represents only 4% of the total market capitalization universe, instead, we decided to simply replace the missing values with transaction costs of 0.50%. We employ this (extreme) arbitrary value because (1) we see from Figure 3 that none of the estimates ever breach a trading cost of 50 bps since 1963, (2) this choice will more strongly impact illiquid stocks with short amount of daily observations (small-capitalization stocks) and (3) Plyakha, Uppal and Vilkov (2015) also choose to set this threshold for transaction costs from 1993 onwards.

In Table 2, we report the transaction costs for our Strategic Beta strategies. We distinguish the implication of transaction for rebalancing the investment style portfolios on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. We also examine the trading costs according to the number of constructed portfolios, namely, six (2x3), nine (3x3), and twenty-seven (3x3x3). The results are presented in annual terms (in %) and show that transaction costs have a linear relationship with the rebalancing frequency according to the Patton and Timmermann (2010) test for decreasing monotonic relationships. The level of transaction costs is also greater for portfolios sorted dependently and suggests that small stocks have higher weights with dependent sorting.

[Table 2 near here]

To summarize, each sorting methodology (independent and dependent) generates twenty-four combinations of portfolios whom will constitute our test assets for performing the heuristic and risk-based optimizations. From these twenty-four combinations of constructing investment styles portfolios, we apply the four different allocation techniques defined in Table 1 of Section I, namely, minimum variance, maximum diversification, risk parity, and equally weighted. In total, the combinations of strategies are equal to 96 for each methodology of portfolio construction.

After adjusting our US sample for well-known biases, delisting return, survivorship bias, etc., and controlling for a variety of protocols to construct Strategic Beta strategies, we expect a *dependent* sorting methodology to outperform a *independent* sorting because the allocation of a dependent sort is not contingent on the correlations between the sorted variables. Such that, the dependent sort shows a strong stability in the stock allocation among the style portfolios. The diversification properties of the style portfolios constructed from the dependent sorting methodology are thus expected to be greater in comparison to an independent sort.

4 Mean-Variance Spanning Test

To test the outperformance of one sorting methodology versus the other, we build on a traditional mean-variance spanning test introduced by Huberman and Kandel (1987) which differentiates whether adding one security to a set of investment improves the efficient frontier. Although it is important for an investor to know whether her set of opportunity increases with a new asset, it might be interesting to know whether the source of improvement comes either from the tangent or global minimum-variance (GMV) portfolio. Indeed, the two-fund separation theorem (Tobin (1958)) tells us that if an investor has access to a risk-free asset, he should only interested in the portfolio with the highest Sharpe ratio. According to his risk-preferences, his optimal allocation should lie on the Capital Market Line (CML) and be composed of a mix of the tangent portfolio (MSR) and the risk-free asset. However, if all capital is invested in risky assets, the optimal portfolio lies on the efficient frontier and is dependent on the investor utility and degree of risk aversion.

Kan and Zhou (2012) establish a mean-variance spanning test based on a step-down approach that allows to differentiate whether the source of improvement in the efficient frontier comes from the tangent and/or the GMV portfolio. This is only when both portfolios are improved that we observe a shift of the entire opportunity set.

The elegance of the step-down spanning tests lies in its ability to determine whether adding *N* elements to a set of *K* assets improves the tangent and/or GMV portfolio. The regression spanning test can be written as

(1) $$R_2^t = \alpha + \beta R_1^t + \varepsilon^t$$

where R_1^t is the benchmark portfolios composed of *K* assets (US Bond and a Portfolio A) and R_2^t is the set of test assets composed of *K*+*N* elements (Portfolio B) . In this paper, *K* is equal to 2 and *N* is equal to 1. Kan and Zhou (2012) present the test in matrix notations as follows:

(2) $$\mathrm{R} = \mathrm{XB} + \mathrm{E}$$

where

R is the test assets returns and *X* is a *K*+1 matrix of the benchmark assets returns. The matrix *X* can be written as follows:

(3) $$\mathrm{X} = \begin{pmatrix} 1 & \mathrm{R}_1^{\mathrm{US\ bond}} & \mathrm{R}_1^{\mathrm{Portfolio\ A}} \\ \vdots & \vdots & \vdots \\ 1 & \mathrm{R}_t^{\mathrm{US\ bond}} & \mathrm{R}_t^{\mathrm{Portfolio\ A}} \end{pmatrix}$$

Finally, *B* in equation (7) is a *K*+1 vector $[\alpha, \beta_{US\ bond}, \beta_{Portfolio\ A}]'$, and *E* is the error term vector $(\varepsilon_1, \dots, \varepsilon_{1t})'$.

The first test of the step-down procedure poses the null hypothesis for the tangent portfolio such that $\alpha = 0_N$ using OLS regression. The tangent portfolio is improved when the null is rejected.

(4) $$H_0^1 : \alpha = 0_N$$

The second test of the step-down procedure poses the null hypothesis for the GMV portfolio. This second test is conditional on the first test, $\alpha = 0_N$, and verifies whether $\delta = 1_N - \beta\, 1_K = 0_N$. It is only when both conditions are rejected that the test suggests an improvement of the GMV portfolio by adding *N* assets to the *K* benchmark assets.

(5) $$H_0^2 : \delta = 1_N - \beta\, 1_K = 0_N \mid \alpha = 0_N$$

The term "|" means conditional.

Kan and Zhou (2012) identify a test for statistical significance of the hypothesis similar to a GRS F-test. The F-test for the first hypothesis (H_0^1) is

(6) $$F_1 = \left(\frac{T - K - N}{N}\right)\left(\frac{\hat{a} - \hat{a}_1}{1 + \hat{a}_1}\right)$$

where *T* is the number of observations, *K* is the number of benchmark assets, *N* is the number of test assets, $\hat{a}_1 = \hat{\mu}_1{}'\hat{V}_{11}^{-1}\hat{\mu}_1$, with $\hat{V}_{11}$ denoting the variance of the benchmark assets, and $\hat{a}$ takes the same notation as $\hat{a}_1$ but refers to the benchmark assets plus the new test asset.

The F-test for the second hypothesis (H_0^2) is

(7) $$F_2 = \left(\frac{T - K - N + 1}{N}\right)\left[\left(\frac{\hat{c} + \hat{d}}{\hat{c}_1 + \hat{d}_1}\right)\left(\frac{1 + \hat{a}_1}{1 + \hat{a}}\right) - 1\right]$$

where $\hat{c}_1 = 1_K{}'\hat{V}_{11}^{-1}1_K$, $\hat{d}_1 = \hat{a}_1\hat{c}_1 - \hat{b}_1{}^2$ are the efficient set (hyperbola) constants with $\hat{b}_1 = \hat{\mu}_1{}'\hat{V}_{11}^{-1}1_K$ for the benchmark assets. In equation (7), $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are the equivalent notations for the benchmark assets plus the new test asset.

We illustrate in Panel A of Figure 4 a significant improvement for the tangency portfolio when a test asset (Portfolio B) is added to the benchmark assets (US bonds and Portfolio A). Panel B indicates a significant improvement for the GMV portfolio when a test asset (Portfolio B) is added to the benchmark assets (US bonds and Portfolio A). The original graphical illustrations of Kan and Zhou (2012, p. 158) explain in more detail how the constants $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are used to determine the geometric locations of the GMV portfolio.

[Figure 4 near here]

Intuitively, a spanning of the test assets ($K+N$) means that the weight attributed to the N test assets in the portfolio is trivial. Put differently, discarding the N test assets does not change significantly the efficient frontier of the K benchmark assets.

A. Testing Characteristic-Sorted Portfolios

In Table 3, we summarize the results of the step-down analysis[6] applied to the strategies developed in this paper. More precisely, Panel A reports the results when the benchmark assets are 30-Year US Treasury Bonds and a risk-optimization technique applied to *independent* portfolios (Portfolio A), whereas the test asset is the same risk-optimization technique applied to *dependent* portfolios (Portfolio B). We perform the same analysis in Panel B but in the reverse order, that is, the benchmark assets are now 30-Year US Treasury Bonds and a risk-based strategy constructed on *dependent* portfolios (Portfolio A), whereas the test asset is now the same risk-based strategy but constructed on *independent* portfolios (Portfolio B).

It is important to note that each of optimizations can have twelve different combinations of construction according to the choices of rebalancing frequency and number of portfolios. The portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis,

[6] The full analysis is reported in Appendices 3 to 6.

and the number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). All results are presented net of transaction costs.

[Table 3 near here]

The results shown in Panel A (Table 3) demonstrate that applying risk-optimization techniques to *dependent* portfolios can significantly improve the tangency portfolio of the benchmark assets for two risk-oriented strategies, that is, maximum diversification (MD) and minimum variance (MV). Indeed, the first hypothesis (H_0^1) is rejected five (nine) out of twelve times for MD on cap-weighted (equal-weighted) portfolios, whereas the rejection for MV optimization is six (five) out of twelve with cap-weighted (equal-weighted) portfolios.

In Panel B, the results demonstrate that all risk-optimization techniques applied to *independent* portfolios never improve the tangency portfolio of the benchmark assets with a confidence level of 95% relative to the same risk-optimization techniques applied on *dependent* portfolios.

In each panel, we also report the joint spanning test to understand whether the whole efficient frontier is shifted up and to the left once Portfolio B is added to the benchmark assets (US bond and Portfolio A). This test assume that returns are normally distributed and should be interpreted as interesting for an investor who is willing to invest her *full* capital in the risky assets when the *p*-value is less than 5% (rejection of the null).

[Table 4 near here]

In Table 4, we report the results of the previous joint spanning tests (last column of Table 3) and examine whether the assumption of a normal distribution for the returns of the strategies affects our results. Kan and Zhou (2012) describe two extensions when returns are not assumed to be normally distributed and exhibit excess kurtosis. Using the moment conditions, they apply a

GMM method to estimate the regression parameters of equation (7). The first test is a joint F-test based on the results from the last column of Table 3. Recall that in this case, the returns are assumed to obey a normal distribution and the test is a joint test on the whole efficient frontier. In the first column of both panels (A and B), the level of rejection is high owing to an improvement in the global minimum variance. The second column is a GMM Wald test for returns under general distributions[7] (W_a). The last column is a GMM Wald test in which returns are assumed to follow an elliptical distribution (W_a^e) and that controls for heteroscedasticity and excess kurtosis. Moving from the first column to the last, we see that rejection rates of spanning only decrease in Panel B, not in Panel A, indicating that there are over-rejection problems in the first column of Panel B (Table 4) because the returns are assumed to obey a normal distribution. Yet, the rejection rates in Panel A appears to remain stable across the assumptions about the return distribution of the strategies.

B. Testing Against the US ETF Universe

We retrieve data about the US ETF universe from Morningstar and classify it according to the Morningstar® Strategic Beta classification tool. Table 5 reports the Strategic Beta definitions from Morningstar Guide[8]. The last column of the table specifies the category used in this paper to categorize the ETF universe. There are four categories: (1) Risk-weighted, (2) Return-oriented, (3) Blended, and (4) Other. We also report in parentheses the number of ETFs that fall in each category.

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[7] For more details, see Kan and Zhou (2012, p. 171) and Chen, Chung Ho, Hsu (2010). See also Chen, Ho, and Wu (2004) for GMM step-down resolution.

[8] The Morningstar Strategic Beta guide can be find on this website.

[Table 5 near here]

Concerning the treatment of the database, we winsorize the 1st and 99th percentile of return at each available month date and then remove returns lower than -100 percent and higher than 100 percent to lower the impact of outliers and reporting issues. Finally, we keep ETFs with more than one year of observations. Our tests are based on monthly returns, and all returns are denominated in US dollars.

Once the database is treated for extreme values, we apply Kan and Zhou's (2012) step-down spanning test on the ETFs (Portfolio A) against our strategies made of portfolios sorted *independently* or *dependently* (Portfolio B). We end up with results for all ETFs according to the 12 means of construction for our strategies specified in Table 4 (Section IV.A). However, for the sake of clarity, we aggregate the results according to the four categories listed above. We only differentiate the results for our strategies constructed on cap-weighted or equal-weighted portfolios. In Figure 5, we report the average outperformance frequency in term of tangent portfolio (H_0^1) for the strategies built on *independent* (red) and *dependent* (blue) portfolios. Results of Panel A is for cap-weighted portfolios, and Panel B is for equal-weighted portfolios.

[Figure 5 near here]

For example, risk-optimization strategies constructed on *independent* cap-weighted portfolios (in blue) outperform, on average, 31% of the risk-oriented ETFs listed in the US (graph on the left), whereas risk-optimization strategies constructed on *dependent* cap-weighted portfolios (in red) outperform, on average, 42% of the risk-oriented ETFs listed in the US (Panel A). The analysis can also be performed for the three other remaining categories, in which a risk-optimization on dependent (*independent*) cap-weighted portfolios outperform 52% (43%) of return-oriented, 70% (64%) of blended and 80% (78%) of other ETFs. The results are similar for risk optimizations on equal-weighted portfolios (graph on the right).

Whereas the results are consistently better for strategies based on *dependent* portfolios, it is also interesting to consider ETFs with equivalent objectives and investment universes as our risk-based strategies for an apples-to-apples comparison. Morningstar® provides data for such information under the name of "US Category Group". On its website[9], Morningstar® defines the criteria as follows: "*The term is used to group funds with similar categories and investing styles; can be used for a more broad-based analysis*" There are six categories available in which a filter for ETFs investing primarily in US stocks is present. Filtering the database on these criteria strongly reduces our sample, but the results regarding the outperformance of our risk-based strategies compared to ETFs providers may be more objective since we focus our investment strategies on the same geographical region (apples-to-apples comparison).

[Figure 6 near here]

The results for ETFs listed on US exchanges and primarily investing in US equity are presented in Figure 6. We see that risk-optimization strategies constructed on *independent* cap-weighted portfolios outperform, on average, 10% of the risk-oriented ETFs (graph on the left). Whereas risk-optimization strategies constructed on *dependent* cap-weighted portfolios outperform, on average, 24% of the risk-oriented ETFs. The analysis can also be performed on the other categories, for which a risk-optimization on dependent (*independent*) cap-weighted portfolios outperform 42% (27%) of return-oriented, 42% (27%) of blended and 50% (50%) of

[9] Morningstar® Add-in website. The information can be found by typing "US_Category_Group" in the search bar.

other ETFs. The results are similar for risk optimizations on equal-weighted portfolios (graph on the right).

In conclusion, a dependent sort to construct investment style portfolios exhibits better risk-return attributes for risk-optimization strategies. Whereas the majority of ETFs providers logically exhibit equivalent or better selectivity skills than our fully passive strategies on all US stock universes. The next sections disentangle the distinctive characteristics of the two sorting methodologies.

5 Disentangling the outperformance: the diversification properties of the opportunity sets

Our spanning tests deliver conclusive results towards better risk-return attributes for dependent portfolios. This section investigates further the diversification properties yielded by the two competing sorting methods. Section A analyses the pair-wise correlations between the portfolios depending on the sorting methodology. Section B estimates the correlation of stocks among the portfolios. Section C review the general case of equal-weighted portfolio variance and the implications according to the sorting methodology. Section D quantifies the differences in return yielded by the diversification properties of each sorting procedure.

A. Pair-Wise Correlation of Style Portfolios

In Figure 7, we illustrate the stock repartition when the number of portfolios is increased either by a larger split of the sample (from a 2x3 to a 3x3) or by adding a new characteristic (3x3x3). Throughout this paper, we followed Lambert et al. (2016) to stratify the US equity universe according to the dimensions of size, value and momentum characteristics. Considering the repartitions of the stocks based on the two sorting methods, we see that using an independent sort results in an imbalanced of stocks across the portfolios, and this effect becomes larger when more groups are constructed.

[Figure 7 near here]

A traditional (rational) view of returns' comovement is that comovement in prices reflects comovement in fundamental values (Barberis, Shleifer and Thaler (2005)). If this assumption holds, the *average* correlation between investment style portfolios should decrease when stocks are categorized according to a methodology that forms groups of stocks with stronger similarity in their fundamental values. We can thus formulate two predictions regarding the *average* correlation between investment style portfolios: first, using a dependent sort to group stock in investment styles provides a finer classification according to stocks' fundamentals such that the average correlation between style portfolios is reduced. Second, with a large number of stocks and the luxury of constructing *N*x*NxN* portfolios, we could create portfolios that are close to uncorrelated with each other. In Table 6, we illustrate that both predictions are verified: the average correlation between the investment style portfolios is lower when stocks are sorted dependently and split into a larger number of groups (i.e. 3x3x3). The results for cap-weighted (equal-weighted) portfolios are reported in Panel A (Panel B). It is well-know that assets with reduced coefficients of correlation start bending the efficient frontier toward the left because of diversification benefits. This reduced correlation might thus explain part of the outperformance in term of the Sharpe ratio for risk optimizations applied to portfolios sorted dependently.

[Table 6 near here]

B. Stocks Correlation Intra Portfolios

Rational investors hold that comovement in prices reflects comovement in fundamental values (Barberis et al. (2005)). We would thus expect that the average stock correlation intra portfolios (after formation) under a dependent sort should be stronger than under an independent sort. However, Figure 8 shows that the average correlation in stocks returns after portfolio formation under an independent sort (red line) is consistently stronger than under a dependent sort (blue line). The post-portfolio formation is based on 252 days and starts from July to end of June *t*+1. For brevity, results are only shown for the 2x3 portfolios[10]. We also report the 25^{th} and 75^{th} percentile distribution of the yearly stocks correlations in shaded areas.

[Figure 8 near here]

We uncover from this puzzling systematic correlation bias that stock correlation is stronger among securities for which the exchange has the largest total market capitalization, i.e., the NYSE, NASDAQ and then AMEX, in this order. We plot in Figures 9 and 10 the reparation of stocks belonging to the NYSE, NASDAQ and AMEX in green, red and blue, respectively. Results for the 2x3 portfolios sorted independently and dependently are shown in Figures 9 and 10, respectively.

[Figure 9 near here]

[Figure 10 near here]

We see from Figure 9 that a large part of the large-capitalization portfolios (the graphs on top) are represented by NYSE stocks. However, approximately 20%, on average, of the small-capitalization portfolios (graphs on the bottom) are also represented by NYSE stocks. In contrast,

[10] The analysis was also performed using the 3x3 and 3x3x3 sorting methodologies and yielded similar findings.

in Figure 10, representation of NYSE stocks is approximately 10% in each small-cap portfolio sorted dependently. This analysis is complemented by the yearly average correlation of stocks listed on each exchange from Figure 11. The graphs let us understand that a portfolio composed of a greater proportion of stocks listed on the NYSE will exhibit a higher average stocks correlation. The average correlation of stocks decreases as we move from the NYSE, to the NASDAQ and AMEX, in this order.

