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Diametral dimensions and some applications to spaces S^ν

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Abstract

The "classic" diametral dimension is a topological invariant which characterizes Schwartz and nuclear locally convex spaces. Besides, there exists a second diametral dimension which is conjectured to be equal to the first one (on Fréchet-Schwartz spaces).

The first part of this thesis is dedicated to the study of this conjecture. We present several positive partial results in metrizable spaces (in particular in Köthe sequence spaces and Hilbertizable spaces) and some properties which provide the equality of the two diametral dimensions (such as the Δ-stability, the existence of prominent bounded sets, and the property $(\overline{\Omega})$). Then, we describe the construction of some non-metrizable locally convex spaces for which the two diametral dimensions are different.

The other purpose of this work is to pursue the topological study of sequence spaces S^ν, originally defined in the context of multifractal analysis. For this, the second part of the present thesis focuses on the study of the two diametral dimensions in spaces S^ν. Finally, we show that some classes of spaces S^ν verify (a variation of) the property $(\overline{\Omega})$.

Résumé

La dimension diamétrale “classique” est un invariant topologique capable de caractériser les espaces localement convexes de Schwartz et nucléaires. En outre, il existe une deuxième dimension diamétrale conjecturée comme étant égale à la première (au niveau des espaces de Fréchet-Schwartz).

La première partie de cette thèse est consacrée à l'étude de cette conjecture. Nous y présentons plusieurs résultats positifs partiels dans les espaces métrisables (notamment au niveau des espaces de suites de Köthe et des espaces hilbertisables) ainsi que plusieurs propriétés permettant d'avoir l'égalité des deux dimensions diamétrales (comme la Δ-stabilité, l'existence de bornés proéminents et la propriété $(\overline{\Omega})$). Ensuite, nous décrivons la construction d'espaces localement convexes non métrisables pour lesquels les deux dimensions diamétrales sont distinctes.

Le second objectif de ce travail est de poursuivre l'étude topologique des espaces de suites S^ν, à l'origine définis dans le cadre de l'analyse multifractale. Pour ce faire, la deuxième partie de la présente thèse se focalise sur l'étude des deux dimensions diamétrales au niveau des espaces S^ν. Enfin, nous montrons que certaines classes d'espaces S^ν vérifient (une variation de) la propriété $(\overline{\Omega})$.

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Contents

Introduction

Multifractal analysis is a branch of mathematics which aims to determine the *regularity* of a *signal*. Among the notions defined to quantify the regularity of a given signal, one can particularly mention the so-called *Hölder exponent*, which leads to the concept of *spectrum of singularities* (see for instance [5, 16, 17]).

Actually, it appeared that the asymptotic behaviour of the wavelet coefficients of a signal (in any wavelet basis) can be used to estimate its spectrum of singularities ([18]). Translating this property in terms of functional spaces, Jaffard introduced in 2004 the *sequence spaces* S^ν ([17]).

Defined in the context of multifractal analysis, these spaces S^ν appeared to be functional spaces: more precisely, they can be endowed with a natural vector metric ([4]), which leads several mathematicians to study them from a functional analysis point of view (see for example [1, 2, 3, 4, 16]). They pointed out different properties, such as the facts that spaces S^ν are complete, separable, locally p-convex in some cases and only locally pseudoconvex in others, Schwartz, and non-nuclear.

To determine the relative "position" of spaces S^ν between Schwartz and nuclear spaces, Aubry and Bastin studied the notion of *diametral dimension*. This tool – denoted by Δ – is in fact a *topological invariant* on the class of topological vector spaces, which can be used to characterize Schwartz and nuclear locally convex spaces ([7, 11, 15, 19, 25, 26, 30, 38, 40]). Moreover, the diametral dimension has been deeply studied for *Köthe sequence spaces* ([11, 14, 19, 27, 30, 31, 32, 33]).

Actually, Aubry and Bastin obtained a formula for the diametral dimension of *locally p-convex spaces* S^ν, which is the same for all these spaces ([2]). Then, two questions appear:

- we can wonder whether the spaces S^ν are isomorphic (when they share the same index of p-convexity) and, for this, we could study some other topological invariants in the context of spaces S^ν;
- this formula is valid for locally p-convex spaces, but we can wonder whether it remains true or not when locally *pseudoconvex* spaces S^ν are concerned.

In another context, Mityagin defined in [25] a second diametral dimension, denoted by Δ_b, using bounded sets in its definition, contrary to Δ. Besides, he claimed that the two diametral dimensions Δ and Δ_b are equal for Fréchet spaces, referring to a forthcoming paper which, as far as the author knows, has never been published. More recently, Terzioğlu developed a proof for the equality of the two diametral dimensions in quasinormable, metrizable locally convex spaces ([34]). Nevertheless, Frerick and Wengenroth found a mistake in this proof (which implies that the considered space is finite-dimensional when it is Montel and has a continuous norm), so the question remains open. We then decided to deal with this problem, with the later purpose to determine $\Delta_b(S^\nu)$. When treating this open question, we also discovered that the property $(\overline{\Omega})$ of Vogt and Wagner (see for example [24]) implies the equality of the two diametral dimensions, which led us to verify whether the spaces S^ν have this property.

Therefore, these questions constitute the main topics of the present thesis. In Part I, we focus on the study of the equality of the two diametral dimensions Δ and Δ_b. Moreover, Part II is dedicated to the pursuit of the study of spaces S^ν through the diametral dimensions and the property $(\overline{\Omega})$. In this context, let us describe more explicitly the contents of these two parts.

Part I begins with a general presentation of the theory of the (classic) diametral dimension Δ. In Chapter 1, we first introduce the notion of Kolmogorov's diameters and its main properties. Then, we use this to define the classic diametral dimension Δ and we provide its fundamental properties, such as the facts that it is a topological invariant and it characterizes Schwartz spaces. Next, we present the main examples of spaces for which the diametral dimension can be computed, namely Köthe sequence spaces, and we particularly focus on two important subclasses of such spaces: *regular spaces* and *smooth sequence spaces.*

In Chapter 2, we define the second diametral dimension Δ_b and give its main properties. We also describe more precisely the question about the equality of Δ and Δ_b and explain why it is directly solved for non-Schwartz spaces. Besides, we present the main known properties in Schwartz metrizable spaces assuring the equality of the two diametral dimensions ([6, 13]): the property of large bounded sets and the equality $\Delta = \Delta^\infty$, which holds for Köthe-Schwartz echelon spaces and Hilbertizable Schwartz spaces.

Chapter 3 presents some properties of the diametral dimension Δ in finite Cartesian products ([27]) and shows how to use them in order to obtain some spaces verifying the equality $\Delta = \Delta_b$. In this chapter, we also study the existence of the so-called prominent bounded sets ([34]), which implies $\Delta = \Delta_b$, and we prove that the property $(\overline{\Omega})$ implies the existence of prominent sets, although the converse is false.

Finally, Chapter 4 describes a family of Schwartz – or even nuclear – non-metrizable, locally convex spaces E with $\Delta(E) \neq \Delta_b(E)$, which implies that the open question about the equality of the diametral dimensions is definitely false in non-metrizable spaces. Besides, we explain why this construction cannot be imitated in metrizable spaces, thanks to a characterization of metrizable spaces E verifying $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ or $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$.

Part II focuses on spaces S^ν. More precisely, Chapter 5 introduces the definition and the topological properties of spaces S^ν and the formula of $\Delta(S^\nu)$ when S^ν is locally p-convex ([1, 2, 3, 4, 16]). Besides, it also explains why $\Delta_b\left(S^\nu\right) = \Delta\left(S^\nu\right)$ thanks to the results given in Part I.

In Chapter 6, we describe a new method ([12]) to obtain the formula of $\Delta\left(S^\nu\right)$ when the associated profile ν is *concave* and which particularly works for some "strict" locally pseudoconvex spaces. Then, we show that this new method can actually be used to prove that such spaces verify the property $\left(\overline{\Omega}\right)$.

Finally, in Chapter 7, we explain how to adapt the previous technique in the context of locally p-convex spaces S^ν. In the same way as in the concave case, this gives the possibility to revisit the developments which lead to the formula of $\Delta\left(S^\nu\right)$ ([2]) and to show that those spaces have the property $\left(\overline{\Omega}\right)$.

At the end of this thesis, Appendix A gathers two important results of the theory of Fréchet spaces, namely Closed Graph Theorem and Grothendieck's Factorization Theorem, and some of their main consequences. It also gives some applications of these fundamental results to the study of the inclusions and equalities between Köthe echelon spaces, which will be used several times in our developments.

Part I

Diametral Dimension(s)

Chapter 1

Preliminaries

In this chapter, we define the "classic" diametral dimension and present its main properties. We also consider the case of the Köthe sequence spaces. But, first of all, we need to introduce the so-called Kolmogorov's diameters, which are used in the definition of the diametral dimension.

1.1 Kolmogorov's diameters

In this section, we fix a vector space E (on the field of all complex numbers $\mathbb{C}$) and two subsets V and U of E, with the condition that there exists $\mu > 0$ for which $V \subseteq \mu U$.

Besides, if $n \in \mathbb{N}_0$, we denote by $\mathcal{L}_n(E)$ the class of all vector subspaces of E with a dimension at most equal to n.

Then, we can define Kolmogorov's diameters as follows:

Definition 1.1.1. The *n-th Kolmogorov's diameter of V with respect to U* is the positive number

$$\delta_n(V, U) := \inf \{\delta > 0 : \exists L \in \mathcal{L}_n(E) \text{ such that } V \subseteq \delta U + L\}.$$

These numbers have many straightforward – but useful – properties. Here is a list of such direct results (for more details, cf. [11, 19]):

Proposition 1.1.2. *For any $n \in \mathbb{N}_0$, we have*

(1) $\delta_{n+1}(V, U) \leq \delta_n(V, U)$;

(2) $0 \leq \delta_n(V, U) \leq \mu$.

In particular, the sequence $(\delta_n(V, U))_{n\in\mathbb{N}_0}$ converges to a positive real number smaller than μ.

Proposition 1.1.3. *If $V_0, U_0 \in \wp(E)$ are such that $V_0 \subset V$ and $U \subset U_0$, then*

$$\delta_n(V_0, U_0) \leq \delta_n(V, U).$$

In particular, we also have

$$\delta_n(V_0, U) \leq \delta_n(V, U) \quad \textit{and} \quad \delta_n(V, U_0) \leq \delta_n(V, U).$$

Proposition 1.1.4. *Let $W \in \wp(E)$ be such that there exists $\nu > 0$ with $W \subset \nu V$. Then, for any $m, n \in \mathbb{N}_0$,*

$$\delta_{n+m}(W, U) \leq \delta_n(W, V)\delta_m(V, U).$$

Proposition 1.1.5. *For $\lambda, \nu > 0$,*

$$\frac{\lambda}{\nu}\delta_n(V, U) = \delta_n(\lambda V, \nu U).$$

Proposition 1.1.6. *If E is finite-dimensional, then*

$$\delta_n(V, U) = 0,$$

for every $n \in \mathbb{N}_0$ with $n \geq \dim(E)$.

Proposition 1.1.7. *If F is another vector space and if $T : E \to F$ is a linear map, then*

$$\delta_n(T(V), T(U)) \leq \delta_n(V, U).$$

In particular, if $T : E \to F$ is an isomorphism of vector spaces,

$$\delta_n(T(V), T(U)) = \delta_n(V, U).$$

We can also mention the following property ([19]):

Proposition 1.1.8. *If U is absolutely convex, then*

$$\delta_n(V, U) = \delta_n(\Gamma(V), U),$$

where $\Gamma(V)$ is the absolutely convex hull of V.

Proof. Indeed, if $\delta > 0$ and $L \in \mathcal{L}_n(E)$ are given, then we have

$$V \subset \delta U + L \Leftrightarrow \Gamma(V) \subset \delta U + L$$

since the set $\delta U + L$ is itself absolutely convex. □

In addition to these basic properties, Kolmogorov's diameters are actually quite close to the notion of precompactness, which will give a characterization of Schwartz locally convex spaces thanks to the "classic" diametral dimension. For this, we recall the following definition:

Definition 1.1.9. The set V is *precompact with respect to U* if, for any $\varepsilon > 0$, there exists a finite subset P of E such that

$$V \subseteq \varepsilon U + P.$$

Moreover, in a locally convex space, a set is *precompact* if it is precompact with respect to any 0-neighbourhood.

Now, we are ready to consider the next result (the proof is extracted from [11, 38]).

Proposition 1.1.10. *Assume that U is absolutely convex and absorbing in E. Then, V is precompact with respect to U if and only if*

$$\lim_{n\to\infty} \delta_n(V,U) = 0.$$

Proof. If V is precompact with respect to U, then, for a given $\varepsilon > 0$, there exists a finite subset P of E such that

$$V \subseteq \varepsilon U + P \subseteq \varepsilon U + \operatorname{span}(P),$$

where $\operatorname{span}(P)$ is the linear span of P, which is of course finite-dimensional. Therefore, $\delta_n(V,U) \leq \varepsilon$ if $n \geq \dim(\operatorname{span}(P))$.

Conversely, suppose that $\lim_{n\to\infty} \delta_n(V,U) = 0$ and fix $\varepsilon > 0$. Because U is balanced, it means there exists a finite-dimensional vector subspace L of E with

$$V \subseteq \frac{\varepsilon}{4}U + L.$$

If p_U is the gauge of U, then we know there exists a finite-dimensional space L_0 for which $L = (L \cap \ker(p_U)) \oplus L_0$. Consequently,

$$V \subseteq \frac{\varepsilon}{4}U + L \cap \ker(p_U) + L_0 \subseteq \frac{\varepsilon}{2}U + L_0.$$

But it is easy to check that this inclusion can be written as $V \subseteq \frac{\varepsilon}{2}U + L_0 \cap (V - \frac{\varepsilon}{2}U)$. Since the set $L_0 \cap (V - \frac{\varepsilon}{2}U)$ is included in $L_0 \cap (\mu + \frac{\varepsilon}{2})U$, it is in particular a bounded set of the finite-dimensional normed space (L_0, p_U) and so a precompact set of this space. Thus, there exists a finite subset P of L_0 with

$$L_0 \cap \Big(V - \frac{\varepsilon}{2}U\Big) \subseteq \frac{\varepsilon}{2}U + P,$$

hence $V \subseteq \varepsilon U + P$. □

Of course, this property leads to a characterization of precompactness in locally convex spaces:

Corollary 1.1.11. *If E is a locally convex space and if K is a bounded set of E, then K is precompact in E if and only if, for every (absolutely convex) 0-neighbourhood W of E, we have*

$$\lim_{n\to\infty} \delta_n(K,W) = 0.$$

Finally, we conclude this section by a proposition and its corollary which are very important to compare the diametral dimension of two locally convex spaces (cf. Proposition 1.2.3 and its applications in Chapter 4 to construct some counterexamples). The first proof is very close to the previous one (and comes from [19, 38]).

Proposition 1.1.12. *If U is absolutely convex and absorbing, then, for any $n \in \mathbb{N}$,*

$$\delta_n(V,U) = \inf\{\delta > 0 : \exists P \in \wp(E) \setminus \{\emptyset\},\ \#P \leq n,\ \text{such that } V \subseteq \delta U + \Gamma(P)\},$$

where $\#P$ is the cardinality of P.

Proof. If we denote the right-hand side of the claimed equality by $\gamma_n(V,U)$, we just have to prove that $\gamma_n(V,U) \leq \delta_n(V,U)$. For this, assume that $\delta > 0$ and $L \in \mathcal{L}_n(E)$ are such that $V \subseteq \delta U + L$ and fix $\varepsilon > 0$.

As before, we take $L_0 \in \mathcal{L}_n(E)$ such as $L = (L \cap \ker(p_U)) \oplus L_0$, so

$$V \subseteq \left(\delta + \frac{\varepsilon}{2}\right) U + L_0.$$

In particular, it implies that

$$V \subseteq \left(\delta + \frac{\varepsilon}{2}\right) U + L_0 \cap \left(V - \left(\delta + \frac{\varepsilon}{2}\right) U\right) \subseteq \left(\delta + \frac{\varepsilon}{2}\right) U + L_0 \cap \left(\mu + \delta + \frac{\varepsilon}{2}\right) U.$$

Because the set

$$K := L_0 \cap \left(\mu + \delta + \frac{\varepsilon}{2}\right) U$$

is a bounded set – and so a precompact set – of the finite-dimensional normed space (L_0, p_U), there exists a finite subset P_0 of L_0 with $K \subseteq \frac{\varepsilon}{2} U + P_0$. Moreover, by a suitable choice of a vector basis P of L_0[1], we have $P_0 \subseteq \Gamma(P)$, $\#P \leq n$, and

$$V \subseteq (\delta + \varepsilon) U + \Gamma(P).$$

Therefore $\gamma_n(V,U) \leq \delta + \varepsilon$. What is more, since the argument works for every $\varepsilon > 0$, we even have $\gamma_n(V,U) \leq \delta$, which gives $\gamma_n(V,U) \leq \delta_n(V,U)$. □

From this last result, we deduce the following property ([19]):

Corollary 1.1.13. *Assume that E is a topological vector space. If U is an absolutely convex, absorbing, closed set and if $n \in \mathbb{N}_0$, then*

$$\delta_n\left(\overline{V}, U\right) = \delta_n(V,U),$$

where $\overline{V}$ is the closure of V in E.

Proof. Firstly, assume that $n = 0$. Then, the claimed equality is straightforward because U is closed.

Secondly, suppose that $n > 0$. Of course, $\delta_n(V,U) \leq \delta_n\left(\overline{V}, U\right)$, so that we just have to prove the other inequality. For this, we use the previous result: we fix $\delta > 0$ and $P \in \wp(E) \setminus \{\emptyset\}$, with $\#P \leq n$, such that $V \subseteq \delta U + \Gamma(P)$.