[Figure 11 near here]

As demonstrated in the previous section, a dependent sort on the whole set of breakpoint names implies a better stratification of the stocks universe compared to an independent sort. A lower proportion of NYSE stocks in a portfolio will mechanically reduce the average stock correlation after portfolio formation.

In the next section, we review the implication of the sorting techniques for the portfolio variance.

C. Portfolio Variance

The variance of a cap-weighted portfolio is given by

$$\sigma_i^2 = \sum_{j}^{n} \sum_{k}^{n} w_j w_k \sigma_j \, \sigma_k \rho_{jk} \tag{8}$$

where n is the number of stocks in the i^{th} portfolio, j is for the j^{th} stocks in the portfolio, k is for the k^{th} in the portfolios, w means weight, σ is the standard deviation, and ρ_{jk} is the correlation between the j^{th} and k^{th} stocks of the portfolio.

Although a generalization of equation (8) is complicated because the stocks weights (w) in cap-weighted portfolios vary over time, Gorton and Rouwenhorst (2006) describe a generalization of equation (9) for equal-weighted portfolios. The equation of the variance for the i^{th} equal-weighted portfolio can be written as follows:

$$\sigma_i^2 = \underbrace{\frac{1}{n}}_{\substack{\text{Weight on the assets}\\ \text{average variance}}} \overline{\text{var}} + \underbrace{\frac{n-1}{n}}_{\substack{\text{Weight on the assets}\\ \text{average covariance}}} \overline{\text{covar}} \qquad (9)$$

where n is the number of stocks in the i^{th} portfolio, $\overline{var}$ is the average variance of the n stocks, and $\overline{covar}$ is the average covariance of these same n stocks.

As n (the number of stocks within a portfolio) becomes larger, the variance of the portfolio (σ_i^2) converges to the average covariance of its constituents, $\sigma_i^2 \approx \overline{covar}$. From Figure 1 (Section I), we know that the number of stocks found in the portfolios sorted independently are, on average, equal to 24, 26, 28, 10, 8, and 4 percent over our 60-year sample period for the portfolios denoted LL, LM, LH, HL, HM, and HH, respectively. In Table 7, we illustrate a scenario in which 6 portfolios (2x3) are constructed according to the market equity and book-to-market equity ratio of a firm with a stock universe of 100 stocks.

[Table 7 near here]

According to Panel A of Table 7, independent sorting assigns greater weights to the average variance of large-capitalization stocks, more specifically to large value stocks (25%). As soon as the level of diversification in the portfolio decreases, higher weights will be given to the idiosyncratic risk of a stock. Panel B shows that even with a small number of securities (100), the weight assigned to stocks' specific risk with dependent sorting remains fairly low (6.25%). Using dependent sorting to classify stocks in style portfolios, the repartition of stocks becomes close to $1/n$ (n is equal to 6) when the breakpoints are the 33^{th} and 66^{th} percentiles of the distribution for all breakpoint names (NYSE, NASDAQ, and AMEX). Even though this concern may be immaterial for the construction of six portfolios with a universe of more than 3,000 stocks (US), the issue might become important for (1) a greater number of portfolios, (2) the early stage of the sample, or (3) less developed markets.

D. Diversification Return

Booth and Fama (1992) introduce the concept of the diversification return as a function of a portfolio geometric average return. According to those authors, a geometric average return is an important performance measure for portfolio management practices because it represents the growth rate that an investor would have earned if she held a portfolio since day one[11]. Denoting the geometric average return as g, volatility as σ, arithmetic average return as μ, Booth and Fama (1992) demonstrate that the measure can be approximated by the following mathematical formula:

$$g_P = \mu_p - \frac{\sigma_p^2}{2} \tag{10}$$

where p means portfolio. Moreover, we also know from Plyakha et al. (2015) that the expected arithmetic return (μ_p) of a portfolio made of N constituents is equal to

$$\mu_p = \sum_i^N E\left(w_{i,t+1} R_{i,t}\right) = \sum_i^N \left[E\left(w_{i,t+1}\right) E\left(R_{i,t}\right) + \mathrm{cov}(w_{i,t+1}, R_{i,t})\right] \tag{11}$$

where $\mu_p = E\left(R_p\right)$ and i refers to the i^{th} security in the portfolio (p).

Finally, from the paper of Erb and Harvey (2006, Table 8), the impact of a simple buy and hold strategy for the portfolio (p) and the consituents' weights (w) according to the market capitalization can be formalized as

[11] Willenbrock (2011) notes the mathematical equation as $(1 + g)^T$, with g denoting the geometric average return and T denoting holding periods.

(12) $$\mu_p - \sum_{i}^{N} \overline{w_i}\, \mu_i$$

However, we know only the average of w_i after realization (*ex-post*). A more natural method to compare the impact of rebalancing *ex-ante* (before realization) should be against an allocation that is equally weighted (DeMiguel et al. (2009)) because this is the only allocation for which we know *ex-ante* $\overline{w_i}$, that is (1/N). The equation thus becomes

(13) $$\mu_p - \frac{1}{N}\sum_{i}^{N} \mu_i$$

Substituting (11) in (13), we have

(14) $$\sum_{i}^{N}\left[E(w_{i,t+1})E(R_{i,t}) + cov(w_{i,t+1}, R_{i,t})\right] - \frac{1}{N}\sum_{i}^{N} E(R_{i,t})$$

Rearranging the terms leads to the following form:

(15) $$\underbrace{\sum_{i}^{N} cov(w_{i,t+1}, R_{i,t})}_{\substack{\text{covariance} \\ \text{return}}} + \underbrace{\sum_{i}^{N}\left(E(w_{i,t+1}) - \frac{1}{N}\right) E(R_{i,t})}_{\substack{\text{Adjustment for} \\ \text{not being EW}}}$$

It is important to note that both terms vanish if we implement an equally weighted strategy that rebalances at each period *t* (in this study, on a monthly basis).

Finally, Willenbrock (2011) formalizes the diversification return as follows:

(16) $$DR = g_P - \sum_{i}^{N} \overline{w_i}\, g_i$$

where *i* stands for the i^{th} security in the portfolio (*p*) and *g* refers to the geometric return.

The concept of diversification in returns emphasizes that the geometric average return of a portfolio is greater than the sum of the geometric average return of its constituents. Substituting (10) in (16), we have

$$\text{(17)} \qquad \mathrm{DR} = \mu_p - \frac{\sigma_p^2}{2} - \frac{1}{N}\sum_{i}^{N}\left(\mu_i - \frac{\sigma_i^2}{2}\right)$$

Alternatively, we can factorize the equation such that,

$$\text{(18)} \qquad \mathrm{DR} = \mu_p - \frac{\sigma_p^2}{2} - \frac{1}{N}\sum_{i}^{N}\mu_i + \frac{1}{N}\sum_{i}^{N}\frac{\sigma_i^2}{2}$$

Rearranging the terms,

$$\text{(19)} \qquad DR = \underbrace{\mu_p - \frac{1}{N}\sum_{i}^{N}\mu_i}_{\text{equation (15)}} + \frac{1}{2}\left(\bar{\sigma}_i^2 - \sigma_p^2\right)$$

We can substitute the terms from equations (15) in (19) and rewrite the diversification return as follows:

$$\text{(20)} \qquad \mathrm{DR} = \underbrace{\sum_{i}^{N}\mathrm{cov}(w_{i,t+1}, R_{i,t})}_{\substack{\text{covariance}\\ \text{return}}} + \underbrace{\sum_{i}^{N}\left(E(w_{i,t+1}) - \frac{1}{N}\right)E(R_{i,t})}_{\substack{\text{Adjustment for}\\ \text{not being EW}}} + \underbrace{\frac{1}{2}\left(\bar{\sigma}_i^2 - \sigma_p^2\right)}_{\substack{\text{Variance Reduction}\\ \text{Benefit}}}$$

Equation (20) provides a benchmark[12] to compare a strategy using dynamic weights with the average constituents that it holds. The benefits of applying risk-optimization strategies (MV, MD, and risk parity) to the passive portfolios can be compared to a naïve diversification allocation (equally weighted). We can isolate whether the incremental return benefit is earned by the weights

[12] This test is implemented on gross returns. The decomposition in equation (20) does not hold when transaction costs are considered.

of the strategy (first and second terms) or the last term, also coined volatility harvesting by Bouchey et al. (2012). The full decomposition of the results can be found in Appendices 7 to 10.

To compare the benefits of diversification between the two sorting methodologies, we adjust the diversification return according to the *risk* of the benchmark sorting methodology. In this paper, we use the independent sorted portfolios as a benchmark. Equation (21) describe the risk-adjustment for comparing the diversification return.

$$\pi_{\mathrm{DR}} = \frac{\sigma_{\mathrm{Independent}}}{\sigma_{\mathrm{dependent}}}\left(\mathrm{DR}_{\mathrm{dependent}}\right) - \left(\mathrm{DR}_{\mathrm{Independent}}\right) \tag{21}$$

In short, if the spread (π_{DR}) is positive, a dependent sort to construct portfolios delivers greater diversification benefits on a risk-adjusted basis than an independent sort. In Table 8, we report the results for four Strategic Beta strategies and the combination of rebalancing frequency, number of portfolios and portfolios allocation (cap-weighted and equal-weighted). The spreads are strictly positive in 92 of 96 strategies. Risk-oriented strategies specifically designed to maximize the objective of diversification (max. div.) yields (logically) the best results.

[Table 8 near here]

Figures 12 and 13 let us visualize the global results for all the risk-based strategies on cap-weighted and equal-weighted portfolios, respectively. On a risk-adjusted basis, we see that the covariance drag (light red columns) is smaller for portfolios sorted dependently. However, for most of the strategies, the adjustment for not being equally weighted (light blue columns) and the variance reduction benefit (light gray columns) are typically greater for dependently sorted portfolios.

[Figure 12 near here]

[Figure 13 near here]

6 Conclusions

Motivated by the need to reduce the number of assets in portfolio optimizations, we implement smart beta strategies on "style" portfolios as the equity building block. This approach not only reduces the issues in estimating a large covariance matrix of returns but also is consistent with the common practice of institutional investors, who tend to reallocate funds across style groupings (see, for instance, Froot and Teo (2008)). We show that the methodology for grouping stocks in different style buckets has strong implications for the performance of the final strategy. To categorize stocks in investment style portfolios, we stratify the universe along the dimensions of size, value and momentum characteristics. We implement two sorting methodologies to construct characteristic-based portfolios: traditional *independent* sorting according to Fama and French (1993) and *dependent* sorting according to Lambert et al. (2016). To demonstrate the implications of the sorting methodologies, we apply mean-variance spanning tests from Kan and Zhou (2012) on the risk-oriented strategies that use characteristic-based portfolios as assets (or Strategic Beta strategies). The results show that dependent sorting of stocks in portfolios provides significantly higher Sharpe ratios for risk-oriented Strategic Beta strategies. The results hold regardless of whether stocks are capitalization-weighted or equal-weighted in portfolios, whether stocks are rebalanced at different frequencies or whether returns are net of transaction costs. Because dependent sorting controls for correlated variables and stratifies the stock universe in well-diversified portfolios (Lambert et al. (2016)), this sorting methodology delivers better diversification benefits for Strategic Beta strategies. To demonstrate this point, we provide a decomposition of the diversification return from Booth and Fama (1992). We uncover that the diversification return is, on a risk-adjusted basis, higher for Strategic Beta strategies implemented on dependent portfolios than on independent portfolios.

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Appendices

Appendix 1

Estimation of the Covariance Matrix

We briefly describe in this section a *shrinkage* methodology used in our applications to estimate the covariance with lower sampling errors following Ledoit and Wolf (2004). In their model, the authors build on Elton and Gruber (1973) who use a constant correlation coefficient to shrink the assets' covariance toward a global average correlation estimator

The constant correlation coefficient is determined using,

$$\hat{\rho} = \frac{1}{N(N-1)}\left(\sum_{i}^{N}\sum_{j}^{N}\hat{\rho}_{ij} - N\right) \tag{22}$$

where N is the number of portfolios - in our applications, either 6, 9 or 27. The term $\hat{\rho}_{ij}$ is the historical correlation estimate between the i^{th} portfolio and the j^{th} portfolio.

Ledoit and Wolf (2004) then obtain an optimal structure for the covariance matrix and reduce the sampling error of a traditional sample covariance matrix (S):

$$\Sigma = \delta F + (1-\delta)S \tag{23}$$

where Σ is the output covariance matrix from the *shrinkage* estimation and δ is the optimal shrinkage intensity[13]. S is the sample covariance matrix from our 60 daily returns, and F is the structured covariance matrix with assets' covariance estimated via the constant correlation estimator[14] in equation (22). In our empirical study, the estimations of the sample and the structured covariance matrices are based on 60-day rolling windows to accommodate for gradual

[13] Matlab code is made available at Prof. Wolf's website.

[14] The covariance of the matrix F is given by $\sigma_{ij} = \hat{\rho}\sigma_i\sigma_j$.

changes in the return distribution and short-term variations. On a practical basis, using monthly return to estimate assets' covariance matrix might be cumbersome because of the short track records of the Exchange-Traded Fund (ETF) universe. Although our paper, we intend to reconstruct a proxy for the market portfolio based on risk-factor-based assets, a real-life application with tradable assets (Idzorek and Kowara (2013)) would impose constraints on the historical information available to replicate our results. For this reason - to stay as close as possible to what real-world applications may offer - we focus our optimizations on 60-day rolling windows.

Appendix 2

Transaction Costs

1. Gibbs estimates

A traditional model to estimate trading costs of a security is documented by Roll (1984). This model only requires information about the daily trade price, prior midpoint of the bid-ask prices and the sign of trade to perform the estimation. Formally, the procedure is written as follows:

$$m_t = m_{t-1} + u_t$$
$$p_t = m_t + cq_t \tag{24}$$

where m_t is the log midpoint of the prior bid-ask price, p_t is the log trade price, q_t is the sign of the trade (+1 for a buy and -1 for a sale), c is the effective cost, and u_t is assumed to be unrelated to the sign of the trade (q_t).

Since we take in equation (24) the logarithm for the price variables, the daily change in price is given by

$$\Delta p_t = p_t - p_{t-1}$$
$$= m_t + cq_t - m_{t-1} - cq_{t-1}$$
$$= c\Delta q_t + u_t \tag{25}$$

Hasbrouck (2009) suggests extending Roll's (1984) model with a market factor and estimating the effective trading costs using Bayesian Gibbs sampling applied to the daily prices of U.S. equity retrieved from CRSP data. The market-factor model[15] is presented as follows:

$$\Delta p_t = c\Delta q_t + \beta_{rm} rm_t + u_t \tag{26}$$

[15] The SAS code is made available on Prof. Hasbrouck's website.

where rm_t is the market return on day t and β_{rm} is the parameter estimate from the regression on the market return.

The Bayesian methodology estimates the effective costs (c) based on a sequence of iterations where the initial prior for c is strictly positive and follows a normal distribution with mean zero and variance equal to 0.05^2, denoted $N^+(\mu = 0, \sigma^2 = 0.05^2)$. According to Hasbrouck (2009), the initial values of the prior should not impact the final estimate of the effective cost of a stock because the first 200 iterations (of 1,000) are disregarded to compute the average of the estimated values for the trading cost (c). The purpose of the Gibbs sampling is to estimate the value of the parameters c and β conditional on the values drawn for the sign of trade (q_t) and the error term (u_t). For each iteration, new values for q_t and u_t are drawn from their respective distributions, and an OLS regression is performed to estimate the new values of c and β.The process is repeated 1,000 times, and the final value for c is the average of the last 800 estimations of the procedure. For more information on the iterative process, we refer to Hasbrouck (2009, p. 1447), who summarizes the procedure in four steps[16].

[16] Further details regarding the application of the estimation technique can also be found in Marshall, Nguyen and Visaltanachoti (2011) and Novy-Marx and Velikov (2016).

Appendix 3

The table reports the results for the step-down regression-based mean-variance spanning test from Kan and Zhou (2012). We display the results when the benchmark assets are 30-Year US Treasury Bonds and the Strategic Beta on ***independent*** cap-weighted portfolios. The test assets have the same Strategic Beta on ***dependent*** cap-weighted portfolios. In total, there are four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk-parity (RP). These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. The number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The first column reports the correlation between Strategic Beta on both sorting methodologies. F1 tests the null hypothesis that additional tests asset do not improve the *ex-post* tangency portfolio. F2 tests the null hypothesis that additional test assets do not improve the *ex-post* global-minimum-variance (GMV) portfolio. We also report the step-down joint-*p*, which tests whether the efficient frontier is improved when we add the test asset to the benchmark assets. W_a^e is the GMM Wald test when returns are assumed to have a multivariate ellicptical distribution; the results are comparable to those for the step-down joint-*p*. W_a is the GMM Wald and is valid under all return distributions. The sample period ranges from July 1963 to December 2015.

						Step-Down Test					GMM Wald Test			
Strategy	Corr.	$\hat{\alpha}$	$\hat{\delta}$	*F*-test	*p*-value	F_1	*p*-value	F_2	*p*-value	Joint-*p*	W_a^e	*p*-value	W_a	*p*-value
						Rebalancing: Monthly								
EW (2x3)	0.98	0.0000	0.0189	1.0130	0.3640	0.01	0.93	2.02	0.16	0.15	1.30	0.52	1.55	0.46
EW (3x3)	0.97	0.0001	0.0310	1.8230	0.1620	0.05	0.83	3.61	0.06	**0.05**	2.08	0.35	2.87	0.24
EW (3x3x3)	0.97	0.0003	0.0288	1.3960	0.2480	0.24	0.63	2.56	0.11	0.07	1.62	0.45	2.42	0.30
MD (2x3)	0.96	0.0012	0.0714	8.3880	0.0000	5.33	**0.02**	11.37	**0.00**	**0.00**	8.38	**0.02**	12.28	**0.00**
MD (3x3)	0.94	0.0013	0.1080	10.7860	0.0000	3.65	0.06	17.85	**0.00**	**0.00**	9.26	**0.01**	17.86	**0.00**
MD (3x3x3)	0.9	0.0022	0.1549	12.4000	0.0000	5.44	**0.02**	19.23	**0.00**	**0.00**	11.27	**0.00**	14.26	**0.00**
MV (2x3)	0.91	0.0019	0.1123	7.6330	0.0010	4.87	**0.03**	10.34	**0.00**	**0.00**	6.16	**0.05**	18.28	**0.00**
MV (3x3)	0.92	0.0018	0.1466	15.0730	0.0000	5.31	**0.02**	24.67	**0.00**	**0.00**	10.72	**0.01**	27.55	**0.00**
MV (3x3x3)	0.93	0.0005	0.0817	5.2970	0.0050	0.54	0.46	10.06	**0.00**	**0.00**	3.41	0.18	10.06	**0.01**
RP (2x3)	0.97	0.0006	0.0415	3.7240	0.0250	1.58	0.21	5.86	**0.02**	**0.00**	3.51	0.17	6.68	**0.04**
RP (3x3)	0.97	0.0006	0.0595	5.2740	0.0050	1.34	0.25	9.21	**0.00**	**0.00**	4.62	0.10	9.08	**0.01**
RP (3x3x3)	0.97	0.0008	0.0613	5.7450	0.0030	2.12	0.15	9.36	**0.00**	**0.00**	5.26	0.07	9.87	**0.01**

Rebalancing: Quarterly														
EW (2x3)	0.98	0.0001	0.0189	1.0060	0.3660	0.02	0.90	2.00	0.16	0.14	1.28	0.53	1.56	0.46
EW (3x3)	0.97	0.0001	0.0310	1.8130	0.1640	0.07	0.80	3.57	0.06	**0.05**	2.06	0.36	2.88	0.24
EW (3x3x3)	0.97	0.0003	0.0287	1.3900	0.2500	0.27	0.61	2.51	0.11	0.07	1.61	0.45	2.43	0.30
MD (2x3)	0.96	0.0012	0.0623	6.6670	0.0010	4.77	**0.03**	8.51	**0.00**	**0.00**	6.73	**0.04**	10.09	**0.01**
MD (3x3)	0.94	0.0013	0.1001	9.1350	0.0000	3.49	0.06	14.72	**0.00**	**0.00**	7.89	**0.02**	16.36	**0.00**
MD (3x3x3)	0.9	0.0023	0.1729	14.6390	0.0000	5.78	**0.02**	23.32	**0.00**	**0.00**	12.81	**0.00**	17.64	**0.00**
MV (2x3)	0.91	0.0015	0.1093	6.5930	0.0010	2.73	0.10	10.43	**0.00**	**0.00**	5.12	0.08	13.64	**0.00**
MV (3x3)	0.92	0.0015	0.1439	14.3150	0.0000	3.79	0.05	24.73	**0.00**	**0.00**	9.49	**0.01**	24.68	**0.00**
MV (3x3x3)	0.93	0.0007	0.0899	6.4010	0.0020	0.81	0.37	12.00	**0.00**	**0.00**	4.41	0.11	9.52	**0.01**
RP (2x3)	0.97	0.0006	0.0384	3.1790	0.0420	1.60	0.21	4.76	**0.03**	**0.01**	3.21	0.20	5.78	0.06
RP (3x3)	0.97	0.0007	0.0573	4.9510	0.0070	1.47	0.23	8.43	**0.00**	**0.00**	4.28	0.12	8.57	**0.01**
RP (3x3x3)	0.97	0.0008	0.0594	5.4180	0.0050	2.04	0.15	8.78	**0.00**	**0.00**	4.92	0.09	9.21	**0.01**
Rebalancing: Semi-Annually														
EW (2x3)	0.98	0.0001	0.0189	1.0040	0.3670	0.02	0.89	1.99	0.16	0.14	1.28	0.53	1.56	0.46
EW (3x3)	0.97	0.0001	0.0310	1.8110	0.1640	0.07	0.79	3.56	0.06	**0.05**	2.05	0.36	2.89	0.24
EW (3x3x3)	0.97	0.0003	0.0288	1.3960	0.2480	0.29	0.59	2.51	0.11	0.07	1.62	0.44	2.46	0.29
MD (2x3)	0.96	0.0011	0.0668	6.4900	0.0020	3.79	0.05	9.15	**0.00**	**0.00**	6.47	**0.04**	9.24	**0.01**
MD (3x3)	0.94	0.0014	0.1081	10.3410	0.0000	3.71	0.06	16.90	**0.00**	**0.00**	9.31	**0.01**	16.55	**0.00**
MD (3x3x3)	0.9	0.0021	0.1697	14.4240	0.0000	4.88	**0.03**	23.82	**0.00**	**0.00**	13.05	**0.00**	15.23	**0.00**
MV (2x3)	0.91	0.0019	0.0858	5.1340	0.0060	4.71	**0.03**	5.52	**0.02**	**0.00**	4.69	0.10	11.05	**0.00**
MV (3x3)	0.93	0.0016	0.1100	8.6880	0.0000	4.14	**0.04**	13.17	**0.00**	**0.00**	5.96	0.05	17.21	**0.00**
MV (3x3x3)	0.94	0.0010	0.0995	8.3200	0.0000	1.82	0.18	14.80	**0.00**	**0.00**	6.51	**0.04**	11.41	**0.00**
RP (2x3)	0.97	0.0006	0.0354	2.8010	0.0620	1.94	0.17	3.66	0.06	**0.01**	3.11	0.21	5.22	0.07
RP (3x3)	0.97	0.0007	0.0561	4.7940	0.0090	1.74	0.19	7.84	**0.01**	**0.00**	4.27	0.12	8.11	**0.02**
RP (3x3x3)	0.97	0.0007	0.0575	5.0630	0.0070	1.92	0.17	8.19	**0.00**	**0.00**	4.63	0.10	8.14	**0.02**
Rebalancing: Annually														
EW (2x3)	0.98	0.0001	0.0187	0.9860	0.3740	0.01	0.91	1.96	0.16	0.15	1.25	0.53	1.53	0.47
EW (3x3)	0.97	0.0001	0.0309	1.7940	0.1670	0.07	0.79	3.52	0.06	**0.05**	2.03	0.36	2.86	0.24
EW (3x3x3)	0.97	0.0003	0.0287	1.3910	0.2500	0.30	0.58	2.48	0.12	0.07	1.62	0.45	2.46	0.29
MD (2x3)	0.96	0.0010	0.0613	5.2740	0.0050	3.11	0.08	7.42	**0.01**	**0.00**	5.31	0.07	8.72	**0.01**
MD (3x3)	0.94	0.0014	0.1035	8.8440	0.0000	3.55	0.06	14.08	**0.00**	**0.00**	7.82	**0.02**	14.28	**0.00**
MD (3x3x3)	0.9	0.0018	0.1475	9.9280	0.0000	3.51	0.06	16.28	**0.00**	**0.00**	9.22	**0.01**	12.92	**0.00**
MV (2x3)	0.91	0.0021	0.1009	6.7670	0.0010	5.79	**0.02**	7.69	**0.01**	**0.00**	5.89	0.05	15.36	**0.00**
MV (3x3)	0.93	0.0017	0.0998	7.8050	0.0000	4.67	**0.03**	10.88	**0.00**	**0.00**	5.57	0.06	14.75	**0.00**
MV (3x3x3)	0.94	0.0009	0.1355	14.8070	0.0000	1.57	0.21	28.02	**0.00**	**0.00**	10.57	**0.01**	17.03	**0.00**
RP (2x3)	0.97	0.0006	0.0367	2.8470	0.0590	1.54	0.22	4.15	**0.04**	**0.01**	3.02	0.22	5.00	0.08
RP (3x3)	0.97	0.0007	0.0593	5.1100	0.0060	1.67	0.20	8.54	**0.00**	**0.00**	4.78	0.09	7.86	**0.02**
RP (3x3x3)	0.97	0.0008	0.0576	4.7380	0.0090	1.83	0.18	7.63	**0.01**	**0.00**	4.65	0.10	7.11	**0.03**

Appendix 4

The table reports the results for the step-down regression-based mean-variance spanning test from Kan and Zhou (2012). We display the results when the benchmark assets are 30-Year US Treasury Bonds and the Strategic Beta on ***independent*** equally weighted portfolios. The test assets have the same Strategic Beta on ***dependent*** equally weighted portfolios. In total, there are four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk-parity (RP). These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. The number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The first column reports the correlation between Strategic Beta on both sorting methodologies. F1 tests the null hypothesis that additional test assets do not improve the *ex-post* tangency portfolio. F2 tests the null hypothesis that additional test assets do not improve the *ex-post* global-minimum-variance (GMV) portfolio. We also report the step-down joint-*p*, which tests whether the efficient frontier is improved when we add the test asset to the benchmark assets. W_a^e is the GMM Wald test when returns are assumed to have a multivariate elliptical distribution; the results are comparable to those for the step-down joint-*p*. W_a is the GMM Wald and is valid under all return distributions. The sample period ranges from July 1963 to December 2015.

						Step-Down Test					GMM Wald Test			
Strategy	Corr.	$\hat{\alpha}$	$\hat{\delta}$	F-test	*p*-value	F_1	*p*-value	F_2	*p*-value	Joint-*p*	W_a^e	*p*-value	W_a	*p*-value
						Rebalancing: Monthly								
EW (2x3)	0.99	0.0000	-0.0107	0.3820	0.6830	0.00	0.98	0.77	0.38	0.38	0.46	0.79	0.62	0.74
EW (3x3)	0.98	0.0004	0.0054	0.3540	0.7020	0.69	0.41	0.02	0.90	0.37	0.66	0.72	0.94	0.62
EW (3x3x3)	0.98	0.0005	0.0079	0.4860	0.6160	0.93	0.34	0.05	0.83	0.28	0.88	0.65	1.31	0.52
MD (2x3)	0.97	0.0010	0.0229	2.1220	0.1210	3.56	0.06	0.68	0.41	**0.02**	3.55	0.17	5.73	0.06
MD (3x3)	0.96	0.0018	0.0625	5.9120	0.0030	7.25	**0.01**	4.53	**0.03**	**0.00**	8.49	**0.01**	13.96	**0.00**
MD (3x3x3)	0.93	0.0032	0.1101	10.4300	0.0000	13.08	**0.00**	7.63	**0.01**	**0.00**	13.61	**0.00**	18.55	**0.00**
MV (2x3)	0.96	0.0012	0.0504	3.7630	0.0240	3.64	0.06	3.87	0.05	**0.00**	5.26	0.07	4.86	0.09
MV (3x3)	0.96	0.0011	0.0935	8.5000	0.0000	2.85	0.09	14.11	**0.00**	**0.00**	6.14	**0.05**	14.27	**0.00**
MV (3x3x3)	0.96	0.0008	0.0654	5.4520	0.0040	1.87	0.17	9.02	**0.00**	**0.00**	4.02	0.13	6.82	**0.03**
RP (2x3)	0.99	0.0003	0.0006	0.3810	0.6830	0.73	0.40	0.04	0.85	0.34	0.76	0.68	0.94	0.63
RP (3x3)	0.98	0.0007	0.0219	1.5260	0.2180	2.16	0.14	0.89	0.35	**0.05**	2.32	0.31	3.68	0.16
RP (3x3x3)	0.98	0.0008	0.0273	2.1170	0.1210	2.74	0.10	1.49	0.22	**0.02**	2.92	0.23	4.87	0.09

							Rebalancing: Quarterly							
EW (2x3)	0.99	0.0000	-0.0107	0.3820	0.6830	0.00	0.98	0.77	0.38	0.38	0.46	0.79	0.62	0.74
EW (3x3)	0.98	0.0004	0.0054	0.3540	0.7020	0.69	0.41	0.02	0.90	0.37	0.66	0.72	0.94	0.62
EW (3x3x3)	0.98	0.0005	0.0078	0.4860	0.6160	0.93	0.34	0.05	0.83	0.28	0.88	0.65	1.31	0.52
MD (2x3)	0.98	0.0009	0.0151	1.6350	0.1960	3.08	0.08	0.19	0.67	0.05	2.98	0.23	4.41	0.11
MD (3x3)	0.96	0.0017	0.0581	5.1810	0.0060	6.40	**0.01**	3.93	**0.05**	**0.00**	7.39	**0.03**	12.21	**0.00**
MD (3x3x3)	0.92	0.0029	0.1104	8.9160	0.0000	9.99	**0.00**	7.73	**0.01**	**0.00**	10.78	**0.01**	16.08	**0.00**
MV (2x3)	0.97	0.0012	0.0548	4.3650	0.0130	3.82	0.05	4.89	**0.03**	**0.00**	7.60	**0.02**	5.07	0.08
MV (3x3)	0.96	0.0014	0.0736	6.3090	0.0020	4.52	**0.03**	8.05	**0.01**	**0.00**	5.85	0.05	12.53	**0.00**
MV (3x3x3)	0.96	0.0011	0.0610	5.3830	0.0050	3.51	0.06	7.23	**0.01**	**0.00**	4.72	0.09	7.07	**0.03**
RP (2x3)	0.99	0.0003	0.0002	0.3940	0.6740	0.74	0.39	0.05	0.82	0.32	0.79	0.68	0.93	0.63
RP (3x3)	0.98	0.0007	0.0207	1.5060	0.2230	2.24	0.14	0.77	0.38	0.05	2.33	0.31	3.57	0.17
RP (3x3x3)	0.98	0.0008	0.0269	2.0870	0.1250	2.69	0.10	1.48	0.22	**0.02**	2.85	0.24	4.69	0.10
							Rebalancing: Semi-Annually							
EW (2x3)	0.99	0.0000	-0.0107	0.3820	0.6820	0.00	0.98	0.77	0.38	0.38	0.46	0.79	0.62	0.74
EW (3x3)	0.98	0.0004	0.0054	0.3540	0.7020	0.69	0.41	0.02	0.90	0.37	0.66	0.72	0.94	0.62
EW (3x3x3)	0.98	0.0005	0.0078	0.4860	0.6160	0.93	0.34	0.05	0.83	0.28	0.88	0.65	1.31	0.52
MD (2x3)	0.98	0.0009	0.0176	1.8590	0.1570	3.43	0.07	0.29	0.59	**0.04**	3.36	0.19	4.71	0.10
MD (3x3)	0.96	0.0016	0.0529	4.6080	0.0100	5.93	**0.02**	3.26	0.07	**0.00**	6.62	**0.04**	10.26	**0.01**
MD (3x3x3)	0.93	0.0031	0.0983	9.1780	0.0000	12.35	**0.00**	5.90	**0.02**	**0.00**	12.41	**0.00**	15.93	**0.00**
MV (2x3)	0.96	0.0016	0.0468	4.1210	0.0170	6.17	**0.01**	2.06	0.15	**0.00**	9.03	**0.01**	6.74	**0.03**
MV (3x3)	0.96	0.0017	0.0960	10.4430	0.0000	6.93	**0.01**	13.82	**0.00**	**0.00**	9.04	**0.01**	20.12	**0.00**
MV (3x3x3)	0.96	0.0009	0.0701	6.3640	0.0020	2.47	0.12	10.23	**0.00**	**0.00**	5.61	0.06	7.98	**0.02**
RP (2x3)	0.99	0.0004	0.0006	0.5540	0.5750	1.05	0.31	0.06	0.81	0.25	1.11	0.58	1.32	0.52
RP (3x3)	0.98	0.0008	0.0208	1.6430	0.1940	2.56	0.11	0.73	0.40	**0.04**	2.60	0.27	3.85	0.15
RP (3x3x3)	0.98	0.0008	0.0274	2.1320	0.1190	2.70	0.10	1.56	0.21	**0.02**	2.88	0.24	4.64	0.10
							Rebalancing: Annually							
EW (2x3)	0.99	0.0000	-0.0107	0.3820	0.6820	0.00	0.98	0.77	0.38	0.38	0.46	0.79	0.62	0.74
EW (3x3)	0.98	0.0004	0.0054	0.3540	0.7020	0.69	0.41	0.02	0.90	0.37	0.66	0.72	0.94	0.62
EW (3x3x3)	0.98	0.0005	0.0078	0.4860	0.6160	0.93	0.34	0.05	0.83	0.28	0.88	0.65	1.31	0.52
MD (2x3)	0.97	0.0011	0.0135	2.3480	0.0960	4.66	**0.03**	0.03	0.86	**0.03**	4.45	0.11	5.81	0.06
MD (3x3)	0.96	0.0021	0.0513	5.7170	0.0030	9.40	**0.00**	2.01	0.16	**0.00**	8.79	**0.01**	12.36	**0.00**
MD (3x3x3)	0.93	0.0029	0.0792	6.7670	0.0010	10.38	**0.00**	3.11	0.08	**0.00**	10.17	**0.01**	13.12	**0.00**
MV (2x3)	0.97	0.0013	0.0349	3.0340	0.0490	4.73	**0.03**	1.33	0.25	**0.01**	4.27	0.12	4.96	0.08
MV (3x3)	0.96	0.0015	0.0786	7.3760	0.0010	5.60	**0.02**	9.09	**0.00**	**0.00**	7.41	**0.03**	9.25	**0.01**
MV (3x3x3)	0.97	0.0003	0.0466	2.9930	0.0510	0.25	0.62	5.74	**0.02**	**0.01**	2.72	0.26	2.60	0.27
RP (2x3)	0.99	0.0004	-0.0017	0.5970	0.5510	1.02	0.31	0.18	0.67	0.21	1.18	0.55	1.30	0.52
RP (3x3)	0.98	0.0008	0.0203	1.7600	0.1730	2.91	0.09	0.61	0.44	**0.04**	2.81	0.25	3.68	0.16
RP (3x3x3)	0.98	0.0009	0.0230	1.9530	0.1430	3.06	0.08	0.85	0.36	**0.03**	2.96	0.23	4.06	0.13

Appendix 5

The table reports the results for the step-down regression-based mean-variance spanning test from Kan and Zhou (2012). We display the results when the benchmark assets are 30-Year US Treasury Bonds and the Strategic Beta on *dependent* cap-weighted portfolios. The test assets have the same Strategic Beta on *independent* cap-weighted portfolios. In total, there are four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk-parity (RP). These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. The number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The first column reports the correlation between Strategic Beta on both sorting methodologies. F1 tests the null hypothesis that additional test assets do not improve the *ex-post* tangency portfolio. F2 tests the null hypothesis that additional test assets do not improve the *ex-post* global-minimum-variance (GMV) portfolio. We also report the step-down joint-*p*, which tests whether the efficient frontier is improved when we add the test asset to the benchmark assets. W_a^e is the GMM Wald test when returns are assumed to have a multivariate ellicptical distribution; the results are comparable to those for the step-down joint-*p*. W_a is the GMM Wald and is valid under all return distributions. The sample period ranges from July 1963 to December 2015.