Then, using the facts that U is closed and $\Gamma(P)$ is compact, $\delta U + \Gamma(P)$ is itself closed and so $\overline{V} \subseteq \delta U + \Gamma(P)$. Thus, we conclude because this means that $\delta_n\left(\overline{V}, U\right) \leq \delta$ and so $\delta_n\left(\overline{V}, U\right) \leq \delta_n(V,U)$. □

[1] If $L_0 = \{0\}$, we simply take $P = \{0\}$.

1.2 Diametral dimension

Thanks to Kolmogorov's diameters – and their properties –, we are now ready to study the "classic" diametral dimension itself. For this, we fix a topological vector space E (again, on the field $\mathbb{C}$) and a basis of 0-neighbourhoods $\mathcal{U}$ in E.

Definition 1.2.1. The *diametral dimension* of E is the set

$$\Delta(E) := \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, V \subseteq U, \text{ such that } (\xi_n \delta_n(V,U))_{n\in\mathbb{N}_0} \in c_0\right\}.$$

It is easy to check that this definition is independent of the choice of the basis of 0-neighbourhoods and that $\Delta(E)$ is a vector space.

This diametral dimension was introduced by Mityagin ([25]) to obtain a *topological invariant*. In order to explain this important property, we first need to consider a result to compare the diametral dimension of two spaces and which will be itself useful for the construction of counterexamples in Section 4.1.

Definition 1.2.2. Let F be another topological vector space. Then a linear map $T : E \to F$ is *nearly open* if, for any 0-neighbourhood U in E, $\overline{T(U)}$ is a 0-neighbourhood in F.

For instance, an open map is of course nearly open. Besides, as explained by Jarchow ([19]), if E and F are locally convex, if F is barrelled, and if $T : E \to F$ is onto, then this map is also nearly open.

Thanks to this notion, we have the following result (cf. [11, 19]):

Proposition 1.2.3. *Let F be another topological vector space.*

1. *If there exists a linear, continuous, and open map $T : E \to F$, then $\Delta(E) \subseteq \Delta(F)$.*
2. *If F is a locally convex space and if there exists a linear, continuous, and nearly open map $T : E \to F$, then $\Delta(E) \subseteq \Delta(F)$.*

Proof. Assume that $T : E \to F$ is a linear and continuous map and fix $\xi \in \Delta(E)$ and U_0 a 0-neighbourhood in F. Using the continuity of T, we know there exists a 0-neighbourhood U in E with $T(U) \subseteq U_0$. Then, by definition of the diametral dimension, there is a 0-neighbourhood V in E, with $V \subseteq U$, such that

$$(\xi_n \delta_n(V,U))_{n\in\mathbb{N}_0} \in c_0.$$

Now, we split the argument according to the two situations described above.

1. Suppose that T is also open. In this case, $V_0 := T(V)$ is itself a 0-neighbourhood in F and so, using Propositions 1.1.3 and 1.1.7, we have

$$\delta_n(V_0, U_0) \le \delta_n(T(V), T(U)) \le \delta_n(V, U).$$

   Consequently, $(\xi_n \delta_n(V_0, U_0))_{n\in\mathbb{N}_0} \in c_0$ and it proves that $\xi \in \Delta(F)$.

2. Assume that F is locally convex and T is nearly open. Then, we can suppose without loss of generality that U_0 is a closed absolutely convex set. Therefore, $V_0 := \overline{T(V)}$ is a 0-neighbourhood in F and, by Corollary 1.1.13 and Propositions 1.1.3 and 1.1.7,

$$\delta_n(V_0, U_0) = \delta_n\left(\overline{T(V)}, U_0\right) = \delta_n(T(V), U_0) \leq \delta_n(T(V), T(U)) \leq \delta_n(V, U).$$

Again, we deduce from this that $\xi \in \Delta(F)$.

Hence the conclusion. □

As announced, this property implies that the diametral dimension is a topological invariant for topological vector spaces ([19]):

Theorem 1.2.4. *If E and F are two isomorphic topological vector spaces, then $\Delta(E) = \Delta(F)$.*

Moreover, we can mention these additional consequences of the previous proposition ([19]):

Corollary 1.2.5. *Let F be another topological vector space and $(E_\alpha)_{\alpha\in A}$ a family of topological vector spaces. Then,*

- $\Delta(E/F) \subseteq \Delta(E)$;
- $\Delta\left(\prod_{\alpha\in A} E_\alpha\right) \subseteq \bigcap_{\alpha\in A} \Delta(E_\alpha)$.

A more detailed description of the diametral dimension for finite Cartesian products will be provided in Section 3.1, when we will study the notion of Δ-stability.

But, now, it could be interesting to consider some examples. The following result is straightforward, thanks to Proposition 1.1.6:

Example 1.2.6. If E is finite-dimensional, then $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$.

In fact, there are also some infinite-dimensional locally convex spaces with such a diametral dimension. In Section 4.1, we will show that any space E with a weak topology verifies $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$. Moreover, we will prove in Section 4.2 that there is only one infinite-dimensional Fréchet space with this property (up to isomorphism): it is the space ω, i.e. the linear space $\mathbb{C}^{\mathbb{N}_0}$ endowed with the topology of pointwise convergence. More generally, we will show that a metrizable space E with $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ is in fact – up to isomorphism – a subspace of ω.

Other interesting examples are the *Köthe sequence spaces*, which will be intensively treated (in terms of diametral dimension) in the next section. Finally, we can also consider the case of *Schwartz spaces*. For this, let us recall their definition:

Definition 1.2.7. The topological vector space E is *Schwartz* if every 0-neighbourhood U in E contains another 0-neighbourhood V which is precompact with respect to U.

As claimed before, the diametral dimension can be used to characterize Schwartz locally convex spaces. For this, we first remark that we have the following property ([19]):

Proposition 1.2.8. *We have*

$$c_0 \subseteq \Delta(E).$$

Proof. Indeed, for any 0-neighbourhood U in E, we have $\delta_n(U,U) \leq 1$ by Proposition 1.1.2. □

Now, we can consider the following characterization (cf. [11, 19]):

Theorem 1.2.9. *Assume that E is locally convex. Then, the following are equivalent:*

(1) E is a Schwartz space;

(2) $l_\infty \subseteq \Delta(E)$;

(3) $c_0 \subsetneq \Delta(E)$.

Proof. The implications (1) ⇒ (2) and (2) ⇒ (3) follow from the definitions of $\Delta(E)$ and Schwartz spaces and from Proposition 1.1.10.

Assume that there exists a sequence $\xi \in \Delta(E) \setminus c_0$ and fix an absolutely convex 0-neighbourhood U in E. Then, there exists an absolutely convex 0-neighbourhood V in E such that

$$(\xi_n \delta_n(V,U))_{n\in\mathbb{N}_0} \in c_0.$$

Moreover, since $\xi \notin c_0$, there exist a subsequence $\left(\xi_{k(n)}\right)_{n\in\mathbb{N}_0}$ of ξ and a constant $C > 0$ with $|\xi_{k(n)}| \geq C$ for every $n \in \mathbb{N}_0$. Consequently,

$$\delta_{k(n)}(V,U) \leq \frac{1}{C}|\xi_{k(n)}|\delta_{k(n)}(V,U).$$

Therefore, the sequence $\left(\delta_{k(n)}(V,U)\right)_{n\in\mathbb{N}_0}$ converges to 0. Consequently, the sequence $(\delta_n(V,U))_{n\in\mathbb{N}_0}$ also converges to 0, because it is convergent (by Proposition 1.1.2) and has a null subsequence. We conclude by Proposition 1.1.10. □

This last result also implies that we have $\Delta(E) = c_0$ for non-Schwartz spaces (such as, for instance, infinite-dimensional normed spaces). This fact, together with Example 1.2.6, shows that the diametral dimension is not a *complete topological invariant* on the class of topological vector spaces: this means that two spaces with the same diametral dimension are not necessarily isomorphic. However, there are some classes of Köthe sequence spaces on which the diametral dimension is complete, such as the class of *power series spaces* or, more generally, the class of *smooth sequence spaces* (see below).

Finally, the diametral dimension can be also used to characterize nuclear locally convex spaces (cf. [11, 19, 26, 38] for more details about these spaces and the links with the diametral dimension):

Theorem 1.2.10. *If E is locally convex, the following are equivalent:*

(1) E is nuclear;

(2) $\forall p > 0$, $((n+1)^p)_{n\in\mathbb{N}_0} \in \Delta(E)$;

(3) $\exists p > 0$ such that $((n+1)^p)_{n\in\mathbb{N}_0} \in \Delta(E)$.

1.3 Applications to Köthe sequence spaces

Some examples of locally convex spaces for which the diametral dimension can be computed are the so-called *Köthe sequence spaces*. These ones have been deeply studied and many results about them can be found for instance in [8, 9, 24].

In 2008, Terzioğlu defined the notion of *admissible normed spaces* ([33]) in order to generalize the definition of Köthe spaces and to gather developments for the diametral dimension of such spaces. In this section, we present Terzioğlu's main results about the diametral dimension of Köthe spaces, which will be particularly used in Section 2.2.

First, we give the definition of admissible spaces. For this, we just introduce two notations. The first one is about the *product of two sequences*: if $\xi, \eta \in \mathbb{C}^{\mathbb{N}_0}$, then the product $\xi\eta$ is defined by

$$(\xi\eta)_n = \xi_n \eta_n \qquad (n \in \mathbb{N}_0).$$

The second notation relates to the *unit sequences*: if $k \in \mathbb{N}_0$, the *unit sequence* e_k is defined by

$$(e_k)_n = \begin{cases} 1 & \text{if} \quad n = k \\ 0 & \text{if} \quad n \neq k \end{cases} \qquad (n \in \mathbb{N}_0).$$

Definition 1.3.1. An *admissible space* is a normed space $(l, \|.\|_l)$ which verifies the following conditions:

(1) $l \subseteq \mathbb{C}^{\mathbb{N}_0}$;

(2) if $\xi \in l_\infty$ and $\eta \in l$, then $\xi\eta \in l$ and

$$\|\xi\eta\|_l \leq \|\xi\|_\infty \|\eta\|_l;$$

(3) for every $k \in \mathbb{N}_0$, $e_k \in l$ and $\|e_k\|_l = 1$.

Of course, the classic spaces l_p (with $p \geq 1$), l_∞, and c_0 are admissible, but we can also cite the *Orlicz sequence spaces* (cf. [21, 23] for the definition and [11] for the "admissibility" of these spaces).

Among the properties of the admissible spaces, we can particularly give the following one ([33]):

Proposition 1.3.2. *If $\xi \in l$ and $\eta \in \mathbb{C}^{\mathbb{N}_0}$ are such that $|\eta_n| \leq |\xi_n|$ for every $n \in \mathbb{N}_0$, then $\eta \in l$ and $\|\eta\|_l \leq \|\xi\|_l$.*

Proof. It is direct. □

To introduce the Köthe sequence spaces, we also need the notion of *Köthe sets* (cf. [9, 19, 33]).

Definition 1.3.3. A set $A \subseteq \mathbb{C}^{\mathbb{N}_0}$ is a *Köthe set* if

(1) $\forall \alpha \in A$, $\forall n \in \mathbb{N}_0$, $\alpha_n \geq 0$;

(2) $\forall n \in \mathbb{N}_0$, $\exists \alpha \in A$ with $\alpha_n > 0$;

(3) $\forall \alpha, \beta \in A$, $\exists \gamma \in A : \sup\{\alpha_n, \beta_n\} \leq \gamma_n \ \forall n \in \mathbb{N}_0$.

From now on, we fix an admissible space l and a Köthe set A. Then, we are ready to introduce the "generalized" Köthe sequence spaces due to Terzioğlu ([33]).

Definition 1.3.4. The *Köthe sequence space* associated to l and A is the space

$$\lambda^l(A) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall \alpha \in A, \alpha\xi \in l \right\},$$

endowed with the locally convex topology defined by the family of seminorms

$$p^l_\alpha : \xi \in \lambda^l(A) \mapsto \|\alpha\xi\|_l \qquad (\alpha \in A).$$

We also define the set U^l_α as the closed unit ball of $\lambda^l(A)$ associated to p^l_α. Moreover, we can use the classic notations when the associated admissible space is l_p (with $p \geq 1$), l_∞, or c_0:

$$\lambda_p(A) := \lambda^{l_p}(A), \quad \lambda_\infty(A) := \lambda^{l_\infty}(A) \quad \text{and} \quad \lambda_0(A) := \lambda^{c_0}(A).$$

It is easy to prove that the space $\lambda^l(A)$ is a complete Hausdorff locally convex space; it is even a Fréchet space when A is countable. Besides, $\lambda^l(A) \subseteq \omega$ continuously.

In fact, thanks to the main properties of admissible spaces, it is possible to find a general formula giving the diametral dimension of any Köthe sequence space.

For this purpose, we need to define an operation of *quotient* between two sequences: if $\alpha, \beta \in \mathbb{C}^{\mathbb{N}_0}$, the quotient-sequence α/β is defined by

$$\left(\frac{\alpha}{\beta}\right)_n := \begin{cases} \alpha_n/\beta_n & \text{if } \beta_n \neq 0; \\ 0 & \text{if } \beta_n = 0. \end{cases}$$

Thanks to this, we can prove a first result about Kolmogorov's diameters in Köthe sequence spaces. This property – and the associated arguments – is very important for us, because one of the main results in Section 2.2 – Proposition 2.2.6 – is actually based on the next proof. We will also imitate these ideas for the diametral dimension of spaces S^ν in Sections 6.1 and 7.1.

Besides, remark that the lower-bound in the following proposition can be also found using Tikhomirov's Theorem (Proposition 6.1.4), which will be completely detailed in Section 6.1. Here, we only present Terzioğlu's developments (the next proof is a slightly modified version of the corresponding proof in [33]).

Proposition 1.3.5. *Let $\alpha, \beta \in A$ be given, for which there exists $\mu > 0$ with $\alpha_m \leq \mu\beta_m$ for every $m \in \mathbb{N}_0$. Besides, assume that J and J' are two subsets of $\mathbb{N}_0$ such that $\#J = n+1$ and $\#J' \leq n$, where $n \in \mathbb{N}_0$. Then, we have*

$$\inf_{j\in J}\left(\frac{\alpha}{\beta}\right)_j \leq \delta_n\left(U^l_\beta, U^l_\alpha\right) \leq \sup_{j\notin J'}\left(\frac{\alpha}{\beta}\right)_j.$$

Proof. 1. For the first inequality, we put $\delta_0 := \inf_{j\in J}\left(\frac{\alpha}{\beta}\right)_j$ and we assume that $\delta_n\left(U^l_\beta, U^l_\alpha\right) < \delta_0$. Remark that this particularly means that $\delta_0 > 0$ and so $\alpha_j > 0$ for any $j \in J$.

Therefore, there exist a $\delta > 0$, with $\delta < \delta_0$, and $L \in \mathcal{L}_n(\lambda^l(A))$ such that

$$U^l_\beta \subseteq \delta U^l_\alpha + L.$$

We also define a projection $P_J : \xi \in \lambda^l(A) \mapsto \sum_{j\in J}\xi_j e_j$ and we denote by G its range (i.e. the space $\operatorname{span}(\{e_j : j \in J\})$). Now, if $j \in J$ and $\xi \in \lambda^l(A)$ are given, we have

$$|\beta_j\xi_j| = \left(\frac{\beta}{\alpha}\right)_j |\alpha_j\xi_j| \leq \frac{1}{\delta_0}|\alpha_j\xi_j|.$$

Consequently, if $\xi \in G$, $p^l_\beta(\xi) \leq \frac{1}{\delta_0}p^l_\alpha(\xi)$, which means that $U^l_\alpha \cap G \subseteq \frac{1}{\delta_0}U^l_\beta \cap G$. Moreover, since $P_J(U^l_\beta) = U^l_\beta \cap G$, we obtain $U^l_\beta \cap G \subseteq \delta U^l_\alpha \cap G + P_J(L)$, so $U^l_\alpha \cap G \subseteq \frac{\delta}{\delta_0}U^l_\alpha \cap G + P_J(L)$. Thus, we deduce from this

$$\begin{aligned}
U^l_\alpha \cap G &\subseteq \frac{\delta}{\delta_0}U^l_\alpha \cap G + P_J(L)\\
&\subseteq \frac{\delta}{\delta_0}\left(\frac{\delta}{\delta_0}U^l_\alpha \cap G + P_J(L)\right) + P_J(L) = \left(\frac{\delta}{\delta_0}\right)^2 U^l_\alpha \cap G + P_J(L)\\
&\subseteq \cdots\\
&\subseteq \left(\frac{\delta}{\delta_0}\right)^N U^l_\alpha \cap G + P_J(L)
\end{aligned}$$

for any $N \in \mathbb{N}$. Therefore,

$$U^l_\alpha \cap G \subseteq \bigcap_{N\in\mathbb{N}}\left(\left(\frac{\delta}{\delta_0}\right)^N U^l_\alpha \cap G + P_J(L)\right) = \overline{P_J(L)}^{(G,p^l_\alpha)},$$

because $\frac{\delta}{\delta_0} < 1$. But we know that $\alpha_j > 0$ for every $j \in J$, so p^l_α is a norm on G and so $\overline{P_J(L)}^{(G,p^l_\alpha)} = P_J(L)$, as a finite-dimensional vector subspace of a Hausdorff space. Therefore,

$$U^l_\alpha \cap G \subseteq P_J(L)$$

and this implies that $G \subseteq P_J(L)$, since G and $P_J(L)$ are both vector spaces. Hence a contradiction, since $\dim G = n+1 > n \geq \dim P_J(L)$.

2. Now, we prove the other inequality. Again, we define a projection $P_{J'} : \xi \in \lambda^l(A) \mapsto \sum_{j\in J'} \xi_j e_j$. Now, for $\xi \in \lambda^l(A)$ and $n \in \mathbb{N}_0$, we have

$$|\alpha_n(\xi - P_{J'}(\xi))_n| = \left(\frac{\alpha}{\beta}\right)_n |\beta_n(\xi - P_{J'}(\xi))_n| \leq \left[\sup_{j\notin J'} \left(\frac{\alpha}{\beta}\right)_j\right] |\beta_n(\xi - P_{J'}(\xi))_n|.$$

This implies that

$$p^l_\alpha(\xi - P_{J'}(\xi)) \leq \left[\sup_{j\notin J'} \left(\frac{\alpha}{\beta}\right)_j\right] p^l_\beta(\xi - P_{J'}(\xi)) \leq \left[\sup_{j\notin J'} \left(\frac{\alpha}{\beta}\right)_j\right] p^l_\beta(\xi).$$

Consequently, if $\xi \in U^l_\beta$, then $\xi - P_{J'}(\xi) \in \left[\sup_{j\notin J'} \left(\frac{\alpha}{\beta}\right)_j\right] U^l_\alpha$ and

$$\xi = \xi - P_{J'}(\xi) + P_{J'}(\xi) \in \left[\sup_{j\notin J'} \left(\frac{\alpha}{\beta}\right)_j\right] U^l_\alpha + P_{J'}(\lambda^l(A)).$$

Hence the conclusion since $\dim(P_{J'}(\lambda^l(A)) \leq n$.

□

These two inequalities already bring the exact values of some Kolmogorov's diameters ([11, 33]):

Corollary 1.3.6. *Let $\alpha, \beta \in A$ for which there is $\mu > 0$ with $\alpha_m \leq \mu\beta_m$ for every $m \in \mathbb{N}_0$. If the sequence α/β is decreasing, then*

$$\delta_n\left(U^l_\beta, U^l_\alpha\right) = \left(\frac{\alpha}{\beta}\right)_n.$$

Proof. It follows from the previous result, taking $J = \{0, ..., n\}$ and $J' = \{0, ..., n-1\}$ if $n > 0$ and $J' = \emptyset$ if $n = 0$. □

Without such an assumption of decrease on α and β, evaluating Kolmogorov's diameters can be very difficult. It is why Terzioğlu provided a characterization of Schwartz Köthe spaces ([33]) – or "Köthe-Schwartz sequence spaces" to respect the traditional terminology –, which actually corresponds to the classic characterization of Schwartz spaces of type $\lambda_p(A)$, $\lambda_\infty(A)$ and $\lambda_0(A)$. From this, we will deduce a general formula for $\Delta\left(\lambda^l(A)\right)$.

Theorem 1.3.7. *The space $\lambda^l(A)$ is Schwartz if and only if, for every $\alpha \in A$, there exists $\beta \in A$, with $\alpha_n \leq \beta_n$ for any $n \in \mathbb{N}_0$, such that $\alpha/\beta \in c_0$.*

Proof. If the space is Schwartz, then, for a given weight $\alpha \in A$, there exists $\beta \in A$, with $\alpha_n \leq \beta_n$ for all n, such that

$$\left(\delta_n\left(U^l_\beta, U^l_\alpha\right)\right)_{n\in\mathbb{N}_0} \in c_0$$

by Proposition 1.1.10. Then $\alpha/\beta \in c_0$, since, otherwise, there exist a constant $C > 0$ and a subsequence $\left((\alpha/\beta)_{k(n)}\right)_{n\in\mathbb{N}_0}$ of α/β with $(\alpha/\beta)_{k(n)} \geq C$ for all $n \in \mathbb{N}_0$. Taking $J = \{k(0), ..., k(n)\}$ in Proposition 1.3.5, we obtain

$$C \leq \inf_{j\in\{k(0),...,k(n)\}} \left(\frac{\alpha}{\beta}\right)_j \leq \delta_n \left(U^l_\beta, U^l_\alpha\right),$$

hence a contradiction.

Now, assume that $\alpha, \beta \in A$ are such that $\alpha_n \leq \beta_n$ for any $n \in \mathbb{N}_0$ and $\alpha/\beta \in c_0$. For a fixed $\varepsilon > 0$, there exists $N \in \mathbb{N}$ for which $(\alpha/\beta)_n \leq \varepsilon$ for all $n \geq N$.

Then, taking $J' = \{0, ..., N-1\}$ in Proposition 1.3.5, we have

$$\delta_n \left(U^l_\beta, U^l_\alpha\right) \leq \delta_N \left(U^l_\beta, U^l_\alpha\right) \leq \sup_{j\geq N} \left(\frac{\alpha}{\beta}\right)_j \leq \varepsilon \qquad (n \geq N).$$

Consequently, by Proposition 1.1.10, U^l_β is precompact with respect to U^l_α. $\square$

This last theorem implicitly means that, when $\lambda^l(A)$ is Schwartz, we can exclusively focus on weights $\alpha, \beta \in A$ such that $\alpha/\beta \in c_0$. Moreover, it is possible to reorganize such a null sequence to obtain a decreasing one – and we recall that Kolmogorov's diameters define decreasing sequences. This idea leads to the following construction ([11, 33]).

Construction 1.3.8. Let $x \in c_0 \cap [0, \infty)^{\mathbb{N}_0}$ be given. Then, the supremum

$$\sup\{x_n : n \in \mathbb{N}_0\}$$

is attained at (at least) one index. We denote by $\pi_0(x)$ this supremum and we put $\varphi(x, 0) := \min\{n \in \mathbb{N}_0 : x_n = \pi_0(x)\}$.

Again, the supremum

$$\pi_1(x) := \sup\{x_n : n \in \mathbb{N}_0 \setminus \{\varphi(x, 0)\}\}$$

is attained and we define $\varphi(x, 1) := \min\{n \in \mathbb{N}_0 \setminus \{\varphi(x, 0)\} : x_n = \pi_1(x)\}$.

Recursively, if $\pi_0(x), ..., \pi_m(x)$ and $\varphi(x, 0), ..., \varphi(x, m)$ are defined, we put

$$\pi_{m+1}(x) := \sup\{x_n : n \in \mathbb{N}_0 \setminus \{\varphi(x, 0), ..., \varphi(x, m)\}\}$$

and $\varphi(x, m+1) := \min\{n \in \mathbb{N}_0 \setminus \{\varphi(x, 0), ..., \varphi(x, m)\} : x_n = \pi_{m+1}(x)\}$.

This recursion defines another sequence $\pi(x) := (\pi_n(x))_{n\in\mathbb{N}_0} = \left(x_{\varphi(x,n)}\right)_{n\in\mathbb{N}_0}$, which actually corresponds to a decreasing reorganization of the sequence x. Therefore, we have just introduced a map

$$\pi : c_0 \cap [0, \infty)^{\mathbb{N}_0} \to c_0 \cap [0, \infty)^{\mathbb{N}_0} : x \mapsto \pi(x),$$

called the *decreasing-reorganization map* in this work.

This decreasing-reorganization map can be used precisely to describe Kolmogorov's diameters in Köthe-Schwartz spaces ([33]):

Proposition 1.3.9. *Let $\alpha, \beta \in A$ be such that there exists $\mu > 0$ with $\alpha_m \leq \mu\beta_m$ for each $m \in \mathbb{N}_0$ and such that $\alpha/\beta \in c_0$. Then,*

$$\delta_n\left(U^l_\beta, U^l_\alpha\right) = \pi_n(\alpha/\beta) \qquad (n \in \mathbb{N}_0).$$

Proof. It is enough to use Proposition 1.3.5 with $J = \{\varphi(\alpha/\beta, 0), ..., \varphi(\alpha/\beta, n)\}$ and $J' = \{\varphi(\alpha/\beta, 0), ..., \varphi(\alpha/\beta, n-1)\}$ if $n > 0$ and $J' = \emptyset$ if $n = 0$. ☐

As a consequence, we deduce this general formula for the diametral dimension of Köthe sequence spaces ([33]):

Theorem 1.3.10.

(1) If $\lambda^l(A)$ is not Schwartz, then $\Delta\left(\lambda^l(A)\right) = c_0$.

(2) If $\lambda^l(A)$ is Schwartz, then

$$\Delta\left(\lambda^l(A)\right) = \Big\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall \alpha \in A, \exists \beta \in A \textit{ with } \alpha_n \leq \beta_n\ \forall n \in \mathbb{N}_0,\ \alpha/\beta \in c_0, \\ \textit{and } \left(\xi_n \pi_n(\alpha/\beta)\right)_{n\in\mathbb{N}_0} \in c_0\Big\}.$$

Proof. That is direct by the previous result. ☐

We would also like to insist on the fact that the diametral dimension of $\lambda^l(A)$ – just like Kolmogorov's diameters in Proposition 1.3.9 – is independent of the admissible space l.

Thanks to the last theorem, we can theoretically find the diametral dimension of any Köthe space. However, it is very difficult in practice to manipulate decreasing reorganizations of positive null sequences and so to obtain a "usable" formula of this diametral dimension. This is why we will conclude this section by considering an interesting subclass of Köthe sequence spaces for which the diametral dimension is easily computable, namely the class of *regular* Köthe spaces.

These regular Köthe spaces are defined thanks to countable Köthe sets. More precisely, we will use the standard definition of Köthe matrices (cf. [9, 24]):

Definition 1.3.11. A countable Köthe set $A = \{a_k : k \in \mathbb{N}_0\}$ is a *Köthe matrix* if, for every $k, n \in \mathbb{N}_0$, we have $0 < a_k(n) \leq a_{k+1}(n)$, where $a_k(n)$ is the component n of a_k.

In this situation, we write $A = (a_k)_{k\in\mathbb{N}_0}$ and we put

$$p^l_k := p^l_{a_k} \qquad \text{and} \qquad U^l_k := U^l_{a_k}.$$

Moreover, when A is a Köthe matrix, we will say that $\lambda^l(A)$ is a *Köthe echelon space*. Some useful properties about the equality and the inclusions between Köthe echelon spaces are presented in the appendix (cf. Section A.2).

Definition 1.3.12. The Köthe matrix $A = (a_k)_{k\in\mathbb{N}_0}$ is *regular* if, for each $k \in \mathbb{N}_0$, the sequence a_k/a_{k+1} is decreasing. We also say that the space $\lambda^l(A)$ is *regular*.

First of all, let us mention an interesting property of non-Schwartz regular spaces ([33]).

Proposition 1.3.13. *If the Köthe matrix $A = (a_k)_{k\in\mathbb{N}_0}$ is regular and if $\lambda^l(A)$ is not Schwartz, then this space is normed.*

Proof. By assumption and by Theorem 1.3.7, there exists $m \in \mathbb{N}_0$ such that, for every $k \geq m$, $a_m/a_k \notin c_0$. But, since a_m/a_k is a decreasing sequence if $k \geq m$, it means there exists $C_k > 0$ with

$$\frac{a_m(n)}{a_k(n)} \geq C_k$$

for every $n \in \mathbb{N}_0$, or, equivalently, $a_m(n) \geq C_k a_k(n)$. Because we also have $a_k(n) \leq a_m(n)$ for all $n \in \mathbb{N}_0$ if $k \leq m$, it implies that the spaces $\lambda^l(A)$ and $\lambda^l(\{a_m\})$ are algebraically and topologically equal (cf. Proposition A.2.1). $\square$

Now, we can prove the following formula for Kolmogorov's diameters in regular spaces ([33]):

Proposition 1.3.14. *If $A = (a_k)_{k\in\mathbb{N}_0}$ is regular and if $k \geq m$, we have*

$$\delta_n\left(U_k^l, U_m^l\right) = \frac{a_m(n)}{a_k(n)}.$$

Proof. It is straightforward by Corollary 1.3.6, since a_m/a_k is a decreasing sequence. $\square$

Thus, we can easily deduce from this the next theorem:

Theorem 1.3.15. *If $A = (a_k)_{k\in\mathbb{N}_0}$ is regular, then*

$$\Delta\left(\lambda^l(A)\right) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall m \in \mathbb{N}_0, \exists k \geq m \text{ such that } \left(\xi_n \frac{a_m(n)}{a_k(n)}\right)_{n\in\mathbb{N}_0} \in c_0\right\}.$$

As announced above, this result provides several examples of spaces with an easily computable diametral dimension. Unfortunately, there exist some Köthe echelon spaces which are *not* regular and not isomorphic to any regular Köthe space. Actually, in Section 3.2, we will use the notion of prominent bounded sets to construct such non-regular spaces.

Among all regular Köthe sequence spaces, we can especially mention the classic *power series spaces.* We recall their definitions (cf. [19, 24]).

Definition 1.3.16. Let $\alpha \in [0,\infty)^{\mathbb{N}_0}$ be an increasing sequence such that $\lim_{n\to\infty} \alpha_n = \infty$. Then, the *power series space of finite type associated to α* is the Köthe space

$$\Lambda_0(\alpha) := \lambda_1(A_0),$$

where A_0 is the Köthe matrix $\left(\left(e^{-\alpha_n/k}\right)_{n\in\mathbb{N}_0}\right)_{k\in\mathbb{N}}$. Similarly, the *power series space of infinite type associated to* α is the space

$$\Lambda_\infty(\alpha) := \lambda_1(A_\infty),$$

where A_∞ is the Köthe matrix $\left(\left(e^{k\alpha_n}\right)_{n\in\mathbb{N}_0}\right)_{k\in\mathbb{N}_0}$[2]. To follow the generalization of Terzioğlu with admissible spaces, we also define the *power series spaces associated to* α *and* l by

$$\Lambda_0^l(\alpha) := \lambda^l(A_0) \qquad \text{and} \qquad \Lambda_\infty^l(\alpha) := \lambda^l(A_\infty).$$

Thanks to the very specific form of the Köthe matrices of power series spaces, the following formulae are easily obtained:

Proposition 1.3.17. *We have*

$$\Delta\left(\Lambda_0^l(\alpha)\right) = \Lambda_0^{c_0}(\alpha) = \Lambda_0^{l_\infty}(\alpha)$$

and

$$\Delta\left(\Lambda_\infty^l(\alpha)\right) = \bigcup_{k\in\mathbb{N}_0}\left\{\xi\in\mathbb{C}^{\mathbb{N}_0} : \left(\xi_n e^{-k\alpha_n}\right)_{n\in\mathbb{N}_0}\in l_\infty\right\} = \left(\Lambda_\infty(\alpha)\right)'.$$

Proof. It follows from Theorem 1.3.15, after some basic computations. More details are available in [11, 19]. Besides, these results can be also proved thanks to smooth sequence spaces (see below), via Propositions 1.4.2 and 1.4.3. □

We recall that the diametral dimension is a complete topological invariant for power series spaces: when they are associated to a same admissible space l, two power series spaces are isomorphic if and only if they have the same diametral dimension. This property is deduced from the very particular expression of the diametral dimension for power series spaces. Moreover, we can even use the diametral dimension to show that

- two power series spaces of the same type (associated to a same admissible space) are isomorphic if and only if they are algebraically (and so topologically) equal;
- two power series spaces of different types are never isomorphic.

All these results and developments can be found in [11, 19], but they can be also seen as a direct corollary of the properties of smooth sequence spaces (cf. Theorem 1.4.5 below).

The power series spaces – and the corresponding Kolmogorov's diameters – will be very important in Section 3.2 to construct some spaces with prominent bounded sets but without the property $(\overline{\Omega})$.

[2]Some authors, like Meise and Vogt ([24]), rather define the power series spaces with Köthe sequence spaces associated to l_2, i.e. $\Lambda_0(\alpha) = \lambda_2(A_0)$ and $\Lambda_\infty(\alpha) = \lambda_2(A_\infty)$.

1.4 Smooth sequence spaces

In this last section of the current chapter, we present another family of Köthe sequence spaces for which the diametral dimension is easily computable, namely the so-called *smooth sequence spaces*. They were introduced by Terzioğlu in [30] and are studied in [27, 30, 31, 32, 34].

In the previous section, the developments for the diametral dimension of Köthe sequence spaces were mainly based on decreasing conditions on quotients of weights. Here, smooth sequence spaces are defined by conditions on the weights themselves and generalize the notions of power series spaces.

More precisely, if we fix a Köthe matrix $A = (a_k)_{k\in\mathbb{N}_0}$ and an admissible space l, we consider the following definitions:

Definition 1.4.1. The space $\lambda^l(A)$ is a *smooth sequence space of finite type* or a *G_1-space* if

(1) $\forall k, n \in \mathbb{N}_0$, $a_k(n+1) \leq a_k(n)$;

(2) $\forall k \in \mathbb{N}_0$, $\exists j \in \mathbb{N}_0$ and $C > 0$ such that $\forall n \in \mathbb{N}_0$ $a_k(n) \leq Ca_j^2(n)$.

It is a *smooth sequence space of infinite type* or a *G_∞-space* if

(1) $\forall k, n \in \mathbb{N}_0$, $a_k(n) \leq a_k(n+1)$;

(2) $\forall k \in \mathbb{N}_0$, $\exists j \in \mathbb{N}_0$ and $C > 0$ such that $\forall n \in \mathbb{N}_0$ $a_k^2(n) \leq Ca_j(n)$.

A *smooth sequence space* is a smooth sequence space of finite or infinite type.

For instance, it is easy to check that power series spaces are smooth sequence spaces: more precisely, a power series space of finite type is a smooth sequence space of finite type and a power series space of infinite type is a smooth sequence space of infinite type.

But the conditions in the previous definitions lead to some results for the diametral dimensions of such spaces, thanks to some techniques which are different from what was presented in the previous section. To prove them, we will simply use Proposition 1.3.5, which will generalize the corresponding results given in [30] in case $l = l_1$.

Proposition 1.4.2. *If $\lambda^l(A)$ is a G_1-space, then*

$$\Delta\left(\lambda^l(A)\right) = \lambda_0(A).$$

Proof. Let $\xi \in \Delta\left(\lambda^l(A)\right)$ and $m \in \mathbb{N}_0$ be given. Then, there exists $k \geq m$ for which $\left(\xi_n\delta_n\left(U_k^l, U_m^l\right)\right)_{n\in\mathbb{N}_0} \in c_0$. But, by Proposition 1.3.5, we have

$$\delta_n\left(U_k^l, U_m^l\right) \geq \inf_{j\leq n}\left(\frac{a_m(j)}{a_k(j)}\right) \geq \frac{1}{a_k(0)}a_m(n),$$

so $(\xi_n a_m(n))_{n\in\mathbb{N}_0} \in c_0$.