						Step-Down Test					GMM Wald Test			
Strategy	Corr.	$\hat{\alpha}$	$\hat{\delta}$	F-test	*p*-value	F_1	*p*-value	F_2	*p*-value	Joint-*p*	W_a^e	*p*-value	W_a	*p*-value
						Rebalancing: Monthly								
EW (2x3)	0.98	0.0004	0.0167	1.0170	0.3620	0.85	0.36	1.18	0.28	0.10	1.41	0.50	1.92	0.38
EW (3x3)	0.97	0.0004	0.0186	0.9470	0.3880	0.91	0.34	0.98	0.32	0.11	1.25	0.53	1.88	0.39
EW (3x3x3)	0.97	0.0003	0.0239	1.1610	0.3140	0.53	0.47	1.79	0.18	0.08	1.40	0.50	1.92	0.38
MD (2x3)	0.96	-0.0004	-0.0001	0.3440	0.7090	0.64	0.43	0.05	0.82	0.35	0.68	0.71	0.76	0.68
MD (3x3)	0.94	0.0000	0.0070	0.0520	0.9490	0.00	0.99	0.10	0.75	0.74	0.05	0.98	0.07	0.97
MD (3x3x3)	0.9	0.0001	0.0453	1.1970	0.3030	0.01	0.91	2.39	0.12	0.11	1.08	0.58	1.11	0.57
MV (2x3)	0.91	0.0003	0.0700	3.3080	0.0370	0.16	0.69	6.46	**0.01**	**0.01**	2.32	0.31	4.73	0.09
MV (3x3)	0.92	-0.0001	0.0064	0.0400	0.9610	0.00	0.95	0.08	0.78	0.74	0.04	0.98	0.05	0.98
MV (3x3x3)	0.93	0.0009	0.0480	2.4370	0.0880	1.78	0.18	3.09	0.08	**0.01**	1.99	0.37	4.88	0.09
RP (2x3)	0.97	0.0000	0.0077	0.1450	0.8650	0.00	0.98	0.29	0.59	0.58	0.14	0.93	0.19	0.91
RP (3x3)	0.97	0.0001	0.0067	0.0860	0.9180	0.06	0.80	0.11	0.74	0.60	0.09	0.96	0.17	0.92
RP (3x3x3)	0.97	0.0000	0.0045	0.0390	0.9620	0.00	0.97	0.08	0.78	0.76	0.04	0.98	0.05	0.98

Rebalancing: Quarterly														
EW (2x3)	0.98	0.0003	0.0167	0.9890	0.3720	0.78	0.38	1.19	0.28	0.10	1.36	0.51	1.84	0.40
EW (3x3)	0.97	0.0004	0.0186	0.9200	0.3990	0.84	0.36	1.00	0.32	0.11	1.20	0.55	1.80	0.41
EW (3x3x3)	0.97	0.0003	0.0239	1.1580	0.3150	0.49	0.48	1.83	0.18	0.09	1.38	0.50	1.88	0.39
MD (2x3)	0.96	-0.0004	0.0073	0.4270	0.6530	0.47	0.50	0.39	0.53	0.26	0.75	0.69	0.77	0.68
MD (3x3)	0.94	0.0000	0.0156	0.2420	0.7850	0.00	0.96	0.48	0.49	0.47	0.21	0.90	0.33	0.85
MD (3x3x3)	0.9	0.0001	0.0350	0.6750	0.5090	0.01	0.91	1.34	0.25	0.22	0.59	0.75	0.65	0.72
MV (2x3)	0.91	0.0008	0.0726	3.5680	0.0290	1.14	0.29	5.99	**0.02**	**0.00**	2.69	0.26	5.79	0.06
MV (3x3)	0.92	0.0003	0.0075	0.0890	0.9150	0.14	0.71	0.04	0.85	0.60	0.12	0.94	0.18	0.91
MV (3x3x3)	0.93	0.0008	0.0390	1.6810	0.1870	1.43	0.23	1.94	0.17	**0.04**	1.53	0.47	3.27	0.20
RP (2x3)	0.97	0.0000	0.0115	0.3180	0.7280	0.00	0.99	0.64	0.43	0.42	0.33	0.85	0.41	0.82
RP (3x3)	0.97	0.0001	0.0081	0.1110	0.8950	0.04	0.85	0.19	0.67	0.57	0.10	0.95	0.18	0.91
RP (3x3x3)	0.97	0.0000	0.0056	0.0580	0.9430	0.00	0.99	0.12	0.73	0.72	0.06	0.97	0.07	0.96
Rebalancing: Semi-Annually														
EW (2x3)	0.98	0.0003	0.0167	0.9820	0.3750	0.77	0.38	1.20	0.27	0.11	1.34	0.51	1.82	0.40
EW (3x3)	0.97	0.0004	0.0186	0.9090	0.4030	0.81	0.37	1.00	0.32	0.12	1.18	0.55	1.77	0.41
EW (3x3x3)	0.97	0.0003	0.0238	1.1470	0.3180	0.47	0.49	1.83	0.18	0.09	1.37	0.50	1.85	0.40
MD (2x3)	0.96	-0.0002	0.0112	0.3390	0.7120	0.15	0.70	0.53	0.47	0.33	0.48	0.79	0.47	0.79
MD (3x3)	0.94	0.0000	0.0104	0.1120	0.8940	0.00	0.98	0.22	0.64	0.63	0.11	0.95	0.14	0.93
MD (3x3x3)	0.9	0.0002	0.0292	0.4640	0.6290	0.06	0.81	0.87	0.35	0.29	0.41	0.82	0.44	0.80
MV (2x3)	0.91	0.0003	0.0823	4.9410	0.0070	0.18	0.68	9.72	**0.00**	**0.00**	3.41	0.18	6.71	**0.04**
MV (3x3)	0.93	0.0003	0.0367	1.0420	0.3530	0.11	0.74	1.98	0.16	0.12	0.65	0.72	1.33	0.52
MV (3x3x3)	0.94	0.0004	0.0214	0.5020	0.6060	0.38	0.54	0.63	0.43	0.23	0.48	0.79	0.95	0.62
RP (2x3)	0.97	-0.0001	0.0157	0.6140	0.5420	0.02	0.90	1.21	0.27	0.24	0.70	0.71	0.73	0.70
RP (3x3)	0.97	0.0000	0.0092	0.1430	0.8670	0.01	0.94	0.28	0.60	0.56	0.13	0.94	0.19	0.91
RP (3x3x3)	0.97	0.0000	0.0070	0.0860	0.9180	0.00	0.99	0.17	0.68	0.68	0.08	0.96	0.10	0.95
Rebalancing: Annually														
EW (2x3)	0.98	0.0003	0.0169	1.0100	0.3650	0.80	0.37	1.23	0.27	0.10	1.38	0.50	1.88	0.39
EW (3x3)	0.97	0.0004	0.0187	0.9190	0.3990	0.81	0.37	1.02	0.31	0.12	1.19	0.55	1.78	0.41
EW (3x3x3)	0.97	0.0003	0.0239	1.1510	0.3170	0.45	0.50	1.85	0.17	0.09	1.37	0.51	1.84	0.40
MD (2x3)	0.96	-0.0001	0.0184	0.5830	0.5580	0.04	0.85	1.13	0.29	0.25	0.65	0.72	0.70	0.71
MD (3x3)	0.94	0.0001	0.0236	0.5000	0.6070	0.01	0.92	0.99	0.32	0.30	0.44	0.80	0.61	0.74
MD (3x3x3)	0.9	0.0006	0.0623	2.1080	0.1220	0.48	0.49	3.74	0.05	**0.03**	1.91	0.39	2.72	0.26
MV (2x3)	0.91	0.0001	0.0748	4.0220	0.0180	0.02	0.88	8.04	**0.01**	**0.00**	2.85	0.24	4.84	0.09
MV (3x3)	0.93	0.0000	0.0393	1.3450	0.2610	0.00	0.98	2.69	0.10	0.10	0.92	0.63	1.65	0.44
MV (3x3x3)	0.94	0.0005	-0.0116	0.4870	0.6150	0.48	0.49	0.49	0.48	0.24	0.82	0.67	0.76	0.69
RP (2x3)	0.97	0.0000	0.0144	0.4750	0.6220	0.00	0.99	0.95	0.33	0.33	0.50	0.78	0.55	0.76
RP (3x3)	0.97	0.0001	0.0088	0.1220	0.8850	0.02	0.90	0.23	0.63	0.57	0.11	0.95	0.16	0.93
RP (3x3x3)	0.97	0.0000	0.0107	0.1810	0.8340	0.01	0.95	0.36	0.55	0.52	0.18	0.92	0.21	0.90

Appendix 6

The table reports the results for the step-down regression-based mean-variance spanning test from Kan and Zhou (2012). We display the results when the benchmark assets are 30-Year US Treasury Bonds and the Strategic Beta on *dependent* equally weighted portfolios. The test assets have the same Strategic Beta on *independent* equally weighted portfolios. In total, there are four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk-parity (RP). These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. The number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The first column reports the correlation between Strategic Beta on both sorting methodologies. F1 tests the null hypothesis that additional test assets do not improve the *ex-post* tangency portfolio. F2 tests the null hypothesis that additional test assets do not improve the *ex-post* global-minimum-variance (GMV) portfolio. We also report the step-down joint-*p*, which tests whether the efficient frontier is improved when we add the test asset to the benchmark assets. W_a^e is the GMM Wald test when returns are assumed to have a multivariate elliptical distribution; the results are comparable to those for the step-down joint-*p*. W_a is the GMM Wald and is valid under all return distributions. The sample period ranges from July 1963 to December 2015.

						Step-Down Test					GMM Wald Test			
Strategy	Corr.	$\hat{\alpha}$	$\hat{\delta}$	F-test	*p*-value	F_1	*p*-value	F_2	*p*-value	Joint-*p*	W_a^e	*p*-value	W_a	*p*-value
						Rebalancing: Monthly								
EW (2x3)	0.99	0.0003	0.0340	4.3770	0.0130	0.76	0.39	8.00	**0.01**	**0.00**	5.06	0.08	7.67	**0.02**
EW (3x3)	0.98	0.0001	0.0335	2.7810	0.0630	0.06	0.81	5.51	**0.02**	**0.02**	3.21	0.20	3.89	0.14
EW (3x3x3)	0.98	0.0001	0.0363	2.8910	0.0560	0.04	0.85	5.75	**0.02**	**0.01**	3.35	0.19	3.88	0.14
MD (2x3)	0.97	-0.0003	0.0288	2.1720	0.1150	0.32	0.57	4.03	**0.05**	**0.03**	3.10	0.21	2.61	0.27
MD (3x3)	0.96	-0.0006	0.0259	1.6790	0.1870	0.88	0.35	2.48	0.12	**0.04**	2.69	0.26	2.40	0.30
MD (3x3x3)	0.93	-0.0010	0.0451	3.0070	0.0500	1.60	0.21	4.41	**0.04**	**0.01**	4.52	0.10	4.38	0.11
MV (2x3)	0.96	-0.0001	0.0280	1.1850	0.3060	0.05	0.83	2.33	0.13	0.11	1.62	0.45	1.28	0.53
MV (3x3)	0.96	0.0001	0.0044	0.0270	0.9730	0.03	0.86	0.02	0.88	0.76	0.03	0.99	0.06	0.97
MV (3x3x3)	0.96	0.0002	0.0150	0.3090	0.7340	0.09	0.76	0.53	0.47	0.36	0.22	0.89	0.37	0.83
RP (2x3)	0.99	0.0000	0.0266	2.5810	0.0770	0.01	0.93	5.16	**0.02**	**0.02**	3.17	0.21	3.36	0.19
RP (3x3)	0.98	-0.0001	0.0253	1.5920	0.2040	0.06	0.81	3.13	0.08	0.06	2.04	0.36	1.90	0.39
RP (3x3x3)	0.98	-0.0002	0.0213	1.2540	0.2860	0.15	0.70	2.36	0.13	0.09	1.58	0.45	1.53	0.47

					Rebalancing: Quarterly									
EW (2x3)	0.99	0.0003	0.0340	4.3770	0.0130	0.76	0.39	8.00	**0.01**	**0.00**	5.06	0.08	7.67	**0.02**
EW (3x3)	0.98	0.0001	0.0335	2.7810	0.0630	0.06	0.81	5.51	**0.02**	**0.02**	3.21	0.20	3.89	0.14
EW (3x3x3)	0.98	0.0001	0.0363	2.8910	0.0560	0.04	0.85	5.75	**0.02**	**0.01**	3.35	0.19	3.88	0.14
MD (2x3)	0.98	-0.0002	0.0325	2.7500	0.0650	0.24	0.63	5.27	**0.02**	**0.01**	3.94	0.14	3.30	0.19
MD (3x3)	0.96	-0.0005	0.0287	1.7300	0.1780	0.63	0.43	2.83	0.09	**0.04**	2.61	0.27	2.34	0.31
MD (3x3x3)	0.92	-0.0006	0.0516	2.6190	0.0740	0.59	0.44	4.66	**0.03**	**0.01**	3.32	0.19	3.28	0.19
MV (2x3)	0.97	-0.0001	0.0214	0.7520	0.4720	0.06	0.82	1.45	0.23	0.19	1.32	0.52	0.78	0.68
MV (3x3)	0.96	-0.0002	0.0158	0.4160	0.6600	0.10	0.76	0.74	0.39	0.30	0.50	0.78	0.58	0.75
MV (3x3x3)	0.96	-0.0001	0.0154	0.4480	0.6390	0.05	0.83	0.85	0.36	0.30	0.48	0.79	0.57	0.75
RP (2x3)	0.99	0.0000	0.0262	2.5830	0.0760	0.01	0.94	5.17	**0.02**	**0.02**	3.15	0.21	3.42	0.18
RP (3x3)	0.98	-0.0001	0.0252	1.6500	0.1930	0.08	0.78	3.23	0.07	0.06	2.13	0.35	2.00	0.37
RP (3x3x3)	0.98	-0.0002	0.0205	1.1980	0.3030	0.16	0.69	2.24	0.14	0.09	1.51	0.47	1.50	0.47
					Rebalancing: Semi-Annually									
EW (2x3)	0.99	0.0003	0.0340	4.3770	0.0130	0.76	0.39	8.00	**0.01**	**0.00**	5.06	0.08	7.67	**0.02**
EW (3x3)	0.98	0.0001	0.0335	2.7810	0.0630	0.06	0.81	5.51	**0.02**	**0.02**	3.21	0.20	3.89	0.14
EW (3x3x3)	0.98	0.0001	0.0363	2.8910	0.0560	0.04	0.85	5.75	**0.02**	**0.01**	3.35	0.19	3.88	0.14
MD (2x3)	0.98	-0.0003	0.0321	2.6780	0.0690	0.31	0.58	5.05	**0.03**	**0.01**	4.00	0.14	3.01	0.22
MD (3x3)	0.96	-0.0004	0.0306	1.8870	0.1520	0.55	0.46	3.23	0.07	**0.03**	2.75	0.25	2.38	0.31
MD (3x3x3)	0.93	-0.0010	0.0475	3.2010	0.0410	1.45	0.23	4.95	**0.03**	**0.01**	4.66	0.10	4.40	0.11
MV (2x3)	0.96	-0.0004	0.0435	2.8340	0.0600	0.31	0.58	5.36	**0.02**	**0.01**	6.39	**0.04**	2.94	0.23
MV (3x3)	0.96	-0.0005	-0.0037	0.2500	0.7790	0.50	0.48	0.00	0.97	0.46	0.47	0.79	0.61	0.74
MV (3x3x3)	0.96	0.0001	0.0080	0.0860	0.9180	0.01	0.93	0.16	0.69	0.64	0.07	0.96	0.09	0.95
RP (2x3)	0.99	0.0000	0.0259	2.6270	0.0730	0.01	0.93	5.25	**0.02**	**0.02**	3.28	0.19	3.27	0.20
RP (3x3)	0.98	-0.0002	0.0252	1.7380	0.1770	0.15	0.70	3.34	0.07	**0.05**	2.29	0.32	2.09	0.35
RP (3x3x3)	0.98	-0.0002	0.0197	1.1290	0.3240	0.17	0.69	2.10	0.15	0.10	1.43	0.49	1.40	0.50
					Rebalancing: Annually									
EW (2x3)	0.99	0.0003	0.0340	4.3770	0.0130	0.76	0.39	8.00	**0.01**	**0.00**	5.06	0.08	7.67	**0.02**
EW (3x3)	0.98	0.0001	0.0335	2.7810	0.0630	0.06	0.81	5.51	**0.02**	**0.02**	3.21	0.20	3.89	0.14
EW (3x3x3)	0.98	0.0001	0.0363	2.8910	0.0560	0.04	0.85	5.75	**0.02**	**0.01**	3.35	0.19	3.88	0.14
MD (2x3)	0.97	-0.0004	0.0379	3.8610	0.0220	0.69	0.41	7.04	**0.01**	**0.00**	5.50	0.06	4.38	0.11
MD (3x3)	0.96	-0.0008	0.0394	3.5040	0.0310	1.61	0.21	5.39	**0.02**	**0.00**	5.12	0.08	4.92	0.09
MD (3x3x3)	0.93	-0.0007	0.0674	4.6050	0.0100	0.80	0.37	8.41	**0.00**	**0.00**	6.04	**0.05**	5.40	0.07
MV (2x3)	0.97	-0.0003	0.0365	2.4020	0.0910	0.22	0.64	4.59	**0.03**	**0.02**	2.68	0.26	2.94	0.23
MV (3x3)	0.96	-0.0003	0.0098	0.3240	0.7230	0.24	0.62	0.41	0.53	0.33	0.52	0.77	0.57	0.75
MV (3x3x3)	0.97	0.0006	0.0188	0.8570	0.4250	1.12	0.29	0.60	0.44	0.13	1.11	0.57	1.25	0.54
RP (2x3)	0.99	0.0000	0.0290	3.1390	0.0440	0.00	0.95	6.28	**0.01**	**0.01**	3.69	0.16	3.75	0.15
RP (3x3)	0.98	-0.0002	0.0263	1.9340	0.1450	0.23	0.64	3.65	0.06	**0.04**	2.49	0.29	2.30	0.32
RP (3x3x3)	0.98	-0.0002	0.0244	1.7060	0.1820	0.25	0.62	3.17	0.08	**0.05**	2.16	0.34	2.05	0.36

Appendix 7

The table reports the diversification return decomposition from equation (19) for the four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk-parity (RP). These strategies are applied on **cap-weighted** portfolios sorted **independently** on characteristics. These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. Finally, the number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The last four columns refer to terms from equation (19). The sample period ranges from July 1963 to December 2015.