Conversely, fix $\xi \in \lambda_0(A)$ and $m \in \mathbb{N}_0$. By definition of G_1-spaces, we know there exist $k \in \mathbb{N}_0$ and $C > 0$ with $a_m(n) \leq C a_k^2(n)$ for all $n \in \mathbb{N}_0$. Using Proposition 1.3.5, we obtain

$$\delta_n\left(U_k^l, U_m^l\right) \leq \sup_{j \geq n} \left(\frac{a_m(j)}{a_k(j)}\right) \leq C \sup_{j \geq n} \left(a_k(j)\right) = C a_k(n).$$

So $\xi \in \Delta\left(\lambda^l(A)\right)$. □

Proposition 1.4.3. *If $\lambda^l(A)$ is a G_∞-space, then*

$$\Delta\left(\lambda^l(A)\right) = \bigcup_{k \in \mathbb{N}_0} \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \xi / a_k \in c_0\right\}.$$

Proof. Suppose that $\xi \in \Delta\left(\lambda^l(A)\right)$. Then, there exists $k \in \mathbb{N}_0$ with $\left(\xi_n \delta_n\left(U_k^l, U_0^l\right)\right)_{n \in \mathbb{N}_0} \in c_0$. From Proposition 1.3.5, we deduce

$$\delta_n\left(U_k^l, U_0^l\right) \geq \inf_{j \leq n} \left(\frac{a_0(j)}{a_k(j)}\right) \geq \frac{a_0(0)}{a_k(n)}.$$

Consequently, $\xi / a_k \in c_0$.

Now, we assume that $\xi \in \mathbb{C}^{\mathbb{N}_0}$ and $k \in \mathbb{N}_0$ are such that $\xi / a_k \in c_0$ and we fix $m \in \mathbb{N}_0$. Without loss of generality, we can suppose that $k \geq m$. Then, we take $j \geq k$ and $C > 0$ for which $a_k^2(n) \leq C a_j(n)$ for any n. Therefore, Proposition 1.3.5 gives

$$\delta_n\left(U_j^l, U_m^l\right) \leq \sup_{t \geq n} \left(\frac{a_m(t)}{a_j(t)}\right) \leq \sup_{t \geq n} \left(\frac{a_k(t)}{a_j(t)}\right) \leq \sup_{t \geq n} \left(\frac{C}{a_k(t)}\right) = \frac{C}{a_k(n)},$$

so $\left(\xi_n \delta_n\left(U_j^l, U_m^l\right)\right)_{n \in \mathbb{N}_0} \in c_0$. Hence $\xi \in \Delta\left(\lambda^l(A)\right)$. □

The G_∞-spaces will be considered in Section 2.2 to obtain some spaces which positively answer the open question about the diametral dimension. For this, the next characterization of Schwartz smooth sequence spaces can be interesting (it is again a generalization of the case $l = l_1$ given in [30]):

Proposition 1.4.4.

(1) If $\lambda^l(A)$ is a G_1-space, then it is Schwartz if and only if $a_k \in c_0$ for every $k \in \mathbb{N}_0$.

(2) If $\lambda^l(A)$ is a G_∞-space, then it is Schwartz if and only if there exists $k \in \mathbb{N}_0$ for which $a_k(n) \to \infty$ if $n \to \infty$.

In both cases, $\lambda^l(A)$ is not Schwartz if and only if $\lambda^l(A) = l$.

Proof. The points (1) and (2) directly follow from Theorem 1.2.9 and from Propositions 1.4.2 and 1.4.3.

Now, assume that $\lambda^l(A)$ is a non-Schwartz G_1-space. It means that there exist $k \in \mathbb{N}_0$ and $C > 0$ with $a_k(n) \geq C$ for every $n \in \mathbb{N}_0$. This implies that $\lambda^l(A) \subseteq l$ (cf. Proposition 1.3.2). Because we also have $a_j(n) \leq a_j(0)$ for all $j, n \in \mathbb{N}_0$, we get $l \subseteq \lambda^l(A)$.

Finally, if $\lambda^l(A)$ is a non-Schwartz G_∞-space, then, for every $k \in \mathbb{N}_0$, there exists $C_k > 0$ such that $a_k(0) \leq a_k(n) \leq C_k$ for any $n \in \mathbb{N}_0$, hence $\lambda^l(A) = l$. □

In the same way as for power series spaces, it is possible to show that the diametral dimension is a *complete* topological invariant for smooth sequence spaces associated to a same admissible space (and it was actually done for $l = l_1$ in [30, 32]). This is described in the following result, which generalizes the corresponding property in power series spaces.

Theorem 1.4.5. *The diametral dimension is a complete topological invariant on the class of smooth sequence spaces associated to l. More precisely, if $B = (b_k)_{k\in\mathbb{N}_0}$ is another Köthe matrix and if l' is another admissible space,*

(1) if $\lambda^l(A)$ and $\lambda^l(B)$ are two smooth sequence spaces of the same type such that $\Delta\left(\lambda^l(A)\right) = \Delta\left(\lambda^l(B)\right)$, then they are algebraically and topologically equal;

(2) if $\lambda^l(A)$ and $\lambda^{l'}(B)$ are two Schwartz smooth sequence spaces of different types, then we always have $\Delta\left(\lambda^l(A)\right) \neq \Delta\left(\lambda^{l'}(B)\right)$.

Proof. We distinguish three situations.

(a) Assume that $\lambda^l(A)$ and $\lambda^l(B)$ are two G_1-spaces with the same diametral dimension. Then, by Proposition 1.4.2, we have

$$\lambda_0(A) = \lambda_0(B).$$

By properties of Köthe echelon spaces (cf. Proposition A.2.1), it implies that $\lambda^l(A) = \lambda^l(B)$ algebraically and topologically.

(b) Now, suppose that $\lambda^l(A)$ and $\lambda^l(B)$ are G_∞-spaces with the same diametral dimension. By the previous property, we can assume that these two spaces are Schwartz (otherwise they are both equal to l). Thus, by Proposition 1.4.3, we obtain

$$\bigcup_{k\in\mathbb{N}_0} \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \xi/a_k \in c_0\right\} = \bigcup_{k\in\mathbb{N}_0} \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \xi/b_k \in c_0\right\}.$$

Because $\lambda^l(A)$ and $\lambda^l(B)$ are Schwartz, it implies that, for every $m \in \mathbb{N}_0$, $a_m, b_m \in \Delta\left(\lambda^l(A)\right) = \Delta\left(\lambda^l(B)\right)$. So there exist $k \geq m$ and $C > 0$ with $a_m(n) \leq Cb_k(n)$ and $b_m(n) \leq Ca_k(n)$ for every $n \in \mathbb{N}_0$. Then, this shows that $\lambda^l(A)$ and $\lambda^l(B)$ algebraically and topologically coincide (cf. Proposition A.2.1).

(c) If now $\lambda^l(A)$ is a Schwartz G_1-space, $\lambda^{l'}(B)$ is a Schwartz G_∞-space, and if they have the same diametral dimension, we have

$$\lambda_0(A) = \bigcup_{k\in\mathbb{N}_0} \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \xi/b_k \in c_0\right\} = \bigcup_{k\in\mathbb{N}_0} \lambda_0(\{1/b_k\}).$$

Then, by Grothendieck's Factorization Theorem (cf. Corollary A.1.5), this implies there exists $k_0 \in \mathbb{N}_0$ with $\lambda_0(A) = \lambda_0(\{1/b_{k_0}\})$. In particular, Closed Graph Theorem (cf. Proposition A.2.1) implies that the spaces $\lambda_0(A)$ and $\lambda_0(\{1/b_{k_0}\})$ have the same topology, which is impossible because the first one is Schwartz and the second one is Banach.

Hence the conclusion. □

Corollary 1.4.6. *Two smooth sequence spaces of the same type and associated to the same admissible space are isomorphic if and only if they are algebraically and topologically equal. Moreover, two Schwartz smooth sequence spaces of different types are never isomorphic.*

Proof. It is straightforward by the previous theorem. □

Thanks to all these notions and examples in Köthe sequence spaces, we are now ready to deal with the open question about the diametral dimension, which is directly studied in the next chapter.

Chapter 2

The open question and some positive results

In this chapter, we present the open question concerning the diametral dimension. We also present some positive partial answers to this question, first in the context of Köthe sequence spaces and after in a more general setting.

2.1 Another diametral dimension

In [25], Mityagin implicitly defined another diametral dimension by

$$\Delta_b(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U},\ \forall B \text{ bounded set in } E, (\xi_n \delta_n(B,U))_{n\in\mathbb{N}_0} \in c_0 \right\},$$

where E is a locally convex space (or, more generally, a topological vector space) and $\mathcal{U}$ is a basis of 0-neighbourhoods[3] in E.

Then, Mityagin claimed that $\Delta(E) = \Delta_b(E)$ for any Fréchet space E, referring to a forthcoming joint paper with Bessaga, Pelczynski, and Rolewicz which, as far as the author knows, was never published. Such a result is even impossible, as we will see below.

In fact, in the original version, Mityagin only considered compact sets B in the definition of $\Delta_b(E)$, rather than bounded sets. But, then, the equality of Δ and Δ_b is already false for infinite-dimensional Banach spaces (they are not Schwartz, so $\Delta(E) = c_0$, but in that case, by Proposition 1.1.10, $\Delta_b(E) \supseteq l_\infty$).

In [34], Terzioğlu claimed that Δ and Δ_b coincide for quasinormable metrizable locally convex spaces. However, Frerick and Wengenroth discovered a gap in his proof, which implies that the space has a bounded 0-neighbourhood when it is Montel and has a continuous norm.

The aims of the current section are to study this second diametral dimension Δ_b and to bring some information about the open question and the topological properties which help to solve it.

First of all, an inclusion between the diametral dimensions is always verified.

[3] Again, this definition is independent of the choice of $\mathcal{U}$.

Proposition 2.1.1. *For any topological vector space E, we have*

$$\Delta(E) \subseteq \Delta_b(E).$$

Proof. It follows from the definition of bounded sets and from Propositions 1.1.3 and 1.1.5. □

Next, we recall that Schwartz locally convex spaces are characterized thanks to the first diametral dimension Δ (cf. Theorem 1.2.9). This property also means that the diametral dimension Δ is "interesting" only when the considered space is Schwartz, otherwise it is reduced to c_0.

Naturally, we can wonder whether a similar characterization exists when we consider Δ_b. Since some notions of precompactness are needed to prove Theorem 1.2.9 (via Proposition 1.1.10), one could expect that Δ_b characterizes properties of type *Montel*. For this, we recall the following definitions:

Definition 2.1.2. A locally convex space is

- *semi-Montel* if all its bounded sets are relatively compact;
- *Montel* if it is semi-Montel and barrelled.

However, in this context, we only need precompactness – and not relative compactness. This is why we define this slight variation of the previous notions:

Definition 2.1.3. A locally convex space is *pseudo-Montel* if all its bounded sets are precompact.

Of course, the notions of being semi-Montel and pseudo-Montel coincide for complete locally convex spaces, and are also equivalent to being Montel in Fréchet spaces.

In fact, this notion of "pseudo-Montel" is linked to Schwartz spaces thanks to the *quasinormability* (which is used by Terzioğlu in [34]).

Definition 2.1.4. A locally convex space E is *quasinormable* if, for every 0-neighbourhood U in E, there exists a 0-neighbourhood V included in U such that, for every $\varepsilon > 0$, there exists a bounded set B with

$$V \subseteq \varepsilon U + B.$$

Then, we can prove the following equivalence, which is well-known in Fréchet spaces.

Proposition 2.1.5. *A locally convex space is Schwartz if and only if it is pseudo-Montel and quasinormable.*

Proof. It is direct by the definitions. □

Now, it is easy to prove the next characterization thanks to Proposition 1.1.10.

Theorem 2.1.6. *Let E be a locally convex space. The following are equivalent:*

(1) E is pseudo-Montel;

(2) $l_\infty \subseteq \Delta_b(E)$;

(3) $c_0 \subsetneq \Delta_b(E)$.

Proof. The argument is exactly the same as for Theorem 1.2.9. □

Now, if we put Theorems 1.2.9 and 2.1.6 together, we deduce this obvious – but useful – property:

Proposition 2.1.7. *Let E be a locally convex space.*

(1) If E is not pseudo-Montel, then $\Delta(E) = \Delta_b(E) = c_0$.

(2) If E is pseudo-Montel but not Schwartz, then $\Delta(E) = c_0 \subsetneq \Delta_b(E)$.

Since there exist Fréchet-Montel spaces which are not Schwartz (cf. for instance [9, 24]), this particularly means that the two diametral dimensions cannot be equal in general for any Fréchet space. Moreover, the last property shows that the open question is completely solved for non-Schwartz locally convex spaces.

Consequently, we just have to study this problem in the context of (Fréchet-)Schwartz spaces. Besides, the assumption of quasinormability considered by Terzioğlu has already been implicitly treated: if a quasinormable space is not pseudo-Montel, the diametral dimensions are equal and if it is pseudo-Montel, then it is Schwartz.

As a conclusion, the open question raised by Mityagin turns to be the following one:

$$\boxed{\textbf{Do we have the equality } \Delta(E) = \Delta_b(E) \textbf{ for every Fréchet-Schwartz space } E\textbf{?}}$$

2.2 A first approach by Köthe spaces

When we began to treat this open question ([6]), we first considered the (regular) Köthe sequence spaces, as they constitute some examples for which the "classic" diametral dimension is easily computable. In fact, we originally obtained a positive result for power series spaces, thanks to two properties which thereafter appeared to be also verified by regular spaces and by G_∞-spaces (see below).

We then translated these properties in the more general context of metrizable locally convex spaces. To present them, we introduce a slight variation of the first diametral dimension.

Definition 2.2.1. If E is a topological vector space and $\mathcal{U}$ is a basis of 0-neighbourhoods in E, we put

$$\Delta^\infty(E) := \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, V \subseteq U, \text{ such that } (\xi_n \delta_n(V,U))_{n\in\mathbb{N}_0} \in l_\infty\right\}.$$

The first assumption we used is that the equality $\Delta(E) = \Delta^\infty(E)$ is verified. Remark that, since $l_\infty \subseteq \Delta^\infty(E)$, such an equality is only possible for Schwartz spaces. Moreover, it is easy to check that it is true for regular Köthe-Schwartz sequence spaces thanks to Proposition 1.3.14 and Theorem 1.3.7. Actually, we will even prove that this equality is verified by any Köthe-Schwartz echelon space (cf. Proposition 2.2.5).

The second assumption corresponds to the next definition – the name of which was chosen to imitate the terminology of Terzioğlu with the prominent bounded sets. From now on, we fix a metrizable locally convex space E with a decreasing basis of 0-neighbourhoods $(U_k)_{k\in\mathbb{N}_0}$.

Definition 2.2.2. The space E has the *property of large bounded sets* if, for every $m \in \mathbb{N}_0$ and every sequence $(r_k)_{k\geq m}$ of strictly positive numbers, there exist $M \geq m$ and a bounded set B with

$$\delta_n(B, U_M) \geq \inf_{k\geq m} (r_k \delta_n(U_k, U_m)) \qquad (n \in \mathbb{N}_0).$$

Thanks to the two notions introduced above, we can prove the following result ([6]):

Theorem 2.2.3. *If E has the property of large bounded sets and verifies $\Delta(E) = \Delta^\infty(E)$, then*

$$\Delta(E) = \Delta_b(E).$$

Proof. Suppose that $\xi \notin \Delta(E)$. Then, there exist $m \in \mathbb{N}_0$ and a strictly increasing sequence $(n(j))_{j\geq m}$ of $\mathbb{N}_0$ with

$$\left|\xi_{n(j)}\right| \delta_{n(j)}(U_j, U_m) \geq 1.$$

Using Proposition 1.1.3, this leads to

$$\left|\xi_{n(j)}\right| \delta_{n(j)}(U_k, U_m) \geq 1 \qquad \forall j, k \in \mathbb{N}_0,\ j \geq k \geq m.$$

After that, for each $k \geq m$, we define[4]

$$r_k := \sup\left\{1,\ \sup_{m\leq j\leq k} \frac{1}{\left|\xi_{n(j)}\right| \delta_{n(j)}(U_k, U_m)}\right\}.$$

So, we have

$$r_k \left|\xi_{n(j)}\right| \delta_{n(j)}(U_k, U_m) \geq 1 \qquad \forall j, k \in \mathbb{N}_0,\ j \geq m \text{ and } k \geq m.$$

Therefore, it also means that

$$\left|\xi_{n(j)}\right| \inf_{k\geq m} \left(r_k \delta_{n(j)}(U_k, U_m)\right) \geq 1$$

[4]When $j \leq k$, remark that we have $\delta_{n(j)}(U_k, U_m) \geq \delta_{n(k)}(U_k, U_m) > 0$.

for every $j \geq m$. However, since E has the property of large bounded sets, there exist $M \geq m$ and a bounded set B for which

$$\delta_n(B, U_M) \geq \inf_{k \geq m} (r_k \delta_n(U_k, U_m)) \qquad (n \in \mathbb{N}_0).$$

This gives the conclusion because, in that case,

$$\left|\xi_{n(j)}\right| \delta_{n(j)}(B, U_M) \geq 1$$

for all $j \in \mathbb{N}_0$, which implies that $\xi \notin \Delta_b(E)$. □

Now, we show that regular spaces and G_∞-spaces verify the two hypotheses in the previous theorem. First, we prove that Δ and Δ^∞ are always equal for Köthe echelon spaces. For this, we fix a Köthe matrix $A = (a_k)_{k \in \mathbb{N}_0}$ and an admissible space l and we consider the next lemma ([6]), which is originally inspired by the proof of Lemma 3.2.9, presented later in this work.

Lemma 2.2.4. *Let $\varepsilon > 0$ be given. If $j, k, m \in \mathbb{N}_0$ and $N \in \mathbb{N}_0$ are such that $j > k > m$, $a_m/a_k \in c_0$, and*

$$\frac{a_m(n)}{a_j(n)} \leq \varepsilon \frac{a_m(n)}{a_k(n)} \qquad \forall n \geq N,$$

then, there exists $N_0 \geq N$ with

$$\delta_n\left(U_j^l, U_m^l\right) \leq \varepsilon \delta_n\left(U_k^l, U_m^l\right) \qquad \forall n \geq N_0.$$

Proof. First of all, since the sequences a_m/a_j and a_m/a_k converge to 0, it means that the sets

$$J_j := \left\{t \in \mathbb{N}_0 : \inf_{0 \leq i \leq N} \left(\frac{a_m(i)}{a_j(i)}\right) \leq \frac{a_m(t)}{a_j(t)}\right\}$$

and

$$J_k := \left\{t \in \mathbb{N}_0 : \inf_{0 \leq i \leq N} \left(\frac{a_m(i)}{a_k(i)}\right) \leq \frac{a_m(t)}{a_k(t)}\right\}$$

are finite. Moreover, we have $\{0, ..., N\} \subseteq J_j \cap J_k$ and the definitions of J_j and J_k even imply that

$$J_j = \{\varphi(a_m/a_j, 0), ..., \varphi(a_m/a_j, \#J_j - 1)\}$$

and

$$J_k = \{\varphi(a_m/a_k, 0), ..., \varphi(a_m/a_k, \#J_k - 1)\}.$$

In this situation, we put $N_0 := \sup\{\#J_j, \#J_k\}$. It is easy to check that we have $N_0 > N$.

Now, let us fix $n \geq N_0$. We define

$$J_j' := \{\varphi(a_m/a_j, 0), ..., \varphi(a_m/a_j, n-1)\} \quad \text{and} \quad J_k' := \{\varphi(a_m/a_k, 0), ..., \varphi(a_m/a_k, n-1)\}.$$

Then, two situations are possible.

1. If $J'_j = J'_k =: J'$, then, by Propositions 1.3.5 and 1.3.9,

$$\delta_n \left(U^l_j, U^l_m \right) = \sup_{i \notin J'} \left(\frac{a_m(i)}{a_j(i)} \right) \leq \varepsilon \sup_{i \notin J'} \left(\frac{a_m(i)}{a_k(i)} \right) = \varepsilon \delta_n \left(U^l_k, U^l_m \right)$$

   because $\{0, ..., N\} \subseteq J_j \cap J_k \subseteq J'$.

2. If $J'_j \neq J'_k$, then there exists $i \in J'_j \setminus J'_k$, because $\# J'_j = \# J'_k = n$. In particular, $i \geq N$ since we have $\{0, ..., N\} \subseteq J_k \subseteq J'_k$. Moreover, by definition of J'_j, there exists $i_0 < n$ with $i = \varphi(a_m/a_j, i_0)$. Thus, by Proposition 1.3.9,

$$\delta_n \left(U^l_j, U^l_m \right) \leq \delta_{i_0} \left(U^l_j, U^l_m \right) = \frac{a_m(i)}{a_j(i)} \leq \varepsilon \frac{a_m(i)}{a_k(i)} \leq \varepsilon \sup_{t \notin J'_k} \left(\frac{a_m(t)}{a_k(t)} \right) = \varepsilon \delta_n \left(U^l_k, U^l_m \right).$$

Hence the conclusion. □

Thanks to this result, we are now ready to prove this property:

Proposition 2.2.5. *If $\lambda^l(A)$ is Schwartz, then*

$$\Delta \left(\lambda^l(A) \right) = \Delta^\infty \left(\lambda^l(A) \right).$$

Proof. Let $\xi \in \Delta^\infty \left(\lambda^l(A) \right)$ and $m \in \mathbb{N}_0$ be given. By assumption, there exist $k > m$ and $C > 0$ such that $a_m/a_k \in c_0$ and, for any $n \in \mathbb{N}_0$,

$$|\xi_n| \delta_n \left(U^l_k, U^l_m \right) \leq C.$$

Besides, there exists $j > k$ such that $a_k/a_j \in c_0$. Therefore, if we fix $\varepsilon > 0$, there exists $N \in \mathbb{N}_0$ for which

$$\frac{a_k(n)}{a_j(n)} \leq \frac{\varepsilon}{C}$$

for each $n \geq N$. Consequently,

$$\frac{a_m(n)}{a_j(n)} = \frac{a_m(n)}{a_k(n)} \frac{a_k(n)}{a_j(n)} \leq \frac{\varepsilon}{C} \frac{a_m(n)}{a_k(n)}$$

for all $n \geq N$. Now, if we use the previous lemma, there exists $N_0 \geq N$ such that, for every $n \geq N_0$,

$$\delta_n \left(U^l_j, U^l_m \right) \leq \frac{\varepsilon}{C} \delta_n \left(U^l_k, U^l_m \right).$$

Then, if $n \geq N_0$,

$$|\xi_n| \delta_n \left(U^l_j, U^l_m \right) \leq \frac{\varepsilon}{C} |\xi_n| \delta_n \left(U^l_k, U^l_m \right) \leq \varepsilon.$$

This shows that $\left(\xi_n \delta_n \left(U^l_j, U^l_m \right) \right)_{n \in \mathbb{N}_0} \in c_0$, so $\xi \in \Delta \left(\lambda^l(A) \right)$. □

After this result, we can consider the property of large bounded sets in the context of Köthe sequence spaces. For this, we first prove the following result, inspired by Proposition 1.3.5. Once again, it can be proved using Tikhomirov's Theorem (Proposition 6.1.4); here, we just present the arguments developed in [6].

Proposition 2.2.6. *Let $(r_k)_{k\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$, $m, n, \in \mathbb{N}_0$, and $J \subseteq \mathbb{N}_0$, with $\#J = n+1$, be given. Then*

$$\inf_{k\geq m}\left(r_k \inf_{j\in J}\left(\frac{a_m(j)}{a_k(j)}\right)\right) \leq \delta_n\left(\bigcap_{k\geq m} r_k U_k^l, U_m^l\right) \leq \inf_{k\geq m}\left(r_k \delta_n\left(U_k^l, U_m^l\right)\right).$$

Proof. 1. Suppose there exist $\delta > 0$, with $\delta < \delta_0 := \inf_{k\geq m}\left(r_k \inf_{j\in J}\left(\frac{a_m(j)}{a_k(j)}\right)\right)$, and $L \in \mathcal{L}_n\left(\lambda^l(A)\right)$ such that

$$\bigcap_{k\geq m} r_k U_k^l \subseteq \delta U_m^l + L.$$

As in Proposition 1.3.5, we define a projection $P_J : \xi \in \lambda^l(A) \mapsto \sum_{j\in J} \xi_j e_j$ and we put $G := P_J\left(\lambda^l(A)\right)$.

Now, if $\xi \in \lambda^l(A)$, $k \geq m$, and $j \in J$, we have

$$|a_k(j)\xi_j| = \frac{a_k(j)}{r_k a_m(j)} r_k |a_m(j)\xi_j| \leq \frac{1}{\delta_0} r_k |a_m(j)\xi_j|,$$

so $p_k^l(\xi) \leq \frac{r_k}{\delta_0} p_m^l(\xi)$ when $\xi \in G$. We deduce from this the inclusion

$$U_m^l \cap G \subseteq \frac{1}{\delta_0}\left[\left(\bigcap_{k\geq m} r_k U_k^l\right) \cap G\right].$$

But we also have

$$\left(\bigcap_{k\geq m} r_k U_k^l\right) \cap G \subseteq \delta U_m^l \cap G + P_J(L),$$

which implies that $U_m^l \cap G \subseteq \frac{\delta}{\delta_0} U_m^l \cap G + P_J(L)$. By the same developments as in the proof of Proposition 1.3.5, this shows that $G \subseteq P_J(L)$, which is impossible because $\dim G = n + 1 > n \geq \dim(P_J(L))$.

2. The second inequality is straightforward thanks to Propositions 1.1.3 and 1.1.5.

Hence the conclusion. □

This property leads to the following result in the context of regular spaces, which shows that regular spaces have the property of large bounded sets:

Proposition 2.2.7. *Assume that A is regular. If $(r_k)_{k\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$, $m, n, \in \mathbb{N}_0$, then*

$$\delta_n \left(\bigcap_{k\geq m} r_k U_k^l, U_m^l \right) = \inf_{k\geq m} \left(r_k \frac{a_m(n)}{a_k(n)} \right).$$

Proof. It is direct by taking $J := \{0, ..., n\}$ in the previous proposition and by Proposition 1.3.14. □

Corollary 2.2.8. *If A is regular, then $\lambda^l(A)$ has the property of large bounded sets.*

Proof. It is clear by the last result: if $m \in \mathbb{N}_0$ and a sequence $(r_k)_{k\geq m}$ of strictly positive numbers are given, then the bounded set

$$B := \bigcap_{k\geq m} r_k U_k^l$$

verifies

$$\delta_n \left(B, U_m^l \right) = \inf_{k\geq m} \left(r_k \delta_n \left(U_k^l, U_m^l \right) \right).$$

for all $n \in \mathbb{N}_0$. □

And so, gathering everything, we obtain a first family of spaces which positively answer our open question:

Theorem 2.2.9. *If A is regular, then*

$$\Delta \left(\lambda^l(A) \right) = \Delta_b \left(\lambda^l(A) \right).$$

Proof. If $\lambda^l(A)$ is not Schwartz, then it is normed by Proposition 1.3.13 and so non-Montel. We know that it implies the equality of the two diametral dimensions.

If now $\lambda^l(A)$ is Schwartz, then it verifies $\Delta\left(\lambda^l(A)\right) = \Delta^\infty\left(\lambda^l(A)\right)$ and has the property of large bounded sets. We conclude by Theorem 2.2.3. □

After regular spaces, we now consider the G_∞-spaces, which also verify the two conditions of Theorem 2.2.3.

Proposition 2.2.10. *If $\lambda^l(A)$ is a G_∞-space, then it has the property of large bounded sets.*

Proof. Let us fix $m \in \mathbb{N}_0$ and a sequence $(r_k)_{k\geq m}$ of strictly positive numbers. If we use the definition of G_∞-spaces, we know that, for every $k \geq m$, there exist $j(k) \geq k$ and $C_k > 0$ for which

$$a_k^2(n) \leq C_k a_{j(k)}(n) \qquad \forall n \in \mathbb{N}_0 .$$

Next, we put

$$s_k := \frac{C_k r_{j(k)}}{a_m(0)}.$$

Then, by Propositions 2.2.6 and 1.3.5, this implies

$$\begin{aligned}
\delta_n\left(\bigcap_{k\geq m} s_k U_k^l, U_m^l\right) &\geq \inf_{k\geq m}\left(s_k \inf_{t\leq n}\left(\frac{a_m(t)}{a_k(t)}\right)\right)\\
&\geq a_m(0)\inf_{k\geq m}\left(s_k\frac{1}{a_k(n)}\right)\\
&= a_m(0)\inf_{k\geq m}\left(s_k\sup_{t\geq n}\left(\frac{1}{a_k(t)}\right)\right)\\
&\geq a_m(0)\inf_{k\geq m}\left(\frac{s_k}{C_k}\sup_{t\geq n}\left(\frac{a_k(t)}{a_{j(k)}(t)}\right)\right)\\
&\geq a_m(0)\inf_{k\geq m}\left(\frac{s_k}{C_k}\sup_{t\geq n}\left(\frac{a_m(t)}{a_{j(k)}(t)}\right)\right)\\
&\geq a_m(0)\inf_{k\geq m}\left(\frac{s_k}{C_k}\delta_n\left(U_{j(k)}^l, U_m^l\right)\right)\\
&= \inf_{k\geq m}\left(r_{j(k)}\delta_n\left(U_{j(k)}^l, U_m^l\right)\right)\\
&\geq \inf_{k\geq m}\left(r_k\delta_n\left(U_k^l, U_m^l\right)\right).
\end{aligned}$$

So, the bounded set $B := \bigcap_{k\geq m} s_k U_k^l$ is such that

$$\delta_n\left(B, U_m^l\right) \geq \inf_{k\geq m}\left(r_k\delta_n\left(U_k^l, U_m^l\right)\right),$$

hence the conclusion. □

Consequently, we have another family of Köthe sequence spaces for which the two diametral dimensions are equal:

Theorem 2.2.11. *If $\lambda^l(A)$ is a G_∞-space, then*

$$\Delta\left(\lambda^l(A)\right) = \Delta_b\left(\lambda^l(A)\right).$$

Proof. If $\lambda^l(A)$ is not Schwartz, then it is the normed space l (by Proposition 1.4.4), so it verifies the equality of the two diametral dimensions.

If it is Schwartz, then the two conditions of Theorem 2.2.3 are verified, so the equality is also true in that case. □

Given these positive results for regular spaces and G_∞-spaces, we can wonder whether the third class of “practical” Köthe sequence spaces for the diametral dimension, namely the G_1-spaces, also verify the equality of Δ and Δ_b. In fact, in Section 3.2, we will see it is the case and even more: such spaces have prominent bounded sets.

But, at this stage, we do not know whether the two diametral dimensions are equal for *any* Köthe sequence space. Therefore, in order to understand what happens for non-regular and "non-smooth" spaces, we develop a little bit more Proposition 2.2.6 thanks to the decreasing-reorganization map.

From now on, we assume that $\lambda^l(A)$ is Schwartz and, without loss of generality, that $a_m/a_k \in c_0$ if $m, k \in \mathbb{N}_0$ and $k > m$. Moreover, in the next result, we consider the following notion : if, for every j, $x^{(j)} \in [0,\infty)^{\mathbb{N}_0}$, then the sequence $\inf_{j\in\mathbb{N}_0} x^{(j)}$ is defined by $\left(\inf_{j\in\mathbb{N}_0} x^{(j)}\right)_n = \inf_{j\in\mathbb{N}_0} x_n^{(j)}$.

Proposition 2.2.12. *Let $(r_k)_{k\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$ and $m, n, \in \mathbb{N}_0$ be given. Then*

$$\pi_n\left(\inf_{k\geq m}\left(r_k\left(a_m/a_k\right)\right)\right) \leq \delta_n\left(\bigcap_{k\geq m} r_k U_k^l, U_m^l\right) \leq \inf_{k\geq m}\left(r_k \pi_n(a_m/a_k)\right).$$

Proof. If x is the sequence defined by $x_j := \inf_{k\geq m}\left(r_k\left(\frac{a_m(j)}{a_k(j)}\right)\right)$ for all $j \in \mathbb{N}_0$, then, for $J = \{\varphi(x,0), ..., \varphi(x,n)\}$, Proposition 2.2.6 gives

$$\begin{aligned}\delta_n\left(\bigcap_{k\geq m} r_k U_k^l, U_m^l\right) &\geq \inf_{k\geq m}\left(r_k \inf_{j\in J}\left(\frac{a_m(j)}{a_k(j)}\right)\right)\\ &= \inf_{j\in J}\left[\inf_{k\geq m}\left(r_k\left(\frac{a_m(j)}{a_k(j)}\right)\right)\right]\\ &= \pi_n\left(\inf_{k\geq m}\left(r_k\left(a_m/a_k\right)\right)\right).\end{aligned}$$

The other inequality follows from Proposition 1.3.9. □

Unfortunately, we have no exact value for Kolmogorov's diameters in the previous result, but only approximations. Nevertheless, for regular spaces, we have

$$\delta_n\left(\bigcap_{k\geq m} r_k U_k^l, U_m^l\right) = \pi_n\left(\inf_{k\geq m}\left(r_k\left(a_m/a_k\right)\right)\right) = \inf_{k\geq m}\left(r_k \pi_n(a_m/a_k)\right)$$

by Propositions 2.2.7 and 1.3.14: so, the lower and upper bounds in the last property are the best approximations for Kolmogorov's diameters of that kind.

In fact, in some cases, we can say even more:

Proposition 2.2.13. *Assume that $l = l_\infty$. If $(r_k)_{k\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$ and $m, n, \in \mathbb{N}_0$ are given, we have*

$$\delta_n\left(\bigcap_{k\geq m} r_k U_k^l, U_m^l\right) = \pi_n\left(\inf_{k\geq m}\left(r_k\left(a_m/a_k\right)\right)\right).$$

Proof. In fact, we have

$$\begin{aligned}\bigcap_{k\geq m} r_k U_k^l &= \left\{\xi \in \lambda^l(A) : \forall k \geq m,\ \sup_{n\in\mathbb{N}_0} |a_k(n)\xi_n| \leq r_k\right\}\\ &= \left\{\xi \in \lambda^l(A) : \sup_{n\in\mathbb{N}_0} \left|\sup_{k\geq m}\left(\frac{a_k(n)}{r_k}\right)\xi_n\right| \leq 1\right\}\\ &= \left\{\xi \in \lambda^l(A) : \sup_{n\in\mathbb{N}_0} |\beta_m(n)\xi_n| \leq 1\right\},\end{aligned}$$

where $\beta_m(n) := \sup_{k\geq m}(a_k(n)/r_k) \in [0,\infty]$. This means that $\bigcap_{k\geq m} r_k U_k^l = U_{\beta_m}^l$. Thus, using Proposition 1.3.9,

$$\delta_n\left(\bigcap_{k\geq m} r_k U_k^l, U_m^l\right) = \pi_n(a_m/\beta_m) = \pi_n\left(\inf_{k\geq m}\left(r_k\left(a_m/a_k\right)\right)\right).$$

Hence the conclusion. □

Comparing these results with what happens for regular spaces in Corollary 2.2.8, we could prove the equality of the two diametral dimensions if we had for instance

$$\inf_{k\geq m}\left(r_k\pi_n(a_m/a_k)\right) \leq \pi_n\left(\inf_{k\geq m}\left(r_k\left(a_m/a_k\right)\right)\right).$$

Unfortunately, we do not know whether this inequality is always true, except in the very particular case of regular spaces.