	Strategy			Constituents Average						
Strategy	Arithmetic Return	Geometric Return	Variance	Geometric Return	Variance	Variance Reduction	Covariance Drag	Adjustment for not being EW	Variance Reduction Benefit	Diversification Return
Independent Sorted Portfolios										
Rebalancing: Monthly										
EW (2x3)	1.084	0.970	0.230	0.947	0.275	0.045	0.000	0.000	0.023	0.0226
EW (3x3)	1.103	0.982	0.242	0.957	0.293	0.051	0.000	0.000	0.025	0.0253
EW (3x3x3)	1.125	0.999	0.250	0.960	0.328	0.078	0.000	0.000	0.039	0.0390
MD (2x3)	1.090	0.979	0.221	0.947	0.275	0.054	-0.005	0.011	0.027	0.0323
MD (3x3)	1.113	1.000	0.228	0.957	0.293	0.065	0.004	0.007	0.033	0.0428
MD (3x3x3)	1.090	0.972	0.236	0.960	0.328	0.093	-0.041	0.006	0.046	0.0120
MV (2x3)	1.167	1.062	0.210	0.947	0.275	0.065	0.013	0.069	0.033	0.1150
MV (3x3)	1.153	1.041	0.225	0.957	0.293	0.068	-0.001	0.051	0.034	0.0840
MV (3x3x3)	1.147	1.044	0.205	0.960	0.328	0.123	-0.018	0.040	0.061	0.0836
RP (2x3)	1.104	0.993	0.223	0.947	0.275	0.052	0.005	0.014	0.026	0.0457
RP (3x3)	1.122	1.005	0.235	0.957	0.293	0.058	0.005	0.014	0.029	0.0482
RP (3x3x3)	1.136	1.017	0.237	0.960	0.328	0.091	-0.001	0.012	0.046	0.0569

Rebalancing: Quarterly										
EW (2x3)	1.088	0.973	0.229	0.947	0.275	0.045	0.003	0.000	0.023	0.0262
EW (3x3)	1.107	0.986	0.242	0.957	0.293	0.051	0.004	0.000	0.025	0.0295
EW (3x3x3)	1.130	1.005	0.250	0.960	0.328	0.078	0.005	0.000	0.039	0.0446
MD (2x3)	1.099	0.988	0.221	0.947	0.275	0.053	0.004	0.011	0.027	0.0413
MD (3x3)	1.122	1.007	0.229	0.957	0.293	0.064	0.012	0.006	0.032	0.0506
MD (3x3x3)	1.112	0.993	0.239	0.960	0.328	0.090	-0.017	0.005	0.045	0.0324
MV (2x3)	1.244	1.139	0.210	0.947	0.275	0.064	0.087	0.073	0.032	0.1918
MV (3x3)	1.202	1.090	0.224	0.957	0.293	0.069	0.049	0.050	0.035	0.1332
MV (3x3x3)	1.158	1.053	0.210	0.960	0.328	0.119	-0.002	0.036	0.059	0.0927
RP (2x3)	1.110	0.999	0.223	0.947	0.275	0.052	0.011	0.015	0.026	0.0519
RP (3x3)	1.130	1.012	0.235	0.957	0.293	0.058	0.012	0.015	0.029	0.0554
RP (3x3x3)	1.141	1.022	0.238	0.960	0.328	0.090	0.004	0.013	0.045	0.0619
Rebalancing: Semi-Annually										
EW (2x3)	1.090	0.976	0.229	0.947	0.275	0.046	0.005	0.001	0.023	0.029
EW (3x3)	1.111	0.989	0.242	0.957	0.293	0.051	0.006	0.001	0.025	0.033
EW (3x3x3)	1.133	1.008	0.250	0.960	0.328	0.078	0.007	0.001	0.039	0.048
MD (2x3)	1.088	0.977	0.222	0.947	0.275	0.052	-0.009	0.012	0.026	0.030
MD (3x3)	1.106	0.990	0.232	0.957	0.293	0.061	-0.004	0.007	0.030	0.033
MD (3x3x3)	1.124	1.002	0.244	0.960	0.328	0.084	-0.004	0.004	0.042	0.041
MV (2x3)	1.201	1.095	0.213	0.947	0.275	0.062	0.047	0.070	0.031	0.148
MV (3x3)	1.230	1.117	0.227	0.957	0.293	0.066	0.081	0.046	0.033	0.160
MV (3x3x3)	1.134	1.024	0.220	0.960	0.328	0.109	-0.022	0.032	0.054	0.064
RP (2x3)	1.104	0.993	0.223	0.947	0.275	0.052	0.004	0.015	0.026	0.046
RP (3x3)	1.125	1.007	0.236	0.957	0.293	0.057	0.007	0.015	0.028	0.051
RP (3x3x3)	1.142	1.022	0.241	0.960	0.328	0.087	0.005	0.013	0.044	0.061
Rebalancing: Annually										
EW (2x3)	1.093	0.978	0.229	0.947	0.275	0.045	0.007	0.002	0.023	0.031
EW (3x3)	1.112	0.991	0.243	0.957	0.293	0.050	0.008	0.002	0.025	0.034
EW (3x3x3)	1.133	1.008	0.250	0.960	0.328	0.078	0.006	0.002	0.039	0.048
MD (2x3)	1.091	0.980	0.222	0.947	0.275	0.053	-0.008	0.014	0.026	0.033
MD (3x3)	1.109	0.994	0.230	0.957	0.293	0.063	-0.005	0.010	0.032	0.037
MD (3x3x3)	1.145	1.027	0.236	0.960	0.328	0.093	0.014	0.006	0.046	0.066
MV (2x3)	1.148	1.041	0.213	0.947	0.275	0.061	0.001	0.063	0.031	0.094
MV (3x3)	1.148	1.035	0.224	0.957	0.293	0.068	0.007	0.037	0.034	0.079
MV (3x3x3)	1.168	1.049	0.238	0.960	0.328	0.090	0.011	0.033	0.045	0.089
RP (2x3)	1.102	0.990	0.224	0.947	0.275	0.051	0.002	0.016	0.025	0.043
RP (3x3)	1.119	1.001	0.236	0.957	0.293	0.057	0.001	0.015	0.029	0.045
RP (3x3x3)	1.137	1.016	0.243	0.960	0.328	0.086	-0.001	0.014	0.043	0.056

Appendix 8

The table reports the diversification return decomposition from equation (19) for the four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk-parity (RP). These strategies are applied on **equal-weighted** portfolios sorted **independently** on characteristics. These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. Finally, the number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The last four columns refer to terms from equation (19). The sample period ranges from July 1963 to December 2015.

	Strategy			Constituents Average						
Strategy	Arithmetic Return	Geometric Return	Variance	Geometric Return	Variance	Variance Reduction	Covariance Drag	Adjustment for not being EW	Variance Reduction Benefit	Diversification Return
Independent Sorted Portfolios										
Rebalancing: Monthly										
EW (2x3)	1.209	1.068	0.283	1.044	0.330	0.047	0.000	0.000	0.024	0.024
EW (3x3)	1.192	1.054	0.276	1.026	0.331	0.054	0.000	0.000	0.027	0.027
EW (3x3x3)	1.208	1.068	0.280	1.027	0.362	0.082	0.000	0.000	0.041	0.041
MD (2x3)	1.255	1.119	0.272	1.044	0.330	0.058	0.001	0.044	0.029	0.075
MD (3x3)	1.251	1.119	0.266	1.026	0.331	0.065	0.009	0.051	0.033	0.092
MD (3x3x3)	1.251	1.114	0.273	1.027	0.362	0.089	-0.021	0.064	0.045	0.088
MV (2x3)	1.355	1.217	0.277	1.044	0.330	0.053	-0.066	0.212	0.027	0.172
MV (3x3)	1.369	1.230	0.278	1.026	0.331	0.052	0.004	0.174	0.026	0.204
MV (3x3x3)	1.255	1.138	0.233	1.027	0.362	0.129	-0.078	0.125	0.064	0.112
RP (2x3)	1.244	1.107	0.275	1.044	0.330	0.055	0.001	0.034	0.028	0.062
RP (3x3)	1.224	1.090	0.268	1.026	0.331	0.062	0.000	0.032	0.031	0.064
RP (3x3x3)	1.230	1.098	0.263	1.027	0.362	0.099	-0.007	0.029	0.050	0.072

					Rebalancing: Quarterly					
EW (2x3)	1.216	1.074	0.283	1.044	0.330	0.047	0.006	0.001	0.024	0.030
EW (3x3)	1.198	1.060	0.276	1.026	0.331	0.054	0.006	0.001	0.027	0.034
EW (3x3x3)	1.215	1.076	0.280	1.027	0.362	0.082	0.007	0.001	0.041	0.049
MD (2x3)	1.262	1.125	0.273	1.044	0.330	0.057	0.009	0.044	0.028	0.081
MD (3x3)	1.258	1.125	0.267	1.026	0.331	0.064	0.017	0.050	0.032	0.099
MD (3x3x3)	1.268	1.132	0.273	1.027	0.362	0.089	-0.003	0.064	0.045	0.105
MV (2x3)	1.393	1.253	0.280	1.044	0.330	0.050	-0.033	0.217	0.025	0.209
MV (3x3)	1.366	1.224	0.285	1.026	0.331	0.046	-0.003	0.178	0.023	0.197
MV (3x3x3)	1.242	1.123	0.238	1.027	0.362	0.125	-0.089	0.123	0.062	0.096
RP (2x3)	1.253	1.115	0.275	1.044	0.330	0.055	0.009	0.035	0.027	0.071
RP (3x3)	1.233	1.098	0.268	1.026	0.331	0.062	0.008	0.033	0.031	0.072
RP (3x3x3)	1.239	1.107	0.264	1.027	0.362	0.098	0.001	0.030	0.049	0.081
					Rebalancing: Semi-Annually					
EW (2x3)	1.221	1.079	0.282	1.044	0.330	0.047	0.009	0.003	0.024	0.035
EW (3x3)	1.203	1.065	0.276	1.026	0.331	0.054	0.009	0.003	0.027	0.038
EW (3x3x3)	1.219	1.080	0.279	1.027	0.362	0.083	0.009	0.003	0.041	0.053
MD (2x3)	1.262	1.125	0.273	1.044	0.330	0.057	0.005	0.048	0.028	0.081
MD (3x3)	1.256	1.123	0.266	1.026	0.331	0.065	0.008	0.056	0.032	0.097
MD (3x3x3)	1.290	1.151	0.277	1.027	0.362	0.085	0.015	0.068	0.042	0.125
MV (2x3)	1.389	1.252	0.275	1.044	0.330	0.055	-0.038	0.218	0.027	0.208
MV (3x3)	1.410	1.270	0.280	1.026	0.331	0.050	0.048	0.170	0.025	0.243
MV (3x3x3)	1.269	1.146	0.247	1.027	0.362	0.115	-0.059	0.121	0.058	0.119
RP (2x3)	1.256	1.119	0.275	1.044	0.330	0.055	0.010	0.037	0.027	0.075
RP (3x3)	1.237	1.103	0.268	1.026	0.331	0.062	0.011	0.035	0.031	0.077
RP (3x3x3)	1.247	1.113	0.266	1.027	0.362	0.096	0.007	0.032	0.048	0.087
					Rebalancing: Annually					
EW (2x3)	1.218	1.077	0.280	1.044	0.330	0.049	0.006	0.003	0.025	0.033
EW (3x3)	1.200	1.063	0.274	1.026	0.331	0.056	0.006	0.003	0.028	0.037
EW (3x3x3)	1.216	1.077	0.277	1.027	0.362	0.085	0.005	0.003	0.043	0.051
MD (2x3)	1.266	1.128	0.276	1.044	0.330	0.054	0.010	0.047	0.027	0.084
MD (3x3)	1.255	1.122	0.267	1.026	0.331	0.063	0.007	0.057	0.032	0.095
MD (3x3x3)	1.291	1.154	0.275	1.027	0.362	0.087	0.010	0.073	0.044	0.127
MV (2x3)	1.407	1.268	0.279	1.044	0.330	0.051	-0.021	0.220	0.026	0.224
MV (3x3)	1.403	1.262	0.282	1.026	0.331	0.049	0.039	0.173	0.024	0.236
MV (3x3x3)	1.373	1.240	0.265	1.027	0.362	0.097	0.020	0.145	0.048	0.213
RP (2x3)	1.259	1.121	0.276	1.044	0.330	0.054	0.011	0.039	0.027	0.077
RP (3x3)	1.242	1.107	0.269	1.026	0.331	0.062	0.013	0.037	0.031	0.081
RP (3x3x3)	1.249	1.115	0.269	1.027	0.362	0.093	0.008	0.034	0.047	0.088

Appendix 9

The table reports the diversification return decomposition from equation (19) for the four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk-parity (RP). These strategies are applied on **cap-weighted** portfolios sorted **dependently** on characteristics. These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. Finally, the number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The last four columns refer to terms from equation (19). The sample period ranges from July 1963 to December 2015.

	Strategy			Constituents Average						
Strategy	Arithmetic Return	Geometric Return	Variance	Geometric Return	Variance	Variance Reduction	Covariance Drag	Adjustment for not being EW	Variance Reduction Benefit	Diversification Return
Dependent Sorted Portfolios										
Rebalancing: Monthly										
EW (2x3)	1.092	0.967	0.250	0.935	0.314	0.064	0.000	0.000	0.032	0.032
EW (3x3)	1.120	0.982	0.276	0.946	0.347	0.071	0.000	0.000	0.036	0.036
EW (3x3x3)	1.160	1.015	0.289	0.960	0.400	0.111	0.000	0.000	0.055	0.055
MD (2x3)	1.181	1.062	0.238	0.935	0.314	0.076	-0.003	0.092	0.038	0.126
MD (3x3)	1.190	1.064	0.251	0.946	0.347	0.096	-0.032	0.101	0.048	0.118
MD (3x3x3)	1.223	1.084	0.278	0.960	0.400	0.122	-0.075	0.138	0.061	0.124
MV (2x3)	1.310	1.182	0.258	0.935	0.314	0.056	-0.025	0.243	0.028	0.246
MV (3x3)	1.263	1.135	0.256	0.946	0.347	0.091	-0.036	0.179	0.046	0.189
MV (3x3x3)	1.179	1.058	0.241	0.960	0.400	0.159	-0.115	0.134	0.080	0.098
RP (2x3)	1.148	1.028	0.241	0.935	0.314	0.073	0.005	0.051	0.036	0.092
RP (3x3)	1.168	1.036	0.264	0.946	0.347	0.083	0.000	0.048	0.042	0.090
RP (3x3x3)	1.194	1.062	0.264	0.960	0.400	0.135	-0.010	0.044	0.068	0.102

					Rebalancing: Quarterly					
EW (2x3)	1.100	0.975	0.250	0.935	0.314	0.064	0.006	0.001	0.032	0.040
EW (3x3)	1.130	0.992	0.276	0.946	0.347	0.071	0.009	0.001	0.035	0.045
EW (3x3x3)	1.173	1.028	0.289	0.960	0.400	0.111	0.012	0.001	0.055	0.068
MD (2x3)	1.190	1.070	0.240	0.935	0.314	0.074	0.006	0.092	0.037	0.135
MD (3x3)	1.205	1.077	0.255	0.946	0.347	0.092	-0.019	0.104	0.046	0.131
MD (3x3x3)	1.250	1.108	0.284	0.960	0.400	0.116	-0.048	0.138	0.058	0.148
MV (2x3)	1.347	1.213	0.266	0.935	0.314	0.048	0.010	0.244	0.024	0.278
MV (3x3)	1.279	1.150	0.259	0.946	0.347	0.088	-0.014	0.173	0.044	0.203
MV (3x3x3)	1.190	1.067	0.246	0.960	0.400	0.153	-0.098	0.129	0.077	0.107
RP (2x3)	1.161	1.039	0.243	0.935	0.314	0.071	0.017	0.052	0.035	0.104
RP (3x3)	1.184	1.051	0.266	0.946	0.347	0.082	0.015	0.049	0.041	0.105
RP (3x3x3)	1.207	1.073	0.268	0.960	0.400	0.132	0.002	0.046	0.066	0.113
					Rebalancing: Semi-Annually					
EW (2x3)	1.107	0.982	0.250	0.935	0.314	0.064	0.011	0.004	0.032	0.046
EW (3x3)	1.137	0.998	0.277	0.946	0.347	0.070	0.013	0.004	0.035	0.052
EW (3x3x3)	1.181	1.037	0.290	0.960	0.400	0.110	0.018	0.004	0.055	0.077
MD (2x3)	1.172	1.051	0.242	0.935	0.314	0.072	-0.015	0.094	0.036	0.116
MD (3x3)	1.190	1.061	0.260	0.946	0.347	0.088	-0.041	0.112	0.044	0.114
MD (3x3x3)	1.246	1.102	0.288	0.960	0.400	0.112	-0.053	0.139	0.056	0.142
MV (2x3)	1.363	1.227	0.271	0.935	0.314	0.043	0.044	0.226	0.021	0.292
MV (3x3)	1.336	1.204	0.263	0.946	0.347	0.084	0.042	0.174	0.042	0.258
MV (3x3x3)	1.188	1.062	0.253	0.960	0.400	0.146	-0.086	0.114	0.073	0.102
RP (2x3)	1.168	1.046	0.245	0.935	0.314	0.069	0.022	0.054	0.034	0.110
RP (3x3)	1.190	1.056	0.268	0.946	0.347	0.079	0.019	0.051	0.039	0.110
RP (3x3x3)	1.213	1.076	0.273	0.960	0.400	0.127	0.006	0.047	0.063	0.117
					Rebalancing: Annually					
EW (2x3)	1.109	0.983	0.252	0.935	0.314	0.062	0.012	0.004	0.031	0.048
EW (3x3)	1.136	0.998	0.278	0.946	0.347	0.069	0.012	0.005	0.035	0.051
EW (3x3x3)	1.175	1.030	0.290	0.960	0.400	0.110	0.011	0.005	0.055	0.071
MD (2x3)	1.170	1.047	0.245	0.935	0.314	0.069	-0.014	0.092	0.034	0.112
MD (3x3)	1.193	1.063	0.260	0.946	0.347	0.087	-0.045	0.118	0.043	0.116
MD (3x3x3)	1.248	1.101	0.295	0.960	0.400	0.105	-0.047	0.136	0.052	0.141
MV (2x3)	1.311	1.177	0.269	0.935	0.314	0.045	0.010	0.209	0.023	0.242
MV (3x3)	1.259	1.128	0.262	0.946	0.347	0.085	-0.018	0.157	0.043	0.182
MV (3x3x3)	1.184	1.049	0.271	0.960	0.400	0.129	-0.088	0.113	0.065	0.089
RP (2x3)	1.159	1.035	0.248	0.935	0.314	0.066	0.016	0.051	0.033	0.100
RP (3x3)	1.178	1.044	0.269	0.946	0.347	0.078	0.009	0.049	0.039	0.097
RP (3x3x3)	1.201	1.063	0.278	0.960	0.400	0.122	-0.003	0.045	0.061	0.103

Appendix 10

The table reports the diversification return decomposition from equation (19) for the four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk-parity (RP). These strategies are applied on **equal-weighted** portfolios sorted **dependently** on characteristics. These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. Finally, the number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The last four columns refer to terms from equation (19). The sample period ranges from July 1963 to December 2015.

	Strategy			Constituents Average						
Strategy	Arithmetic Return	Geometric Return	Variance	Geometric Return	Variance	Variance Reduction	Covariance Drag	Adjustment for not being EW	Variance Reduction Benefit	Diversification Return
Dependent Sorted Portfolios										
Rebalancing: Monthly										
EW (2x3)	1.262	1.095	0.333	1.070	0.384	0.051	0.000	0.000	0.026	0.026
EW (3x3)	1.274	1.109	0.331	1.076	0.396	0.066	0.000	0.000	0.033	0.033
EW (3x3x3)	1.299	1.130	0.338	1.077	0.443	0.105	0.000	0.000	0.052	0.052
MD (2x3)	1.378	1.221	0.312	1.070	0.384	0.072	-0.014	0.130	0.036	0.152
MD (3x3)	1.432	1.277	0.310	1.076	0.396	0.086	-0.034	0.192	0.043	0.201
MD (3x3x3)	1.540	1.373	0.335	1.077	0.443	0.108	-0.050	0.291	0.054	0.295
MV (2x3)	1.456	1.306	0.302	1.070	0.384	0.083	-0.104	0.298	0.041	0.236
MV (3x3)	1.440	1.289	0.303	1.076	0.396	0.094	-0.101	0.267	0.047	0.213
MV (3x3x3)	1.319	1.189	0.260	1.077	0.443	0.183	-0.155	0.175	0.092	0.111
RP (2x3)	1.318	1.160	0.317	1.070	0.384	0.067	-0.009	0.065	0.034	0.090
RP (3x3)	1.326	1.168	0.315	1.076	0.396	0.081	-0.017	0.069	0.041	0.092
RP (3x3x3)	1.332	1.179	0.306	1.077	0.443	0.137	-0.029	0.063	0.069	0.102

				Rebalancing: Quarterly						
EW (2x3)	1.271	1.104	0.333	1.070	0.384	0.051	0.007	0.002	0.026	0.034
EW (3x3)	1.285	1.120	0.331	1.076	0.396	0.065	0.009	0.002	0.033	0.044
EW (3x3x3)	1.314	1.145	0.338	1.077	0.443	0.105	0.014	0.002	0.052	0.068
MD (2x3)	1.381	1.224	0.314	1.070	0.384	0.070	-0.014	0.133	0.035	0.154
MD (3x3)	1.432	1.276	0.313	1.076	0.396	0.083	-0.036	0.194	0.042	0.200
MD (3x3x3)	1.544	1.371	0.346	1.077	0.443	0.097	-0.051	0.297	0.048	0.294
MV (2x3)	1.490	1.338	0.303	1.070	0.384	0.081	-0.079	0.307	0.041	0.268
MV (3x3)	1.468	1.315	0.305	1.076	0.396	0.091	-0.074	0.268	0.045	0.239
MV (3x3x3)	1.330	1.199	0.262	1.077	0.443	0.181	-0.146	0.177	0.090	0.122
RP (2x3)	1.330	1.171	0.318	1.070	0.384	0.067	0.000	0.068	0.033	0.101
RP (3x3)	1.340	1.182	0.316	1.076	0.396	0.080	-0.005	0.071	0.040	0.106
RP (3x3x3)	1.347	1.193	0.308	1.077	0.443	0.135	-0.016	0.065	0.067	0.116
				Rebalancing: Semi-Annually						
EW (2x3)	1.276	1.110	0.333	1.070	0.384	0.051	0.009	0.005	0.026	0.040
EW (3x3)	1.292	1.126	0.331	1.076	0.396	0.065	0.011	0.006	0.033	0.051
EW (3x3x3)	1.320	1.151	0.338	1.077	0.443	0.105	0.015	0.007	0.052	0.074
MD (2x3)	1.384	1.227	0.315	1.070	0.384	0.070	-0.016	0.138	0.035	0.157
MD (3x3)	1.427	1.269	0.316	1.076	0.396	0.080	-0.052	0.206	0.040	0.194
MD (3x3x3)	1.592	1.416	0.351	1.077	0.443	0.092	0.000	0.294	0.046	0.339
MV (2x3)	1.533	1.382	0.302	1.070	0.384	0.082	-0.038	0.309	0.041	0.312
MV (3x3)	1.527	1.376	0.302	1.076	0.396	0.094	-0.023	0.276	0.047	0.300
MV (3x3x3)	1.333	1.198	0.270	1.077	0.443	0.173	-0.138	0.172	0.086	0.121
RP (2x3)	1.339	1.180	0.318	1.070	0.384	0.066	0.006	0.071	0.033	0.111
RP (3x3)	1.351	1.193	0.317	1.076	0.396	0.079	0.002	0.076	0.040	0.117
RP (3x3x3)	1.357	1.201	0.313	1.077	0.443	0.130	-0.010	0.068	0.065	0.124
				Rebalancing: Annually						
EW (2x3)	1.276	1.110	0.331	1.070	0.384	0.053	0.007	0.006	0.027	0.040
EW (3x3)	1.288	1.124	0.329	1.076	0.396	0.068	0.007	0.007	0.034	0.048
EW (3x3x3)	1.312	1.145	0.335	1.077	0.443	0.108	0.006	0.007	0.054	0.068
MD (2x3)	1.416	1.255	0.321	1.070	0.384	0.063	0.009	0.145	0.031	0.185
MD (3x3)	1.485	1.323	0.325	1.076	0.396	0.071	-0.015	0.226	0.036	0.247
MD (3x3x3)	1.595	1.414	0.362	1.077	0.443	0.081	-0.006	0.303	0.040	0.337
MV (2x3)	1.537	1.383	0.308	1.070	0.384	0.076	-0.022	0.297	0.038	0.313
MV (3x3)	1.523	1.369	0.309	1.076	0.396	0.088	-0.018	0.267	0.044	0.293
MV (3x3x3)	1.405	1.254	0.301	1.077	0.443	0.142	-0.070	0.177	0.071	0.177
RP (2x3)	1.348	1.188	0.320	1.070	0.384	0.064	0.012	0.073	0.032	0.118
RP (3x3)	1.363	1.204	0.319	1.076	0.396	0.078	0.011	0.079	0.039	0.128
RP (3x3x3)	1.369	1.210	0.318	1.077	0.443	0.125	0.000	0.071	0.063	0.133

Tables

Table 1

List of the Smart Beta Strategies' Objective Functions

The table decomposes the Smart Beta strategies' objective function and the constraints applied on the constituents' weights. The first column refers to the common name of the strategy. The second column specifies the main authors who analyze the strategy. The third column reports the objective function for minimization or maximization, whereas the last column displays the unleveraged long-only constraint applied to the constituents' weight.

Strategy	Referenced Authors	Objective function	Constraints
Minimum Variance (MV)	Clarke, de Silva and Thorley (2013)	$\min f(w) = \sum_{i}^{N} \sum_{j}^{N} w_i \hat{\sigma}_{ij} w_j$	$w_i \in [0,1]$ $\sum_{i=1}^{N} w_i = 1$
Maximum Diversification (MD)	Choueifaty and Coignard (2008)	$max\ f(w) = \frac{\sum_i^N w_i\ \hat{\sigma}_i}{\sqrt{\sum_i^N \sum_j^N w_i \hat{\sigma}_{ij} w_j}}$	
Risk parity (RP)	Maillard, Roncalli, and Teiletche (2010)	$min\ f(w) = \sum_{i}^{N} \sum_{j}^{N} (w_i \times (\Sigma_p w_i) - w_j \times (\Sigma_p w_j))^2$	
Equally weighted (EW)	DeMiguel, Garlappi and Uppal (2009)	1/N	

Table 2

Strategic Beta and Transaction Costs

The table reports the annual transaction costs (in %) for the four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk parity (RP). These strategies are applied on portfolios sorted independently or dependently. These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. Finally, the number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). We also report the Patton and Timmermann (2010) test for decreasing monotonic relationships[17]. The sample period ranges from July 1963 to December 2015.

Rebalancing Frequency (in months)	1	3	6	12	Decreasing MR p-value	1	3	6	12	Decreasing MR p-value
	Independent Sort					Dependent Sort				
Panel A: Cap-weighted										
EW (2x3)	0.05	0.02	0.02	0.01	**0.00**	0.10	0.07	0.06	0.05	**0.02**
EW (3x3)	0.05	0.04	0.02	0.01	**0.00**	0.13	0.09	0.06	0.05	**0.00**
EW (3x3x3)	0.06	0.04	0.04	0.02	**0.00**	0.13	0.09	0.06	0.05	**0.00**
MD (2x3)	0.07	0.04	0.02	0.02	**0.00**	0.22	0.13	0.09	0.07	**0.00**
MD (3x3)	0.13	0.06	0.04	0.02	**0.00**	0.39	0.21	0.13	0.09	**0.00**
MD (3x3x3)	0.22	0.11	0.07	0.04	**0.00**	0.56	0.30	0.17	0.10	**0.00**
MV (2x3)	0.30	0.14	0.09	0.05	**0.00**	0.56	0.27	0.17	0.10	**0.00**
MV (3x3)	0.51	0.18	0.10	0.05	**0.00**	1.01	0.33	0.18	0.11	**0.00**
MV (3x3x3)	0.55	0.20	0.10	0.05	**0.00**	1.01	0.34	0.18	0.09	**0.00**
RP (2x3)	0.06	0.04	0.02	0.01	**0.00**	0.13	0.09	0.06	0.06	**0.00**
RP (3x3)	0.06	0.04	0.02	0.02	**0.00**	0.15	0.10	0.07	0.06	**0.00**
RP (3x3x3)	0.07	0.05	0.04	0.02	**0.00**	0.17	0.11	0.07	0.05	**0.00**
Panel B: Equal-weighted										
EW (2x3)	0.13	0.13	0.13	0.13	**0.00**	0.20	0.20	0.20	0.20	**0.00**
EW (3x3)	0.11	0.11	0.11	0.11	**0.00**	0.21	0.21	0.21	0.21	**0.00**
EW (3x3x3)	0.13	0.13	0.13	0.13	**0.00**	0.20	0.20	0.20	0.20	**0.00**
MD (2x3)	0.18	0.17	0.15	0.14	**0.00**	0.38	0.31	0.27	0.27	**0.01**
MD (3x3)	0.24	0.20	0.17	0.15	**0.00**	0.58	0.44	0.36	0.34	**0.00**
MD (3x3x3)	0.43	0.28	0.22	0.18	**0.00**	0.84	0.62	0.46	0.41	**0.00**
MV (2x3)	0.66	0.39	0.28	0.24	**0.00**	0.97	0.58	0.43	0.36	**0.00**
MV (3x3)	0.92	0.43	0.30	0.24	**0.00**	1.44	0.68	0.48	0.38	**0.00**
MV (3x3x3)	1.06	0.41	0.28	0.22	**0.00**	1.49	0.60	0.41	0.33	**0.00**
RP (2x3)	0.15	0.14	0.14	0.14	0.45	0.24	0.22	0.22	0.22	0.80
RP (3x3)	0.14	0.14	0.13	0.13	0.32	0.25	0.24	0.24	0.24	0.65
RP (3x3x3)	0.15	0.14	0.14	0.14	**0.00**	0.25	0.24	0.22	0.21	**0.00**

[17] Matlab code is made available on Prof. Patton's website.

Table 3

Traditional Step-Down Approach of Kan and Zhou (2012)

The table summarizes the results for the step-down regression-based mean-variance spanning test[18] from Kan and Zhou (2012). We display the results for the four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk parity (RP). The Strategic Beta strategies are applied on portfolios sorted independently or dependently. These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. The number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). In total, each Strategic Beta can be constructed in 12 manners (denominator). H_0^1 tests the null hypothesis that additional assets (Portfolio B) do not improve the *ex-post* tangency portfolio. H_0^2 tests the null hypothesis that additional assets (Portfolio B) do not improve the *ex-post* global-minimum-variance (GMV) portfolio. We report the frequency at which H_0^1 and H_0^2 are rejected with a confidence interval of 95%. The step-down joint-*p* tests whether the efficient frontier is improved when Portfolio B is added to the benchmark assets. The sample period ranges from July 1963 to December 2015.

	H_0^1: Tangency portfolio	H_0^2: GMV portfolio	Step-down Joint-*p*	H_0^1: Tangency portfolio	H_0^2: GMV portfolio	Step-down Joint-*p*
	Panel A: Bench.=(US+Independent) Portfolio B = Dependent			Panel B: Bench.=(US+Dependent) Portfolio B = Independent		
	Cap-Weighted Portfolios					
EW	0/12	0/12	4/12	0/12	0/12	0/12
RP	0/12	11/12	12/12	0/12	0/12	0/12
MD	5/12	12/12	12/12	0/12	0/12	1/12
MV	6/12	12/12	12/12	0/12	4/12	6/12
	Equal-Weighted Portfolios					
EW	0/12	0/12	0/12	0/12	12/12	12/12
RP	0/12	0/12	7/12	0/12	4/12	7/12
MD	9/12	5/12	11/12	0/12	9/12	12/12
MV	5/12	9/12	12/12	0/12	2/12	2/12

[18] Matlab code is made available on Prof. Zhou's website.

Table 4

GMM Approach of Kan and Zhou (2012)

The table summarizes the results for the GMM regression-based mean-variance spanning test from Kan and Zhou (2012). We display the results for the four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk parity (RP). The Strategic Beta strategies are applied on portfolios sorted independently or dependently. These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. The number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). In total, each Strategic Beta can be constructed in 12 (denominator) ways. The step-down joint-*p* tests whether the efficient frontier is improved when we add Portfolio B to the benchmark assets. W_a^e is the GMM Wald test when returns are assumed to have a multivariate ellicptical distribution. W_a is the GMM Wald and is valid under all return distributions. We report the frequency at which the tests are rejected with a confidence interval of 95%. The sample period ranges from July 1963 to December 2015.

	Step-down Joint-*p*	W_a	W_a^e	Step-down Joint-*p*	W_a	W_a^e
	Panel A: Bench.=(US+Independent) Portfolio B = Dependent			Panel B; Bench.=(US+Dependent) Portfolio B = Independent		
	Cap-Weighted Portfolios					
EW	4/12	0/12	0/12	0/12	0/12	0/12
RP	12/12	9/12	0/12	0/12	0/12	0/12
MD	12/12	12/12	11/12	1/12	0/12	0/12
MV	12/12	12/12	5/12	6/12	1/12	0/12
	Equally Weighted Portfolios					
EW	0/12	0/12	0/12	12/12	4/12	0/12
RP	7/12	0/12	0/12	7/12	0/12	0/12
MD	11/12	8/12	8/12	12/12	0/12	1/12
MV	12/12	8/12	5/12	2/12	0/12	1/12

Table 5

Morningstar® Strategic Beta Classification

The table identifies the Strategic Beta classifications provided by the data provider Morningstar®. The first column report whether the categorization of Strategic Beta is applicable to an ETF. The second column identifies the specific attribute of the ETF strategy, and the last column specifies the broader category used in this paper to categorize the ETF universe. There are four categories: (1) risk-weighted, (2) return-oriented, (3) other, and (4) blended. We also mention in parentheses the number of ETFs that fall under the broader category groups.

Strategic Beta	Strategic Beta Attributes	Strategic Beta Attribute Group
YES	Dividend Screened/Weighted Value Growth Fundamentally Weighted Multifactor Momentum Buyback/Shareholder Yield Earnings-Weighted Quality Expected Returns Size Revenue-Weighted	Return-Oriented (287)
	Minimum Variance Low/High Beta Risk-Weighted	Risk-Oriented (20)
	Non-Traditional Commodity Equal-Weighted Non-Traditional Fixed-Income Multi-asset	Other (82)
NO	Not Applicable	Blended (1013)

Table 6

Correlation between Characteristic-Sorted Portfolios

The table reports the average correlation (in %) for the characteristic-sorted portfolios constructed using independent and dependent sorting methodologies. The third column specifies the difference in the average correlation between the independent and dependent sorting. Correlations are estimated based on daily returns, and the sample period ranges from 01/07/1963 to 31/12/2015.

	Independent Sorting (1)	Dependent Sorting (2)	Difference (1)-(2)
#Number of portfolios			
		Panel A: Cap-weighted Portfolios	
2x3	84.99	78.00	6.99
3x3	84.99	75.81	9.18
3x3x3	78.38	66.80	11.58
		Panel B: Equal-weighted Portfolios	
2x3	87.13	82.64	4.49
3x3	85.62	78.01	7.61
3x3x3	78.63	69.12	9.51

Table 7

Source of Variance of Equally Weighted Portfolios

The table presents the impact on equally weighted portfolios' variance according to the number of stocks that comprise the portfolios. We illustrate the weights assigned based on the average variance of assets with a stock universe comprising 100 stocks. The results of the Fama and French (1993) methodology are presented based on Figure 2.

	Panel A: Independent Sort			Panel B: Dependent Sort		
Portfolios (2x3)	n stocks	Weights on $\overline{var}$	Weights on $\overline{covar}$	n stocks	Weights on $\overline{var}$	Weights on $\overline{covar}$
LL (Small Growth)	24	4.00%	96.00%	16	6.25%	93.75%
LM (Small Neutral)	26	4.00%	96.00%	16	6.25%	93.75%
LH (Small Value)	28	4.00%	96.00%	16	6.25%	93.75%
HL (Large Growth)	10	10.00%	90.00%	16	6.25%	93.75%
HM (Large Neutral)	8	13.00%	87.00%	16	6.25%	93.75%
HH (Large Value)	4	25.00%	75.00%	16	6.25%	93.75%

Table 8

Spread in Risk-adjusted Diversification Returns

The table reports the spread of diversification return from equation (21) for the four different Strategic Beta strategies: equal-weighted (EW), maximum diversification (MD), minimum variance (MV), and risk parity (RP). These strategies are applied on portfolios sorted independently or dependently. These portfolios can be rebalanced on a monthly (1), quarterly (3), semi-annually (6) or annual (12) basis. Finally, the number of portfolios is either six (2x3), nine (3x3) or twenty-seven (3x3x3). The sample period ranges from July 1963 to December 2015.