More generally, another way to proceed would be to translate the property of large bounded sets in the context of Köthe spaces. More precisely, using Proposition 2.2.12, we would like to have the following property: if $m \in \mathbb{N}_0$ is given and if $(r_k)_{k\geq m}$ is a sequence of strictly positive numbers, there exist $M \geq m$ and another sequence of strictly positive numbers $(s_k)_{k\geq M}$ with

$$\pi_n\left(\inf_{k\geq M}\left(s_k\left(a_M/a_k\right)\right)\right) \geq \inf_{k\geq m}\left(r_k\pi_n(a_m/a_k)\right)$$

for every $n \in \mathbb{N}_0$.

Nevertheless, we do not know whether this property is always verified when the considered space is not regular or not G_∞, so that it is unknown whether each Köthe echelon space has the property of large bounded sets. Because of that, the question to know whether there exist some spaces without the property of large bounded sets remains open.

Consequently, to obtain more general results for the equality of the two diametral dimensions – at least for Köthe spaces –, we have to use other ideas and concepts. It is actually the topic of the next section.

2.3 Generalization to Schwartz metrizable spaces and applications to Hilbertizable spaces

Previously, we defined the property of large bounded sets to obtain the equality of the two diametral dimensions, with the additional assumption that Δ and Δ^∞ coincide. Nevertheless, in this definition, we use the existence of particular bounded sets B which, in some sense, comes "from the outside". In this situation, we can wonder whether it would be possible to adapt developments in Theorem 2.2.3 with an explicit construction of bounded sets with convenient properties. This idea finally led to a new argument ([13]), which is however still based on the assumption $\Delta = \Delta^\infty$.

In order to present this argument, we first define a new variation of diametral dimensions.

Definition 2.3.1. If E is a locally convex space (or a topological vector space) and $\mathcal{U}$ is a basis of 0-neighbourhoods in E, then we set

$$\Delta_b^\infty(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded set in } E, (\xi_n \delta_n(B,U))_{n\in\mathbb{N}_0} \in l_\infty \right\}.$$

Here is a diagram summarizing the inclusions between the diametral dimensions and their variations:

$$\begin{array}{ccc} \Delta(E) & \subseteq & \Delta_b(E) \\ \rotatebox{90}{$\supseteq$} & & \rotatebox{90}{$\supseteq$} \\ \Delta^\infty(E) & \subseteq & \Delta_b^\infty(E) \end{array}$$

Before presenting the links between the variations of the two diametral dimensions, we give another description of precompactness.

Remark 2.3.2. Let E be a vector space. If U is an absolutely convex subset of E, then $V \subseteq E$ is precompact with respect to U if and only if, for all $\varepsilon > 0$, there exists a finite set $P \subseteq V$ such that $V \subseteq \varepsilon U + P$.

Proof. Assume that V is precompact with respect to U and fix $\varepsilon > 0$. Then, there exists a finite subset P of E with

$$V \subseteq \frac{\varepsilon}{2}U + P = \bigcup_{p\in P} \left(p + \frac{\varepsilon}{2}U\right).$$

In that case, we can suppose that, for every $p \in P$, we have $V \cap \left(p + \frac{\varepsilon}{2}U\right) \neq \emptyset$. In particular, we can choose $v(p) \in V \cap \left(p + \frac{\varepsilon}{2}U\right)$ for each $p \in P$. Then, if we set $P' := \{v(p) : p \in P\}$, we have $P' \subseteq V$ and $P \subseteq \frac{\varepsilon}{2}U + P'$. Therefore, we obtain

$$V \subseteq \varepsilon U + P'.$$

Hence the conclusion. $\square$

Now, we can prove the following property ([13]), based on the variations of the diametral dimensions:

Theorem 2.3.3. *Let E be a Schwartz metrizable locally convex space. Then $\Delta^\infty(E) = \Delta_b^\infty(E)$.*

Proof. Let $(U_k)_{k\in\mathbb{N}_0}$ be a decreasing basis of absolutely convex 0-neighbourhoods in E such that U_{k+1} is precompact with respect to U_k for any k.

Then, we fix $\xi \in \Delta_b^\infty(E)$ and we assume there exists $m \in \mathbb{N}_0$ such that, for every $k \geq m$, the sequence $(\xi_n\delta_n(U_k, U_m))_{n\in\mathbb{N}_0}$ is unbounded.

In particular, this means there exists a strictly increasing sequence $(n(k))_{k\geq m}$ of $\mathbb{N}_0$ with

$$\left|\xi_{n(k)}\right| \delta_{n(k)}(U_k, U_m) > k$$

for each $k \geq m$. But, using precompactness between 0-neighbourhoods, we know there exist finite sets $P_k \subseteq U_k$ $(k > m)$ with

$$U_k \subseteq \frac{1}{\left|\xi_{n(k)}\right|} U_m + P_k.$$

In this situation, the set $B := \bigcup_{k>m} P_k$ is bounded in E. Indeed, for every $K > m$, we have $\bigcup_{k\geq K} P_k \subseteq U_K$ and the set $\bigcup_{k=m+1}^{K} P_k$ is finite.

Therefore, since $\xi \in \Delta_b^\infty(E)$, there exists $C > 0$ with $|\xi_n|\delta_n(B, U_m) < C$ for every $n \in \mathbb{N}_0$.

Now, we fix $k \geq \sup\{m+1, C+1\}$. Then, there exists $L \in \mathcal{L}_{n(k)}(E)$ for which

$$B \subseteq \frac{C}{\left|\xi_{n(k)}\right|} U_m + L.$$

We deduce from this

$$U_k \subseteq \frac{1}{\left|\xi_{n(k)}\right|} U_m + P_k \subseteq \frac{1}{\left|\xi_{n(k)}\right|} U_m + B \subseteq \frac{1}{\left|\xi_{n(k)}\right|} U_m + \frac{C}{\left|\xi_{n(k)}\right|} U_m + L \subseteq \frac{C+1}{\left|\xi_{n(k)}\right|} U_m + L,$$

so $\delta_{n(k)}\left(U_k, U_m\right) \leq \frac{C+1}{\left|\xi_{n(k)}\right|}$. Thus, gathering everything, we finally have

$$k < \left|\xi_{n(k)}\right| \delta_{n(k)}(U_k, U_m) \leq C + 1 \leq k,$$

which is of course impossible. □

This theorem intuitively means that the two diametral dimensions Δ and Δ_b are quite close to each other for Schwartz metrizable spaces, since two "slight" variations of them are equal. More precisely, we obtain a generalization of Theorem 2.2.3 ([13]):

Theorem 2.3.4. *If E is a Schwartz metrizable locally convex space and if $\Delta(E) = \Delta^\infty(E)$, then*

$$\Delta(E) = \Delta_b(E) = \Delta^\infty(E) = \Delta_b^\infty(E).$$

Proof. It follows from last result and from the inclusions between the diametral dimensions and their variations. □

For instance, using Proposition 2.2.5, this directly gives:

Theorem 2.3.5. *If l is an admissible space, if A is a Köthe matrix, and if $\lambda^l(A)$ is Schwartz, then*

$$\Delta\left(\lambda^l(A)\right) = \Delta_b\left(\lambda^l(A)\right).$$

So, the main examples of Schwartz, metrizable, locally convex spaces for which the diametral dimension is computable verify $\Delta = \Delta_b$. Because of this observation, it seems very difficult to find potential counterexamples to our open question in Schwartz metrizable spaces (but not impossible for non-metrizable spaces, as we will see in Section 4.1).

In this situation, we have to check if Δ and Δ^∞ coincide in Schwartz metrizable spaces. Unfortunately, this question is still open today.

Nevertheless, there exists another specific class of Schwartz metrizable spaces for which Δ and Δ^∞ are equal, namely the *Hilbertizable spaces.*

First, we consider the definition of these spaces:

Definition 2.3.6. Let E be a locally convex space.

(1) A seminorm p on E is a *Hilbert seminorm* if there exists a semi-scalar product $< .,. >$ on E such that $p(x) = \sqrt{< x, x >}$ for each $x \in E$.

(2) The space E is *Hilbertizable* if its topology can be defined thanks to a fundamental system of Hilbert seminorms.

A Hilbertizable Fréchet space is sometimes called a *Fréchet-Hilbert space.*

Of course, Hilbert spaces are Hilbertizable, but it is also the case, for instance, for Köthe spaces of type $\lambda_2(A)$ or even for nuclear spaces (see e.g. [24, 38]).

When we consider Hilbertizable spaces, it is possible to translate the inclusions between 0-neighbourhoods in terms of operators between Hilbert spaces. To understand this fact, we recall the following classic notion:

Definition 2.3.7. Let p be a continuous seminorm on the locally convex convex space E. Then, $\ker(p)$ is a vector subspace of E, so that we can consider the vector space

$$E/\ker(p)$$

and the associated quotient map $\Phi_p : E \to E/\ker(p)$. Since $p(x + y) = p(x)$ for all $x \in E, y \in \ker(p)$, we can define a norm $\|.\|_p$ on $E/\ker(p)$ by $\|\Phi_p(x)\|_p = p(x)$ if $x \in E$. In particular, if U is the closed unit ball associated to p in E, then the topology of $(E/\ker(p), \|.\|_p)$ is defined by the unit ball $\Phi_p(U)$.

With these notions, we define the *local Banach space for the seminorm p* as the Banach space

$$E_p := (E/\ker(p), \|.\|_p)^\wedge,$$

where the symbol $\wedge$ refers to the completion. In fact, the topology of E_p is defined by a norm which is an extension of $\|.\|_p$ to E_p; for simplicity, we will keep the notation $\|.\|_p$ for this extension.

If p is a Hilbert seminorm, then E_p is called the *local Hilbert space for the seminorm* p and is obviously a Hilbert space.

We also define the canonical map $\iota^p : E \to E_p : x \mapsto x + \ker(p)$. Moreover, if q is another seminorm on E with $p \leq q$, there is a natural imbedding

$$E/\ker(q) \to E/\ker(p) : x + \ker(q) \mapsto x + \ker(p),$$

which is of course uniquely extended to a map $\iota_q^p : E_q \to E_p$.

Actually, in Kolmogorov's diameters, it is possible to replace 0-neighbourhoods by unit balls of local Banach spaces. This is described in the lemma and the corollary below ([19, 30]).

But before, we would like to insist on the fact that, in what follows, we will consider three different spaces, namely E, $E/\ker(p)$, and E_p. Consequently, the values of Kolmogorov's diameters can change from a space to the other, even if we consider the same sets. This is why we introduced both maps $\Phi_p : E \to E/\ker(p)$ and $\iota^p : E \to E_p$, although they algebraically coincide: when we consider sets of type $\Phi_p(U)$ (resp. $\iota^p(U)$), this implicitly means that we consider diameters in the space $E/\ker(p)$ (resp. in E_p).

Lemma 2.3.8 ([19]). *If p and q are two seminorms on the locally convex space E, with $p \leq q$ and with respective closed unit balls U and V, then*

$$\delta_n(\Phi_p(V), \Phi_p(U)) = \delta_n(V, U).$$

Proof. Using Proposition 1.1.7, it is clear that we have $\delta_n(\Phi_p(V), \Phi_p(U)) \leq \delta_n(V, U)$. For the other inequality, we take $\delta > 0$ and $L \in \mathcal{L}_n(E)$ such that

$$\Phi_p(V) \subseteq \delta\Phi_p(U) + \Phi_p(L).$$

This particularly implies that $V \subseteq \delta U + L + \ker(p) \subseteq \delta U + L$. Hence $\delta_n(V, U) \leq \delta$ and so $\delta_n(V, U) \leq \delta_n(\Phi_p(V), \Phi_p(U))$. □

Using the fact that $B_p := \overline{\iota^p(U)}$ is the closed unit ball of E_p, we obtain the next corollary (the proof of which is inspired by some arguments from [30]):

Corollary 2.3.9. *With the same notations as in the previous lemma, we have*

$$\delta_n(V, U) = \delta_n\left(\iota_q^p(B_q), B_p\right).$$

Proof. First, by Proposition 1.1.3 and Corollary 1.1.13 and by the fact that $\iota_q^p(B_q) \subseteq \overline{\iota^p(V)}$, we have

$$\begin{aligned}\delta_n\left(\iota_q^p(B_q), B_p\right) &\leq \delta_n\left(\overline{\iota^p(V)}, \overline{\iota^p(U)}\right)\\ &= \delta_n\left(\iota^p(V), \overline{\iota^p(U)}\right)\\ &\leq \delta_n\left(\iota^p(V), \iota^p(U)\right)\\ &\leq \delta_n\left(\Phi_p(V), \Phi_p(U)\right)\\ &= \delta_n(V, U).\end{aligned}$$

Next, for the other inequality, we use Proposition 1.1.12: we take $\delta > 0$, $m \leq n$, and $x_1, ..., x_m \in E_p$ with[5]

$$\iota_q^p(B_q) \subseteq \delta B_p + \Gamma\left(\{x_1, ..., x_m\}\right).$$

We fix $\varepsilon > 0$. Using the density of $E/\ker(p)$ in E_p, we can find $y_1, ..., y_m \in E/\ker(p)$ with $x_j \in y_j + \varepsilon B_p$ if $j \leq m$. So $\iota_q^p(B_q) \subseteq (\delta + \varepsilon)B_p + \Gamma\left(\{y_1, ..., y_m\}\right)$, which implies

$$\iota^p(V) \subseteq (\delta + \varepsilon)\overline{\iota^p(U)} + \Gamma\left(\{y_1, ..., y_m\}\right)$$

because $\iota^p(V) \subseteq \iota_q^p(B_q)$.

Let $v \in \Phi_p(V) = \iota^p(V)$. Then, there exist a sequence $(u_j)_{j\in\mathbb{N}_0}$ of $\iota^p(U) = \Phi_p(U)$ and $y \in \Gamma\left(\{y_1, ..., y_m\}\right)$ for which

$$u_j \to \frac{v - y}{\delta + \varepsilon} \text{ in } E_p \text{ if } j \to \infty.$$

Because u_j and $\frac{v-y}{\delta+\varepsilon}$ belong to $E/\ker(p)$, this implies that the sequence $(u_j)_{j\in\mathbb{N}_0}$ converges to $\frac{v-y}{\delta+\varepsilon}$ in $E/\ker(p)$. Besides, $\Phi_p(U)$ is closed in $E/\ker(p)$, so $u := \frac{v-y}{\delta+\varepsilon} \in \Phi_p(U)$ and

$$v = (\delta + \varepsilon)u + y \in (\delta + \varepsilon)\Phi_p(U) + \Gamma\left(\{y_1, ..., y_m\}\right).$$

Consequently, we obtain $\delta_n(V, U) = \delta_n(\Phi_p(V), \Phi_p(U)) \leq \delta + \varepsilon$. Finally, taking the limit as $\varepsilon \to 0^+$, this leads to $\delta_n(V, U) \leq \delta$ and so $\delta_n(V, U) \leq \delta_n\left(\iota_q^p(B_q), B_p\right)$. □

In summary, this last result means that Komogorov's diameters are, in some sense, a description of the behaviour of the linking maps ι_q^p between local Banach spaces. In particular, when we consider Schwartz Hilbertizable spaces, these linking maps turn to be compact operators between Hilbert spaces. In this situation, the use of the very specific properties of such operators will bring an interesting description for Kolmogorov's diameters, since they appear to be the *singular numbers* of these operators.

Therefore, we need to make some recalls about compact operators between Hilbert spaces and singular numbers.

[5]If $n = 0$, we can for example take $m = 1$ and $x_1 = 0$.

Definition 2.3.10. Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two normed spaces and $T \in L(E,F)$ be given. Then, the *n-th singular number* of T is the number

$$s_n(T) := \inf \left\{ \|T - S\|_{L(E,F)} : S \in L(E,F), \dim\left(S(E)\right) \leq n \right\},$$

where $\|.\|_{L(E,F)}$ is the norm of $L(E,F)$, i.e. $\|T\|_{L(E,F)} = \sup\{\|T(x)\|_F : x \in E, \|x\|_E \leq 1\}$.

When we consider Hilbert spaces, these singular numbers are used to describe a compact operator as a series. For this, if $(H, <.,.>)$ is a Hilbert space, we recall that a sequence $(h_m)_{m\in\mathbb{N}_0}$ of H is an *orthonormal system* if

$$< h_m, h_n >= \begin{cases} 1 \text{ if } m = n \\ 0 \text{ otherwise.} \end{cases}$$

One property of orthonormal systems is the following one: if $\|.\| = \sqrt{<.,.>}$ is the norm of H, then $\sum_{m=0}^{\infty} | < x, h_m > |^2 \leq \|x\|^2$ for every $x \in H$ (since this system defines an orthogonal projection on H, see [24] for more details).

In fact, a very important result in the theory of compact operators between Hilbert spaces is the following one (for the proof, see e.g. [24]):

Proposition 2.3.11. *Let H and G be two infinite-dimensional Hilbert spaces and $T \in L(H,G)$ be a compact operator. Then, there exist a decreasing null sequence $(s_m)_{m\in\mathbb{N}_0}$ of $[0,\infty)$ and orthonormal systems $(h_m)_{m\in\mathbb{N}_0}$ in H and $(g_m)_{m\in\mathbb{N}_0}$ in G for which*

$$T = \sum_{m=0}^{\infty} s_m < ., h_m > g_m,$$

where the limit holds in $L(H,G)$ and $<.,.>$ is the scalar product of H.

The orthonormal systems $(h_m)_{m\in\mathbb{N}_0}$ and $(g_m)_{m\in\mathbb{N}_0}$ are not unique, but the decreasing sequence $(s_m)_{m\in\mathbb{N}_0}$ is uniquely determined by T and we actually have

$$s_m = s_m(T).$$

The description in the previous proposition is called a *Schmidt representation* of T. A very similar result can be proved when H and/or G is (are) finite-dimensional: in this situation, if $M := \dim(T(H))$, there exist *finite* orthonormal systems[6] $h_0, ..., h_{M-1}$ in H and $g_0, ..., g_{M-1}$ in G for which

$$T = \sum_{m=0}^{M-1} s_m(T) < ., h_m > g_m.$$

Thanks to Schmidt representations, Vogt proved the following result ([38]):

[6]If $M > 0$; otherwise, we simply have $T = 0$.

Proposition 2.3.12. *If H and G are Hilbert spaces with respective closed unit balls V and U and if $T \in L(H,G)$ is a compact operator, then*

$$\delta_n(T(V), U) = s_n(T)$$

for any $n \in \mathbb{N}_0$.

Proof. We consider a Schmidt representation of T

$$T = \sum_{m=0}^{\infty} s_m(T) < ., h_m > g_m,$$

where $< ., . >$ is the scalar product of H (if both spaces are infinite-dimensional; otherwise, we keep this notation by putting $h_m = 0$ and $g_m = 0$ if $m \geq M := \dim(T(H))$).

We define $L := \mathrm{span}(\{g_0, ..., g_{n-1}\})$ if $n > 0$ and $L = \{0\}$ if $n = 0$. If $\|.\|$ is the norm of G and if $x \in V$, then

$$\begin{aligned} \left\| \sum_{m=n}^{\infty} s_m(T) < x, h_m > g_m \right\|^2 &= \sum_{m=n}^{\infty} s_m^2(T) | < x, h_m > |^2 \\ &\leq s_n^2(T) \sum_{m=n}^{\infty} | < x, h_m > |^2 \\ &\leq s_n^2(T). \end{aligned}$$

Thus,

$$T(x) = \sum_{m=n}^{\infty} s_m(T) < x, h_m > g_m + \sum_{m=0}^{n-1} s_m(T) < x, h_m > g_m \in s_n(T)U + L,$$

which implies that $\delta_n(T(V), U) \leq s_n(T)$.

It remains to prove the other inequality. This is clear when $s_n(T) = 0$, so we can assume that $s_n(T) > 0$. Now, we fix $\delta > 0$ and $L \in \mathcal{L}_n(G)$ such that

$$T(V) \subseteq \delta U + L.$$

Now, we can find $g \in L^{\perp} \cap \mathrm{span}\{g_0, ..., g_n\} \setminus \{0\}$, where $L^{\perp}$ is the orthogonal complement of L in G.

Indeed, it is direct if $L = \{0\}$; otherwise, we choose a basis $l_1, ..., l_d$ ($d \leq n$) of L. Then, finding g is equivalent to solving the linear system

$$< l_m, g >= 0 \ (m = 1, ..., d)$$

with $n + 1$ unknowns ($< g, g_0 >, ..., < g, g_n >$), which of course admits a non-trivial solution.

Next, we take $\lambda_0, ..., \lambda_n \in \mathbb{C}$ such that $g = \sum_{m=0}^{n} \lambda_m s_m(T) g_m$. In particular, $x := \sum_{m=0}^{n} \lambda_m h_m \in H$ is such that $T(x) = g$. Remark that $x \neq 0$ because $g \neq 0$. Consequently, without loss of generality, we can assume that the norm of x is equal to 1.

In this situation, if $l \in L$ is given,

$$\|T(x) - l\|^2 = \|g - l\|^2 = \|g\|^2 + \|l\|^2 \geq \sum_{m=0}^{n} |\lambda_m|^2 s_m^2(T) \geq s_n^2(T) \sum_{m=0}^{n} |\lambda_m|^2 = s_n^2(T).$$

Since $x \in V$, this inequality implies that $\delta \geq s_n(T)$. Thus $\delta_n(T(V), U) \geq s_n(T)$. $\square$

Thanks to this description with singular numbers, we will obtain a way to compare the velocity of convergence to 0 of Kolmogorov's diameters ([13]).

For this, we will abbreviate $\delta_n(T(V), U)$ by $\delta_n(T)$ when $T : E \to F$ is a continuous operator between the normed spaces E and F, with respective closed unit balls V and U. Moreover, we will use classic Landau notations: if $\xi, \eta \in \mathbb{C}^{\mathbb{N}_0}$, then we write $\eta_n = o(\xi_n)$ when

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N}_0 : \forall n \geq N,\ |\eta_n| \leq \varepsilon |\xi_n|.$$

Proposition 2.3.13. *Let F, G, and H be three Hilbert spaces and $T : F \to G$, $S : G \to H$ be compact operators. Then*

$$\delta_n(S \circ T) = o(\delta_n(S)) \qquad \textit{and} \qquad \delta_n(S \circ T) = o(\delta_n(T)).$$

Proof. Using the same convention as in the proof of Proposition 2.3.12, we consider a Schmidt representation of S

$$S = \sum_{m=0}^{\infty} s_m(S) < ., g_m > h_m,$$

where $< ., . >$ is the scalar product of G and $s_m(S) = \delta_m(S)$. Of course, if $x \in F$, we have

$$(S \circ T)(x) = \sum_{m=n}^{\infty} s_m(S) < T(x), g_m > h_m + \sum_{m=0}^{n-1} s_m(S) < T(x), g_m > h_m.$$

Consequently, if $\|.\|_X$ is the norm of X $(X \in \{F, G, H\})$, this implies that

$$\begin{aligned}
\delta_n(S \circ T)^2 &\leq \sup \left\{ \left\| \sum_{m=n}^{\infty} s_m(S) < T(x), g_m > h_m \right\|_H^2 : \|x\|_F \leq 1 \right\} \\
&= \sup \left\{ \sum_{m=n}^{\infty} s_m^2(S) \,|< T(x), g_m >|^2 : \|x\|_F \leq 1 \right\} \\
&\leq s_n^2(S) \ \sup \left\{ \sum_{m=n}^{\infty} |< y, g_m >|^2 : y \in K \right\},
\end{aligned}$$

where $K := \overline{T(\{x \in F : \|x\|_F \leq 1\})}$ is a compact set in G. Now, for $n \in \mathbb{N}_0$, we define the map

$$r_n : y \in G \mapsto \left(\sum_{m=n}^{\infty} |< y, g_m >|^2 \right)^{1/2} = \left\| \Phi^{(n)}(y) \right\|_G,$$

where $\Phi^{(n)} : G \to G$ is the orthogonal projection onto the closed span of $\{g_m : m \geq n\}$ in G. In this situation, the sequence $(r_n)_{n\in\mathbb{N}_0}$ is equicontinuous on K (by properties of orthonormal systems) and converges pointwise to 0: by Arzelà-Ascoli Theorem, it uniformly converges to 0 on K. Thus, we obtain $\delta_n(S \circ T) = o(\delta_n(S))$.

For the other statement, we recall a property of singular numbers: if $T^* : G \to F$ is the adjoint map of T, then $s_n(T^*) = s_n(T)$ ([24]). Therefore,

$$\delta_n(S \circ T) = \delta_n((S \circ T)^*) = \delta_n(T^* \circ S^*) = o(\delta_n(T^*)) = o(\delta_n(T)).$$

Hence the conclusion. □

So, in some sense, the previous proposition means that the composition of two compact operators in Hilbert spaces is "strictly more compact" than both of them. Although this assertion could seem quite intuitive, we do not know whether this is true when we consider Banach spaces.

Nevertheless, thanks to this last result, we are ready to prove that Δ and Δ^∞ coincide for Hilbertizable spaces:

Proposition 2.3.14. *If E is a Hilbertizable Schwartz locally convex space, then*

$$\Delta(E) = \Delta^\infty(E).$$

In particular, it is true for nuclear spaces.

Proof. Let $\mathcal{P}$ be a fundamental system of Hilbert seminorms in E and let $\xi \in \Delta^\infty(E)$ and $p \in \mathcal{P}$ be given. We denote by U the closed unit ball of p.

Then, there exists $q \in \mathcal{P}$, with $p \leq q$, such that its closed unit ball V is precompact with respect to U and verifies

$$(\xi_n \delta_n(V, U))_{n\in\mathbb{N}_0} \in l_\infty.$$

Moreover, we can choose $r \in \mathcal{P}$, with $q \leq r$, for which its closed unit ball W is precompact with respect to V.

So, the operators

$$\iota_q^p : E_q \to E_p \qquad \text{and} \qquad \iota_r^q : E_r \to E_q$$

are compact and such that $\delta_n(V, U) = \delta_n(\iota_q^p)$ and $\delta_n(W, V) = \delta_n(\iota_r^q)$ by Corollary 2.3.9. Then, by Proposition 2.3.13, we have

$$\xi_n \delta_n(\iota_r^p) = \xi_n \delta_n(\iota_q^p \circ \iota_r^q) = o(\xi_n \delta_n(\iota_q^p)),$$

so $(\xi_n \delta_n(W, U))_{n\in\mathbb{N}_0} \in c_0$. From this, we deduce that $\xi \in \Delta(E)$. □

Remark that such a result can be used to prove Proposition 2.2.5. Indeed, if $\lambda^l(A)$ is a Schwartz space (where l is an admissible space and A is a Köthe set), we know that Kolmogorov's diameters of 0-neighbourhoods in that space are independent of l (cf. Proposition 1.3.9). But, since $\lambda_2(A)$ is Hilbertizable, we have

$$\Delta\left(\lambda^l(A)\right) = \Delta\left(\lambda_2(A)\right) = \Delta^\infty\left(\lambda_2(A)\right) = \Delta^\infty\left(\lambda^l(A)\right).$$

Finally, Proposition 2.3.14 leads to:

Theorem 2.3.15. *If E is a Hilbertizable Schwartz metrizable locally convex space, then*

$$\Delta(E) = \Delta_b(E) = \Delta^\infty(E) = \Delta_b^\infty(E).$$

In particular, it is true for nuclear metrizable spaces.

Proof. It is direct by Theorem 2.3.4 and Proposition 2.3.14. □

Remark that proving Proposition 2.3.13 for Banach spaces would similarly give the equality of Δ and Δ^∞ for Schwartz spaces, and so the equality of Δ and Δ_b for Schwartz metrizable spaces. Unfortunately, this question about compact operators in Banach spaces remains open.

Nonetheless, we found an important class of metrizable spaces for which the equality of the diametral dimensions is true, namely the class of nuclear metrizable spaces. However, we will see in Section 4.1 that the nuclearity itself is not sufficient to have this equality.

Given the proof of Proposition 2.3.14, we can also wonder whether "Hilbertizability" can be used to compare Δ_b and Δ_b^∞ when the space is pseudo-Montel. In fact, it is the case when we consider Fréchet spaces. But, for this, we similarly have to translate inclusions of type

$$B \hookrightarrow U$$

in terms of compact operators between Hilbert spaces (where B is a bounded set and U is a 0-neighbourhood). This leads us to recall the following notions:

Definition 2.3.16. Let E be a Hausdorff locally convex space and B be an absolutely convex bounded set of E. Then, we define the normed space

$$E_B := (\mathrm{span}(B), p_B)$$

where p_B is the gauge of B. If the space E_B is a Banach space, then B is called a *Banach disk*. If E_B is a Hilbert space, then B is a *Hilbert disk*.

So, in this situation, we have to determine if we can only consider Hilbert disks for the description of Δ_b, i.e. if they constitute a fundamental system of bounded sets. We recall that a family $\mathcal{B}$ of bounded sets of a locally convex space E is a *fundamental system of bounded sets* if, for every bounded set B of E, there exists $D \in \mathcal{B}$ and $\mu > 0$ with $B \subseteq \mu D$.

For instance, it is well known that the family of Banach disks constitute a fundamental system of bounded sets in Fréchet spaces. More precisely, we have the following result, extracted from [24]:

Proposition 2.3.17. *In a Hausdorff locally convex space E, every absolutely convex, closed, sequentially complete, and bounded set B is a Banach disk. In particular, if E is sequentially complete, the family of Banach disks in E is a fundamental system of bounded sets.*

Proof. Of course, the inclusion

$$E_B \hookrightarrow E$$

is continuous, which means that the topology $\mathcal{S}$ of E_B is finer than the topology $\mathcal{T}$ induced by E on E_B. Moreover, E_B has a basis of 0-neighbourhoods made of $\mathcal{T}$-closed sets (because B is closed in E). Then, the lemma below implies that B is $\mathcal{S}$-sequentially complete and thus E_B is Banach.

The particular case is then straightforward because, when E is sequentially complete, every absolutely convex, closed, bounded set in E is a Banach disk. □

Lemma 2.3.18. *Let E be a vector space and $\mathcal{T}$ and $\mathcal{S}$ be two vector topologies on E. If $\mathcal{S}$ is finer than $\mathcal{T}$ and has a basis of 0-neighbourhoods made of $\mathcal{T}$-closed sets, then every $\mathcal{T}$-sequentially complete subset of E is $\mathcal{S}$-sequentially complete.*

Proof. Assume that A is a $\mathcal{T}$-sequentially complete subset of E and fix an $\mathcal{S}$-Cauchy sequence $(x_n)_{n\in\mathbb{N}_0}$ in A. In particular, $(x_n)_{n\in\mathbb{N}_0}$ is $\mathcal{T}$-Cauchy and so converges to a vector $x \in A$ for $\mathcal{T}$. Our purpose is to show that $(x_n)_{n\in\mathbb{N}_0}$ converges to x for $\mathcal{S}$.

For this, we fix a $\mathcal{T}$-closed, $\mathcal{S}$-0-neighbourhood U. By assumption, there exists $N \in \mathbb{N}_0$ such that $x_p - x_q \in U$ if $p, q \geq N$. By taking the limit as $p \to \infty$ in $\mathcal{T}$, we deduce that $x - x_q \in U$ if $q \geq N$ (because U is $\mathcal{T}$-closed). Hence the conclusion. □

When we consider Fréchet-Hilbert spaces, we can say even more, as explained in the next result (which comes from the proof of Lemma 29.16 in [24]).

Proposition 2.3.19. *If E is a Fréchet-Hilbert space, then the family of Hilbert disks in E is a fundamental system of bounded sets in E.*

Proof. Let $(p_k)_{k\in\mathbb{N}}$ be an increasing fundamental system of Hilbert seminorms in E and B be a bounded set in E.

Now, for every $k \in \mathbb{N}$, we define $\lambda_k := \sup_{x\in B} p_k(x)$ and

$$\|x\|^2 := \sum_{k=1}^{\infty} 2^{-k} \lambda_k^{-2} p_k(x)^2$$

when $x \in E$. Then, $C := \{x \in E : \|x\| \leq 1\}$ is an absolutely convex, closed, bounded set in E, so a Banach disk by Proposition 2.3.17. By construction, it is even a Hilbert disk containing B. □

Next, if we follow the proof of Proposition 2.3.14 for Δ_b, we will consider inclusions of type $B \hookrightarrow D$, where B and D are two bounded sets. Because of the different topologies which coexist in that situation, the following notion will be useful:

Definition 2.3.20. A locally convex space E satisfies the *strict Mackey condition* if, for every bounded set B in E, there exists an absolutely convex bounded set $D \supseteq B$ such that p_D induces on B the topology of E.

In fact, we have the following result ([10]):

Proposition 2.3.21. *If E is a metrizable locally convex space, then it satisfies the strict Mackey condition.*

Proof. Let $(U_k)_{k\in\mathbb{N}_0}$ be a decreasing basis of absolutely convex 0-neighbourhoods in E and let B be a bounded set in E. Obviously, every inclusion $E_D \hookrightarrow E$, with a bounded set D, is continuous. Therefore, it is enough to prove that there exists an absolutely convex bounded set $D \supseteq B$ such that, for every $\lambda > 0$, there exists $k \in \mathbb{N}_0$ with

$$B \cap U_k \subseteq \lambda D.$$

For every $k \in \mathbb{N}_0$, there exists $r_k > 0$ with $B \subseteq r_k U_k$. Then, we choose a sequence $(s_k)_{k\in\mathbb{N}_0}$, with $r_k \leq s_k$ and $(r_k/s_k)_{k\in\mathbb{N}_0} \in c_0$. Then, we put

$$D := \bigcap_{k\in\mathbb{N}_0} s_k U_k.$$

The set D has the claimed property. Indeed, let $\lambda > 0$ be given. There is $K \in \mathbb{N}$ for which $r_k \leq \lambda s_k$ if $k \geq K$, so

$$B \subseteq \bigcap_{k\geq K} \lambda s_k U_k.$$

What is more, since $\bigcap_{k=0}^{K-1} \lambda s_k U_k$ is a 0-neighbourhood in E, there exists $k_0 \in \mathbb{N}_0$ for which $U_{k_0} \subseteq \bigcap_{k=0}^{K-1} \lambda s_k U_k$. Consequently,

$$B \cap U_{k_0} \subseteq \bigcap_{k\in\mathbb{N}_0} \lambda s_k U_k = \lambda D.$$

Hence the conclusion. □

With these notions, we finally obtain the following property:

Proposition 2.3.22. *If E is a Montel Fréchet-Hilbert space, then*

$$\Delta_b(E) = \Delta_b^\infty(E).$$

Proof. Let B be a Hilbert disk in E, p be a Hilbert seminorm on E, and U be the closed unit ball of p. Then, by definition of the strict Mackey condition, we can find another Hilbert disk $D \supseteq B$ for which E_D induces on B the same topology as E. In particular, the inclusions

$$\iota_B^D : E_B \hookrightarrow E_D \qquad \text{and} \qquad \iota_D^p : E_D \to E_p$$

are compact since E is Montel. By a direct adaptation of Corollary 2.3.9, we can see that $\delta_n\left(\iota_D^p\right) = \delta_n(D,U)$. Thus, applying Proposition 2.3.13 to the inclusions $\iota_B^D : E_B \hookrightarrow E_D$ and $\iota_D^p : E_D \to E_p$, we obtain

$$\delta_n(B,U) = o(\delta_n(D,U)).$$

This gives the conclusion. □

Remark that this last property is not a corollary of Theorem 2.3.4. Indeed, there exist some Fréchet-Montel Köthe spaces which are not Schwartz (cf. [9, 24]): therefore, taking l_2 as associated admissible space, it means there are Montel Fréchet-Hilbert spaces which are not Schwartz.