Rebalancing →	Monthly	Quarterly	Semi-Annually	Annually
↓ Strategy	Panel A: Cap-Weighted Portfolios			
EW (2x3)	0.8%	1.2%	1.5%	1.4%
EW (3x3)	0.8%	1.3%	1.6%	1.4%
EW (3x3x3)	1.3%	1.9%	2.3%	1.8%
MD (2x3)	8.9%	8.8%	8.1%	7.3%
MD (3x3)	7.0%	7.3%	7.5%	7.2%
MD (3x3x3)	10.2%	10.4%	9.0%	6.0%
MV (2x3)	10.7%	5.5%	11.1%	12.1%
MV (3x3)	9.3%	5.6%	7.9%	9.0%
MV (3x3x3)	0.7%	0.6%	3.1%	-0.5%
RP (2x3)	4.3%	4.7%	6.0%	5.2%
RP (3x3)	3.6%	4.3%	5.2%	4.7%
RP (3x3x3)	4.0%	4.5%	4.8%	4.0%
	Panel B: Equal-Weighted Portfolios			
EW (2x3)	0.0%	0.2%	0.2%	0.4%
EW (3x3)	0.3%	0.7%	0.8%	0.7%
EW (3x3x3)	0.7%	1.3%	1.4%	1.1%
MD (2x3)	6.7%	6.3%	6.5%	8.8%
MD (3x3)	9.4%	8.6%	8.1%	12.9%
MD (3x3x3)	17.9%	15.6%	17.7%	16.7%
MV (2x3)	5.4%	4.9%	9.0%	7.4%
MV (3x3)	0.0%	3.4%	4.6%	4.4%
MV (3x3x3)	-0.6%	2.0%	-0.4%	-4.7%
RP (2x3)	2.1%	2.3%	2.8%	3.2%
RP (3x3)	2.1%	2.6%	3.1%	3.7%
RP (3x3x3)	2.3%	2.7%	2.7%	3.4%

Figures

Figure 1

Stock Distribution with Independent vs Dependent Sorting

The figure displays the stock distribution into the 2x3 characteristic-sorted portfolios on size (low and high) and book-to-market equity ratio (low, medium and high) for the independent (left) and dependent (right) sorting methodologies. The independent sorting uses the NYSE as a reference for breakpoints, while the dependent sorting uses all name breakpoints (NYSE, NASDAQ, and AMEX). The period ranges from July 1963 to December 2015.

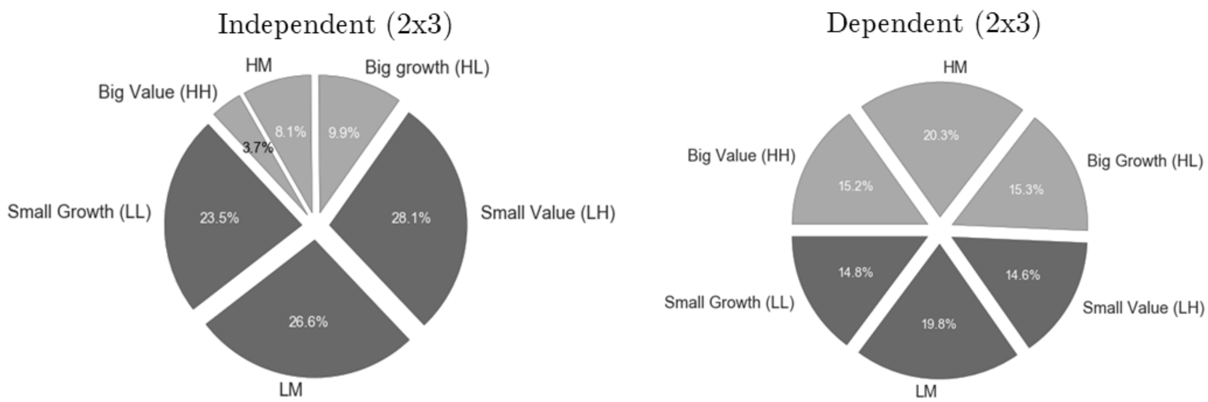


Figure 2

Characteristic-Sorted Portfolios Risk/Return tradeoff

The figure displays the panels of opportunity sets made of the investment style portfolios based on the sorting methodology. The *x*-axis reports the annualized standard deviation (in %), and the *y*-axis reports the annualized average return (in %). Portfolios constructed according to the independent and dependent sorts are displayed in red and blue, respectively. Graphs on the left (right) present the results for cap-weighted (equally weighted) portfolios. We display the opportunity set when the US stock universe is split into six (2x3), nine (3x3) and twenty-seven (3x3x3) groups based on the size and value for the first two splits and the size, value and momentum characteristics of a firm for the triple sort (3x3x3). For the sake of clarity, we only display the portfolio names for the double sorts (2x3 and 3x3). The first letter specifies the size, and the second letter refers to the value characteristic. L, M, and H refer to "Low", "Medium", and "High", respectively.

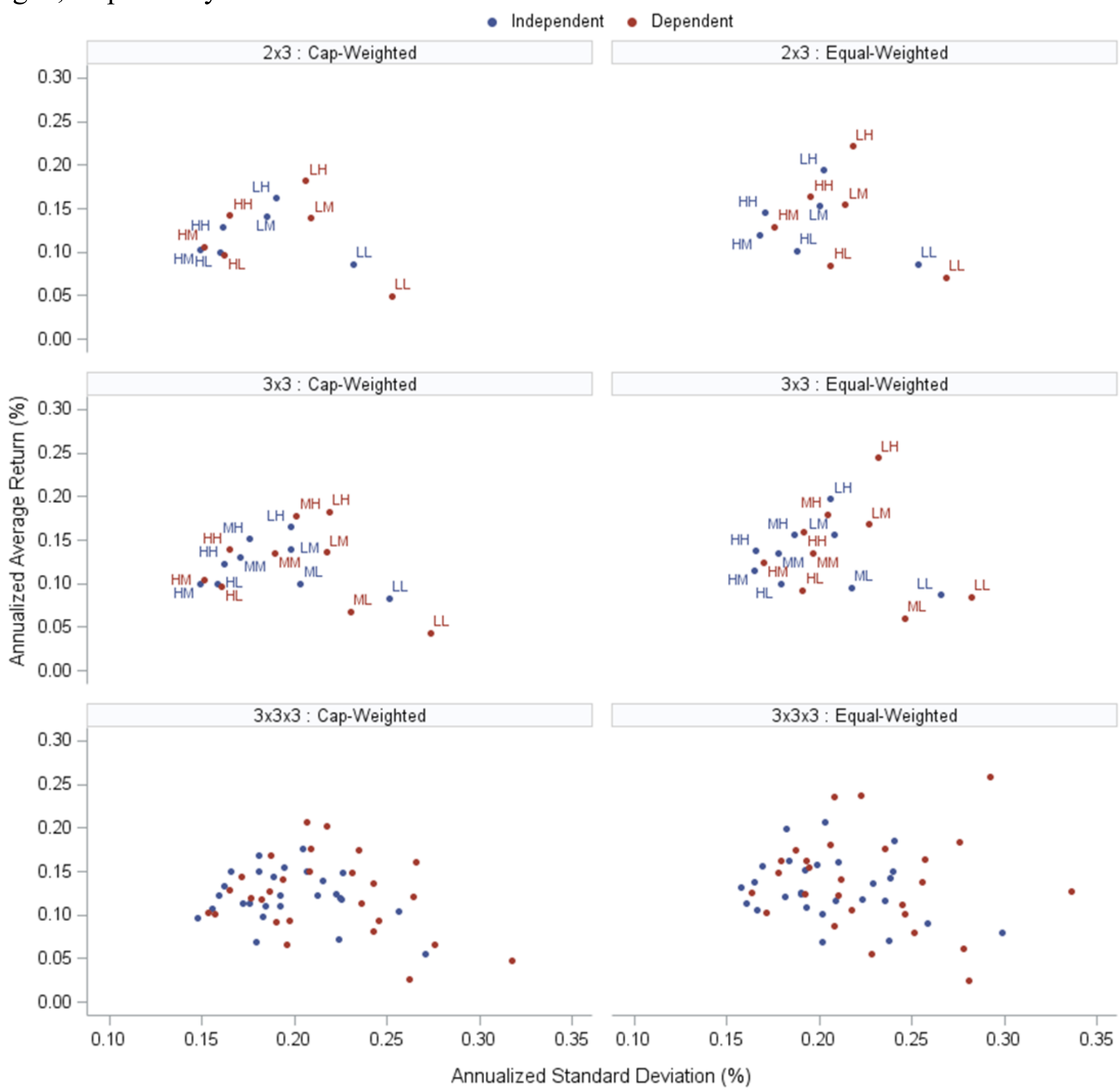


Figure 3

Variation of Transaction Cost Estimates Following Hasbrouck (2009)

The figure presents a boxplot of the distribution of individual stocks transaction costs estimated as in Hasbrouck (2009). The sample period ranges from 1963 to 2015. The whiskers represent the distribution of the 5th to 95th percentile, and the upper and lower edges of the boxes correspond to the 25th and 75th percentiles. The gray dots represent outliers.

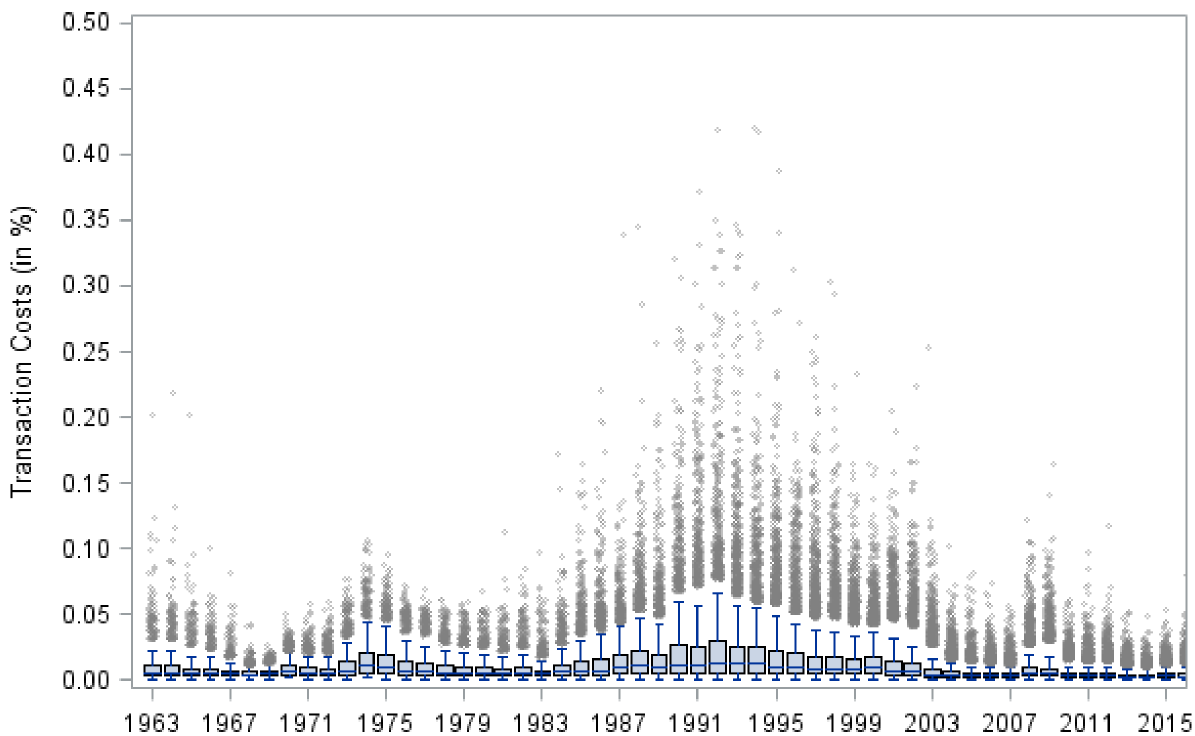
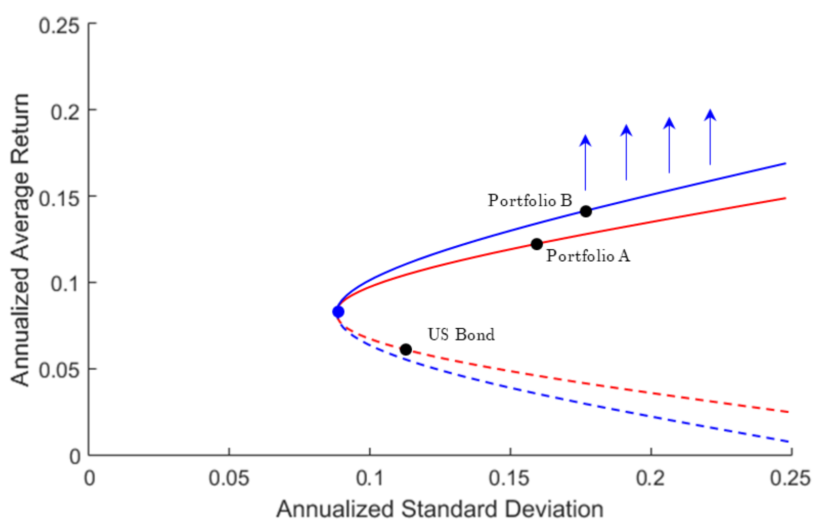


Figure 4

Improving the Tangency (Panel A) and GMV (Panel B) Portfolios

The figure displays the spanning illustration for opportunity sets made of a benchmark asset, i.e., the 30-Year US Treasury Bond and Portfolio A, and a test asset, i.e. the benchmark assets plus Portfolio B. The *x*-axis reports the annualized standard deviation (in %), and the *y*-axis reports the annualized average return (in %). This example is fictitious but illustrates in Panel A (Panel B) an improvement of the tangency (GMV) portfolio after adding Portfolio B to the benchmark assets.

Panel A: Tangency Portfolio



Panel B: GMV Portfolio

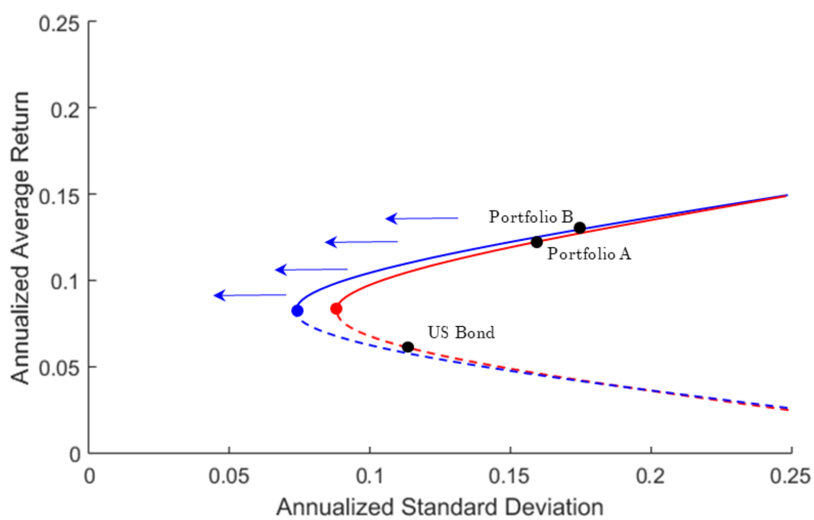


Figure 5

Improvement in the tangency portfolio for ETFs listed on US Exchanges

The panels show the percentage of ETFs listed on US exchanges for which there is a significant improvement in the tangency portfolio (H_0^1) after adding risk-based strategies built upon *independent* (red) and *dependent* (blue) portfolios. The results shown on the left are for cap-weighted portfolios, and those on the right are for equally weighted portfolios.

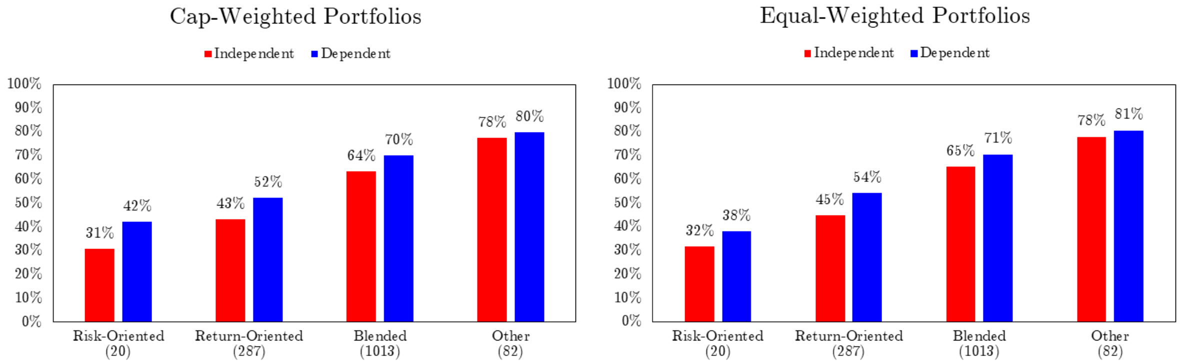


Figure 6

Improvement in the tangency portfolio for ETFs listed on US Exchanges and a Category Group named U.S. Equity

The panels show the percentage of ETFs listed on US exchanges and primarily investing in US equities only for which there is significant improvement of the tangency portfolio (H_0^1) after adding risk-based strategies built upon *independent* (red) and *dependent* (blue) portfolios. The results shown on the left are for cap-weighted portfolios, and those on the right are for equally weighted portfolios.

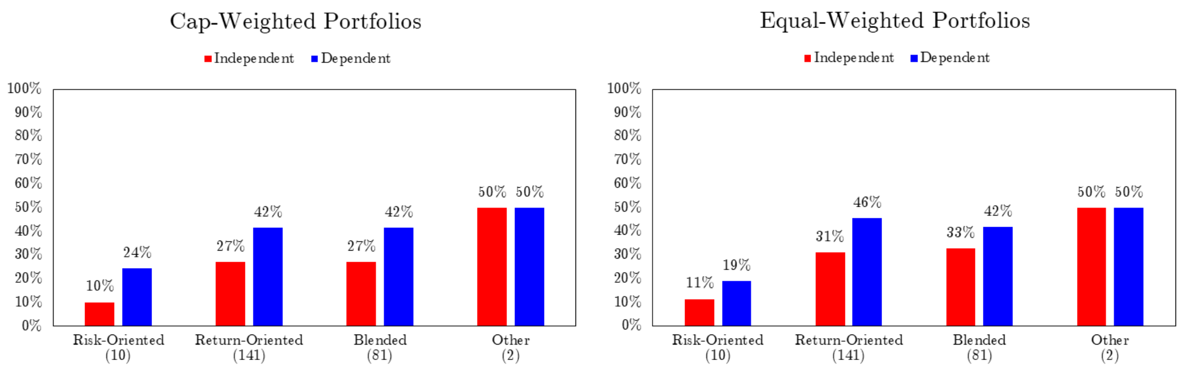


Figure 7

Stock Distribution with Independent vs Dependent Sorting

These plots show the stock distribution among the 3x3 characteristic-sorted portfolios on size (low, medium and high) the book-to-market equity ratio (low, medium and high) for the independent and dependent sorting methodologies. We also report the average percentage of stock repartition among the 3x3x3 characteristic-sorted portfolios when momentum is added as a third variable. For clarity, we group the 27 portfolios according to their size classifications (small, medium, and big). The period ranges from July 1963 to December 2015.

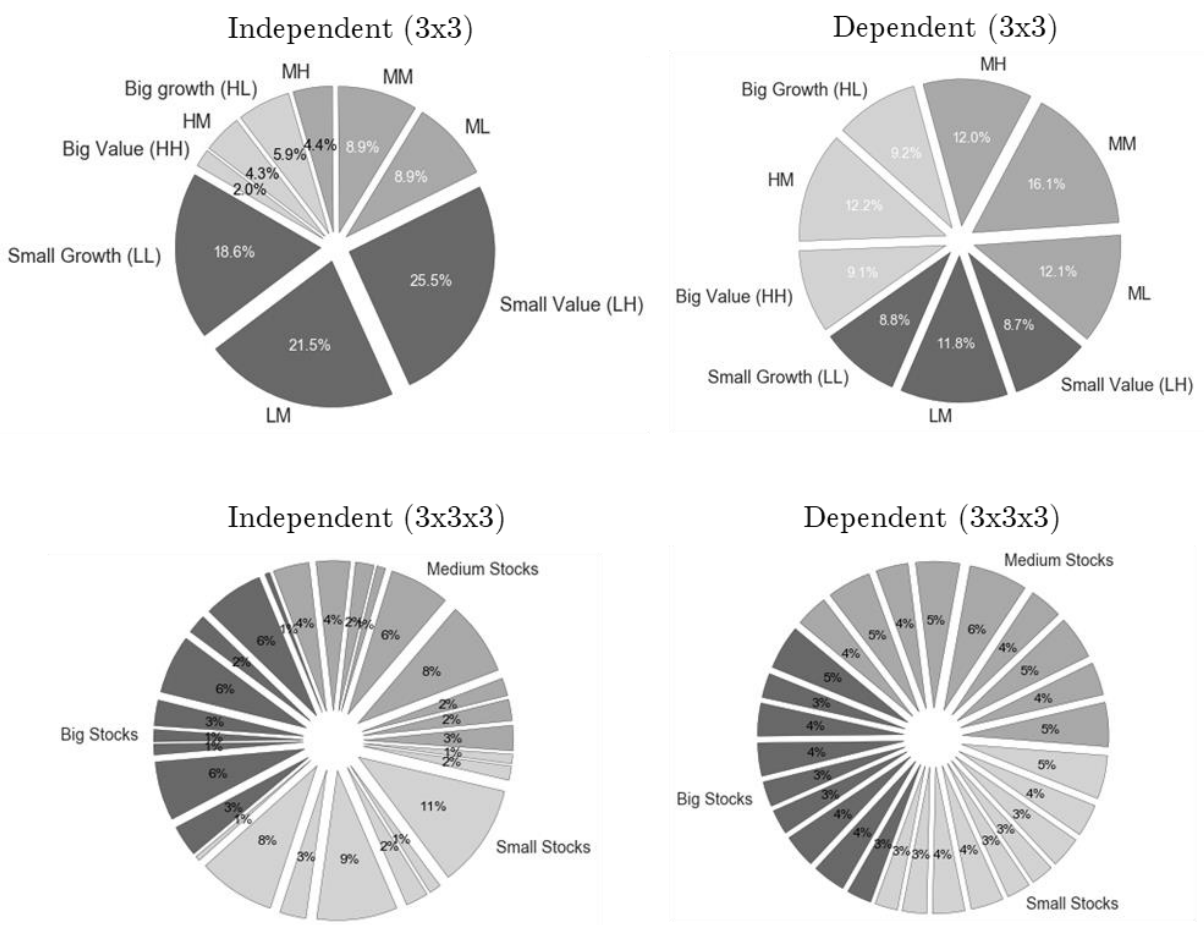


Figure 8

Yearly Average Stock Returns Correlation after Portfolio Formation

The figure shows that the average correlation in stocks returns after portfolio formation under an *independent* sort (red line) and a *dependent* sort (blue line). The post-portfolio-formation period is comprised of 252 days and starts from July to end of June *t*+1. We also represent by the shaded areas the 25-75th percentile distribution of the yearly stock correlations. The results are displayed for the 2x3 portfolios from 1963 to 2015.

Figure 9

Stock Distribution in Portfolios Sorted Independently According to the Listed Exchange

The figure reports the repartitions of stocks belonging to the NYSE, NASDAQ and AMEX in green, red and blue, respectively. The results for the 2x3 portfolios sorted independently are shown.

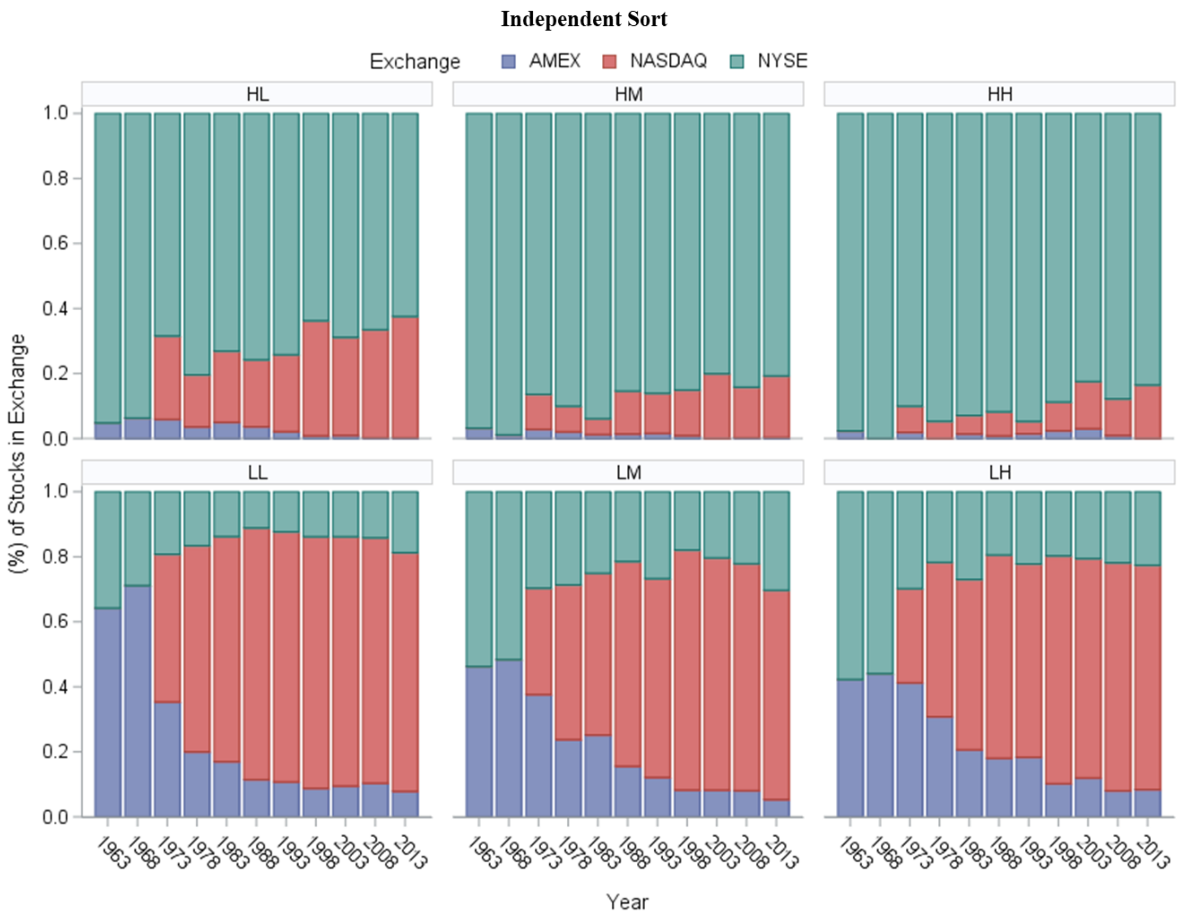


Figure 10

Stock Distribution in Portfolios Sorted Dependently According to the Listed Exchange

The figure reports the repartitions of stocks belonging to the NYSE, NASDAQ and AMEX in green, red and blue, respectively. The results for the 2x3 portfolios sorted dependently are shown.

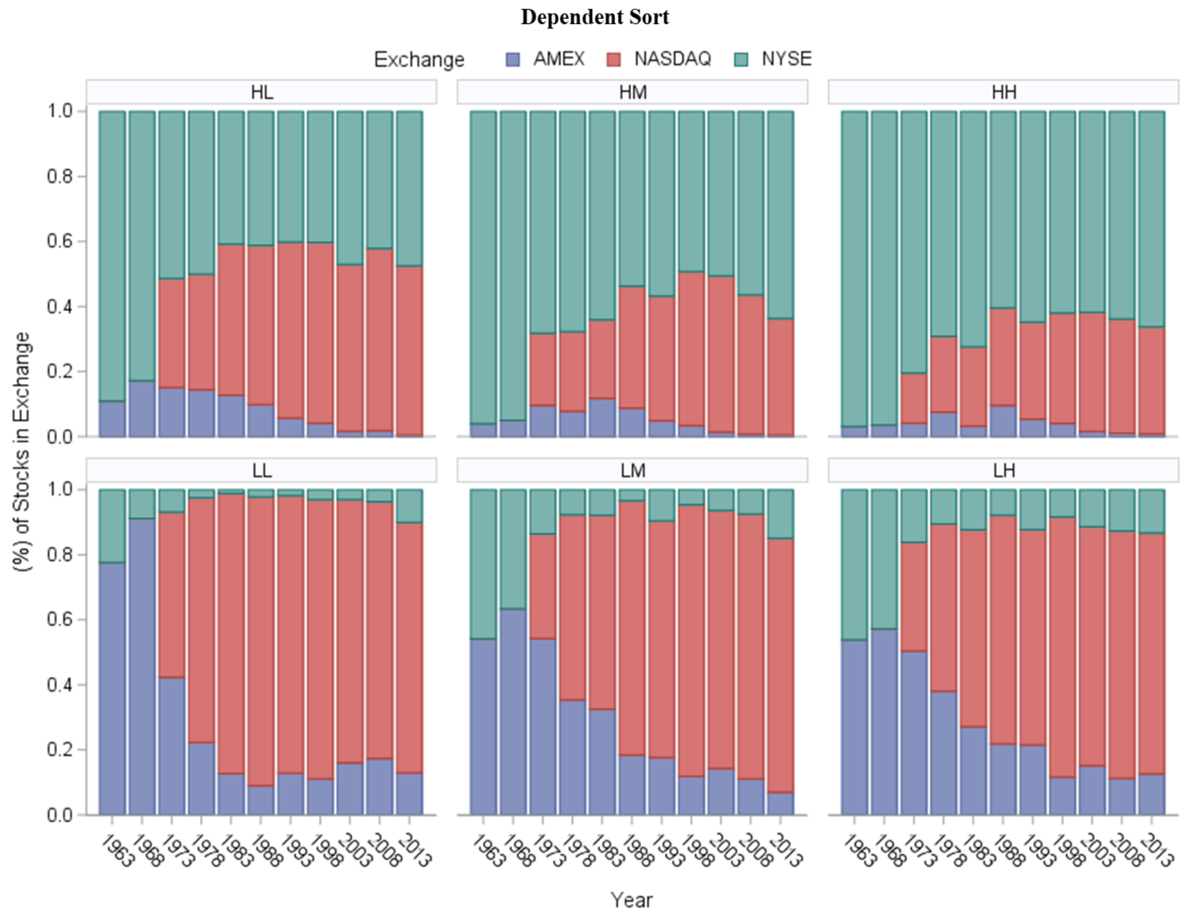


Figure 11

Yearly Average Stock Returns Correlation after Portfolio Formation: US Exchanges

The figure shows the average correlation in stock returns after portfolio formation according to the three main US exchanges, that is, the NYSE, NASDAQ, and AMEX. The post-portfolio-formation period comprises 252 days ranging from July *t* to the end of June *t*+1. We also represent with the shaded areas the 25-75th percentile distribution of the yearly stock correlations. The results for the period ranging from 1963 to 2015 are shown. The results for stocks listed on the NASDAQ only start in 1973.

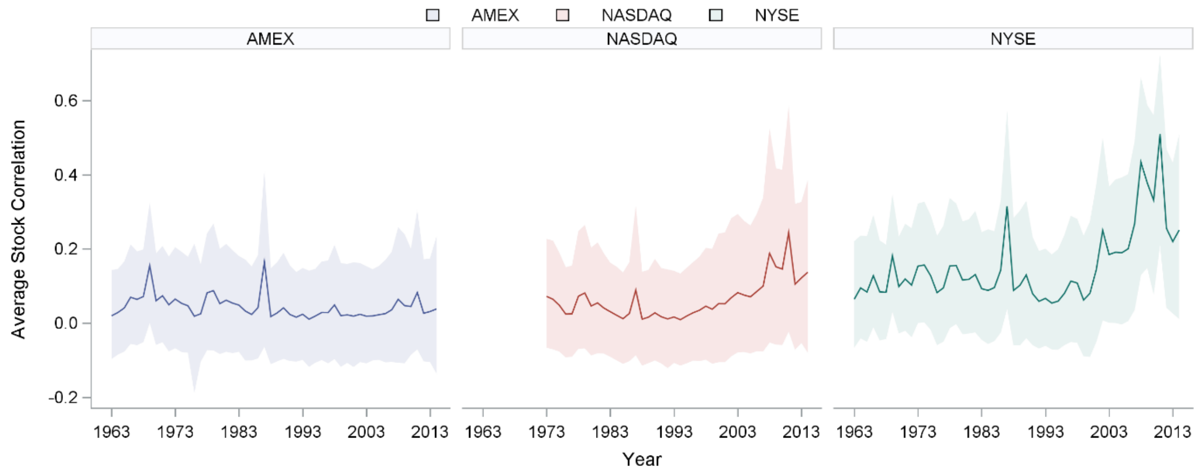


Figure 12

Risk-Adjusted Spread in Diversification Return: Cap-Weighted Portfolios

The figure reports the risk-adjusted spread of diversification return from equation (20). Depicted by the blue columns, the risk-adjusted diversification return is the sum of the covariance drag (red), adjustment for not following equal-weighted rebalancing scheme in each period *t* (light blue), and the variance reduction benefits (light gray) of holding a mix of the constituents rather than individual equities. We report the results of four rebalancing schemes, i.e., monthly, quarterly, semi-annually, and annually. The risk-based strategies are equal weighted (EW), maximum diversification (MD), minimum variance (MV), and risk parity (RP). The results are for cap-weighted portfolios, and the sample period ranges from July 1963 to December 2015.

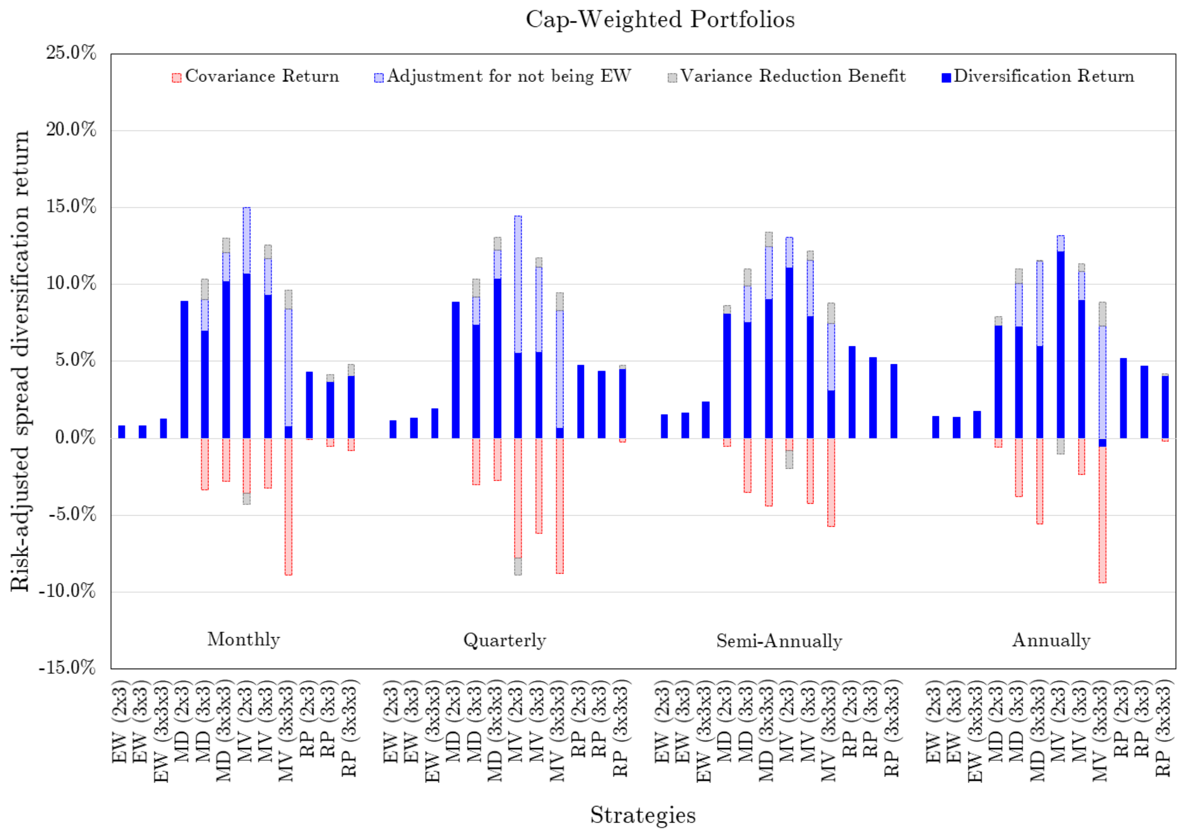


Figure 13

Risk-Adjusted Spread in Diversification Return: Equally Weighted Portfolios

The figure reports the risk-adjusted diversification return from equation (20). Depicted by the blue columns, the risk-adjusted diversification return is the sum of the covariance drag (red), an adjustment for not following equal-weighted rebalancing scheme in each period *t* (light blue), and the variance reduction benefits (light gray) of holding a mix of the constituents rather than individual equities. We report the results of four rebalancing schemes, i.e., monthly, quarterly, semi-annually, and annually. The risk-based strategies are equal weighted (EW), maximum diversification (MD), minimum variance (MV), and risk parity (RP). The results are for equally weighted portfolios, and the sample period ranges from July 1963 to December 2015.