Up to now, the main positive results about the equality of the two diametral dimensions are Theorem 2.3.4 and its two consequences, namely Theorems 2.3.5 and 2.3.15. Unfortunately, the question remains open for general Schwartz metrizable spaces. As explained before, a possible method to solve this problem would be to prove Proposition 2.3.13 in Banach spaces.

This situation pushed us into considering some other tools which assure the equality of the two diametral dimensions. These tools – the Δ-stability and prominent bounded sets – are in fact the main topics of the next chapter.

Chapter 3

Some other tools for the equality of the two diametral dimensions

As explained previously, we present in this chapter two tools which assure the equality of the two diametral dimensions. We begin with the Δ-stability, which is defined thanks to some properties of the classic diametral dimension for finite Cartesian products. After that, we focus on the notion of prominent bounded sets ([34]).

3.1 Finite Cartesian Products and Δ-stability

Originally, the Δ-stability was introduced to have the equality of the two diametral dimensions when we consider some Cartesian products of Schwartz locally convex spaces. But, thereafter, the Δ-stability appeared to be, in some sense, a natural condition for general Schwartz spaces to have the equality of Δ and Δ^∞ (cf. Proposition 3.1.11 below), and so to have the equality of Δ and Δ_b.

First, to understand what happens for the diametral dimension when we consider the Cartesian product of two spaces, we present the following lemma ([27]):

Lemma 3.1.1. *Let E and F be two vector spaces and $U_1, V_1 \subseteq E$ and $U_2, V_2 \subseteq F$ be such that U_1 and U_2 are balanced and respectively absorb V_1 and V_2. Then*

$$\delta_{m+n}(V_1 \times V_2, U_1 \times U_2) \leq \sup\{\delta_m(V_1, U_1), \delta_n(V_2, U_2)\}$$

for all $m, n \in \mathbb{N}_0$.

Proof. Assume that $\delta_1, \delta_2 > 0$ and $L_1 \in \mathcal{L}_m(E)$, $L_2 \in \mathcal{L}_n(F)$ verify

$$V_1 \subseteq \delta_1 U_1 + L_1 \qquad \text{and} \qquad V_2 \subseteq \delta_2 U_2 + L_2.$$

Then, of course, we have $V_1 \times V_2 \subseteq \sup\{\delta_1, \delta_2\}(U_1 \times U_2) + L_1 \times L_2$, so that

$$\delta_{m+n}(V_1 \times V_2, U_1 \times U_2) \leq \sup\{\delta_1, \delta_2\}.$$

Hence the conclusion. □

This result is needed to obtain some descriptions of the diametral dimension of products of type $E \times F$. But, before, we have to introduce a new notation ([27]).

Definition 3.1.2. Let $x, y \in \mathbb{C}^{\mathbb{N}_0}$ be given. Then we define the sequence $x * y$ by

$$(x * y)_n := \begin{cases} x_m \text{ if } n = 2m; \\ y_m \text{ if } n = 2m+1. \end{cases}$$

In other words, $x * y$ is the sequence $x_0, y_0, x_1, y_1, \ldots$ In the following, we will say that $x * y$ is the *cross-product* of x and y.

Similarly, if $X, Y \subseteq \mathbb{C}^{\mathbb{N}_0}$, we put $X * Y = \{x * y : x \in X, y \in Y\}$. Thanks to this notion and Lemma 3.1.1, we obtain these inclusions ([27]):

Proposition 3.1.3. *If E and F are both locally convex spaces, then*

$$(\Delta(E) \cap \Delta(F)) * (\Delta(E) \cap \Delta(F)) \subseteq \Delta(E \times F) \subseteq \Delta(E) \cap \Delta(F).$$

Proof. For the first inclusion, we fix $\xi, \eta \in \Delta(E) \cap \Delta(F)$ and we just have to show that $\xi * \eta \in \Delta(E \times F)$. Moreover, if we define the sequence γ by $\gamma_n = |\xi_n| + |\eta_n|$, then $\gamma \in \Delta(E) \cap \Delta(F)$. Since $\sup\{|\xi_n|, |\eta_n|\} \leq \gamma_n$ for every n, it is enough to prove that $\gamma * \gamma \in \Delta(E \times F)$.

Let U_1 and U_2 be two absolutely convex 0-neighbourhoods, respectively in E and F. Then, by definition, there exist two other 0-neighbourhoods $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$ for which

$$(\gamma_n \delta_n(V_1, U_1))_{n \in \mathbb{N}_0} \in c_0 \qquad \text{and} \qquad (\gamma_n \delta_n(V_2, U_2))_{n \in \mathbb{N}_0} \in c_0.$$

Then, by Proposition 1.1.2 and Lemma 3.1.1, we have

$$\gamma_n \delta_{2n+1}(U_1 \times U_2, V_1 \times V_2) \leq \gamma_n \delta_{2n}(U_1 \times U_2, V_1 \times V_2) \leq \gamma_n \sup\{\delta_n(V_1, U_1), \delta_n(V_2, U_2)\},$$

so $((\gamma * \gamma)_n \delta_n(U_1 \times U_2, V_1 \times V_2))_{n \in \mathbb{N}_0} \in c_0$. This implies that $\gamma * \gamma \in \Delta(E \times F)$.

The other inclusion directly follows from Corollary 1.2.5. □

Remark that such a result can be used, for instance, to prove that the Cartesian product of two Schwartz (resp. nuclear) locally convex spaces is also Schwartz (resp. nuclear).

Besides, if we take $E = F$, this property implies that

$$\Delta(E) * \Delta(E) \subseteq \Delta(E \times E) \subseteq \Delta(E).$$

Based on this, Ramanujan and Terzioğlu claimed a more precise result in [27]:

Conjecture 3.1.4. If E is a locally convex space, then

$$\Delta(E \times E) = \Delta(E) * \Delta(E).$$

This conjecture is derived from the following result, also asserted in [27]: if U, V are two absolutely convex 0-neighbourhoods in a locally convex space E such that U absorbs V, then

$$\delta_n(V, U) = \delta_{2n}(V \times V, U \times U) = \delta_{2n+1}(V \times V, U \times U).$$

Unfortunately, it seems there is a gap in the related proof. Indeed, it is claimed that, if $L \in \mathcal{L}_{2n+1}(E \times E)$ is given and if $p_1 : (x, y) \in E \times E \mapsto x$ and $p_2 : (x, y) \in E \times E \mapsto y$ are the corresponding projections, then the dimensions of $p_1(L)$ and $p_2(L)$ cannot be both strictly greater than n. However, it is not possible, even if E is finite-dimensional: for example, if $E = \mathbb{C}$, $L = \operatorname{span}\{(1, 1)\}$, and $n = 0$, then $\dim(p_1(L)) = \dim(p_2(L)) = 1 > 0$.

Consequently, Conjecture 3.1.4 remains open for general locally convex spaces. Nonetheless, we will show in the following that it is true for "classic" Köthe spaces (see Proposition 3.1.14 for more precision).

Given Proposition 3.1.3, we can wonder whether similar phenomena occur for the diametral dimension Δ_b. Actually, it is the case:

Proposition 3.1.5. *If E and F are locally convex spaces, then*

$$(\Delta_b(E) \cap \Delta_b(F)) * (\Delta_b(E) \cap \Delta_b(F)) \subseteq \Delta_b(E \times F) \subseteq \Delta_b(E) \cap \Delta_b(F).$$

Proof. For the first inclusion, it is enough to proceed in the same way as in the proof of Proposition 3.1.3. For the second one, we fix $\xi \in \Delta_b(E \times F)$, two (non-empty) bounded sets B_1 and B_2, respectively in E and in F, and two absolutely convex 0-neighbourhoods U_1 and U_2, respectively in E and in F.

By definition, $(\xi_n \delta_n(B_1 \times B_2, U_1 \times U_2))_{n \in \mathbb{N}_0} \in c_0$. But, if $p_1 : E \times F \to E$ is the projection on E, we have by Proposition 1.1.7

$$\delta_n(B_1, U_1) = \delta_n\left(p_1(B_1 \times B_2), p_1(U_1 \times U_2)\right) \leq \delta_n(B_1 \times B_2, U_1 \times U_2).$$

So $(\xi_n \delta_n(B_1, U_1))_{n \in \mathbb{N}_0} \in c_0$, which means that $\xi \in \Delta_b(E)$. Symmetrically, we also have $\xi \in \Delta_b(F)$, which provides the conclusion. □

In this situation, we can wonder whether there exist simple conditions which assure the equality of $\Delta(E \times F)$ and $\Delta_b(E \times F)$. An idea would be to consider spaces for which the inclusions in Proposition 3.1.3 become equalities. Translating these conditions for products of the same space, this leads to the notion of Δ-stability.

Definition 3.1.6. A locally convex space E is Δ-*stable* if it verifies the inclusion

$$\Delta(E) \subseteq \Delta(E) * \Delta(E).$$

By Proposition 3.1.3, we can say that E is Δ-stable if and only if

$$\Delta(E) = \Delta(E \times E) = \Delta(E) * \Delta(E).$$

We can say even more: for a space E verifying Conjecture 3.1.4, E is Δ-stable if and only if $\Delta(E) = \Delta(E \times E)$. So, under the assumption that Conjecture 3.1.4 is true (and it is for “classic” Köthe spaces), the Δ-stability corresponds to spaces E for which E and $E \times E$ have the same diametral dimension. This highlights some links with the so-called *stable spaces* ([24]).

Definition 3.1.7. A locally convex space E is *stable* if it is isomorphic to $E \times E$.

Therefore, our previous explanations lead to:

Proposition 3.1.8. *A stable locally convex space verifying Conjecture 3.1.4 is Δ-stable.*

However, the Δ-stability does not imply the stability: for instance, the space $\mathbb{C}$, endowed with the euclidean topology, is Δ-stable by Example 1.2.6 and verifies Conjecture 3.1.4, but is obviously non-stable.

Now, we will see how to use Δ-stability to obtain the equality of Δ and Δ_b. First, we have the following result:

Proposition 3.1.9. *Let E and F be two locally convex spaces.*

(1) If E and F are Δ-stable, then

$$\Delta(E \times F) = \Delta(E) \cap \Delta(F).$$

(2) If E is Δ-stable and $\Delta(E) \subseteq \Delta(F)$, then

$$\Delta(E \times F) = \Delta(E).$$

Proof. It is clear by Proposition 3.1.3. □

Corollary 3.1.10. *Let E and F be two locally convex spaces.*

(1) If E and F are Δ-stable and verify $\Delta(E) = \Delta_b(E)$ and $\Delta(F) = \Delta_b(E)$, then

$$\Delta_b(E \times F) = \Delta(E \times F) = \Delta(E) \cap \Delta(F).$$

(2) If E is Δ-stable and verifies $\Delta(E) = \Delta_b(E)$ and if $\Delta(E) \subseteq \Delta(F)$, then

$$\Delta_b(E \times F) = \Delta(E \times F) = \Delta(E).$$

Proof. (1) By Propositions 3.1.5 and 3.1.9, we have

$$\Delta_b(E \times F) \subseteq \Delta_b(E) \cap \Delta_b(F) = \Delta(E) \cap \Delta(F) = \Delta(E \times F).$$

(2) We use Propositions 3.1.5 and 3.1.9 again and this gives

$$\Delta_b(E \times F) \subseteq \Delta_b(E) \cap \Delta_b(F) \subseteq \Delta_b(E) = \Delta(E) = \Delta(E \times F).$$

Hence the conclusion. □

In summary, we have just seen where the Δ-stability comes from and how to use it to obtain some equalities of type $\Delta(E \times F) = \Delta_b(E \times F)$. Nevertheless, this property can be independently and quite naturally used to obtain the equality of Δ and Δ^∞.

Assume that E is a Schwartz locally convex space. To prove that $\Delta^\infty(E) \subseteq \Delta(E)$, we have to fix $\xi \in \Delta^\infty(E)$ and an absolutely convex 0-neighbourhood U in E. Then, natural arguments are the following ones:

- we take another absolutely convex 0-neighbourhood $V \subseteq U$ and $C > 0$ for which $|\xi_n|\delta_n(V, U) \leq C$ for every $n \in \mathbb{N}_0$;
- we take a third 0-neighbourhood $W \subseteq V$ for which $(\delta_n(W, V))_{n\in\mathbb{N}_0} \in c_0$.

Then, we could hope that $(\xi_n\delta_n(W, U))_{n\in\mathbb{N}_0} \in c_0$. But, for this, we have to compare $\delta_n(W, U)$ with $\delta_n(W, V)$ and $\delta_n(V, U)$, which was possible in Köthe spaces and Hilbertizable spaces. Unfortunately, in general locally convex spaces, we only have Proposition 1.1.4 to do this:

$$\delta_{2n}(W, U) \leq \delta_n(W, V)\delta_n(V, U).$$

So, this inequality proves that $|\xi_n|\delta_{2n}(W, U) \leq C\delta_n(W, V)$, so $(\xi_n\delta_{2n}(W, U))_{n\in\mathbb{N}_0} \in c_0$. Similarly, $(\xi_n\delta_{2n+1}(W, U))_{n\in\mathbb{N}_0} \in c_0$. This particularly means that $\xi * \xi \in \Delta(E)$. Hence, in this situation, to be sure that $\xi \in \Delta(E)$, the inclusion $\Delta(E) \subseteq \Delta(E) * \Delta(E)$ seems indispensable: this is exactly Δ-stability.

Thus, the previous "intuitive" arguments prove the following result:

Proposition 3.1.11. *If E is a Δ-stable Schwartz locally convex space, then*

$$\Delta(E) = \Delta^\infty(E).$$

In particular, if E is also metrizable, then $\Delta(E) = \Delta_b(E)$.

Combining Theorem 2.3.4, Corollary 3.1.10 and Proposition 3.1.11, we particularly obtain:

Corollary 3.1.12. *(1) If E and F are two metrizable Δ-stable Schwartz spaces, then*

$$\Delta_b(E \times F) = \Delta(E \times F) = \Delta(E) \cap \Delta(F).$$

(2) If E is a metrizable Δ-stable Schwartz space and if F is a locally convex space such that $\Delta(E) \subseteq \Delta(F)$, then

$$\Delta_b(E \times F) = \Delta(E \times F) = \Delta(E).$$

Given Proposition 3.1.11, we can wonder whether every Schwartz locally convex space is Δ-stable. This is why we will study this property in the context of Köthe spaces. Actually, we will see that there are some non-Δ-stable power series spaces, which will simultaneously show that the Δ-stability is not a necessary condition to have the equality of Δ and Δ^∞.

From now on, we fix an admissible space l and a Köthe set A. In fact, our results will be only valid for "classic" admissible spaces: so, we will assume that l is equal to l_p (for $p \geq 1$), l_∞, or c_0. First, we prove a result describing the products of two Köthe spaces (and which is a generalization of the corresponding property for $l = l_1$ in [32]).

Proposition 3.1.13. *If B is another Köthe set, then $\lambda^l(A) \times \lambda^l(B)$ is isomorphic to $\lambda^l(A * B)$.*

Proof. We define the linear and injective map

$$T : \lambda^l(A) \times \lambda^l(B) \to \mathbb{C}^{\mathbb{N}_0} : (\xi, \eta) \mapsto \xi * \eta.$$

(1) If $l = l_p$, we have, for $\xi \in \lambda^l(A)$, $\eta \in \lambda^l(B)$, $\alpha \in A$, and $\beta \in B$,

$$\begin{aligned} p^l_{\alpha*\beta}(\xi * \eta) &= \left(\sum_{n=0}^{\infty} [(\alpha * \beta)_n |(\xi * \eta)_n|]^p \right)^{1/p} \\ &= \left(\sum_{n=0}^{\infty} (\alpha_n |\xi_n|)^p + \sum_{n=0}^{\infty} (\beta_n |\eta_n|)^p \right)^{1/p} \\ &\leq 2^{1/p} \sup \left\{ p^l_\alpha(\xi), p^l_\beta(\eta) \right\}. \end{aligned}$$

Thus, $T(\lambda^l(A) \times \lambda^l(B))$ is included in $\lambda^l(A * B)$ and $T : \lambda^l(A) \times \lambda^l(B) \to \lambda^l(A * B)$ is continuous. Besides, if $\xi * \eta \in \lambda^l(A * B)$, we have

$$\sup \left\{ p^l_\alpha(\xi), p^l_\beta(\eta) \right\} \leq p^l_{\alpha*\beta}(\xi * \eta).$$

Consequently, $T : \lambda^l(A) \times \lambda^l(B) \to \lambda^l(A * B)$ is surjective and open. It is therefore an isomorphism between $\lambda^l(A) \times \lambda^l(B)$ and $\lambda^l(A * B)$.

(2) If $l = l_\infty$ or $l = c_0$ and if $\xi \in \lambda^l(A)$, $\eta \in \lambda^l(B)$, $\alpha \in A$, and $\beta \in B$, we get

$$\begin{aligned} p^l_{\alpha*\beta}(\xi * \eta) &= \sup_{n \in \mathbb{N}_0} ((\alpha * \beta)_n |(\xi * \eta)_n|) \\ &= \sup \left\{ \sup_{n \in \mathbb{N}_0} (\alpha_n |\xi_n|), \sup_{n \in \mathbb{N}_0} (\beta_n |\eta_n|) \right\} \\ &= \sup \left\{ p^l_\alpha(\xi), p^l_\beta(\eta) \right\}. \end{aligned}$$

We conclude in the same way as in (1).

□

Unfortunately, we do not know whether such an isomorphism remains valid when we consider a very general admissible space. However, this last result implies that $\lambda^l(A)$ verifies Conjecture 3.1.4:

Proposition 3.1.14. *We have*

$$\Delta\left(\lambda^l(A)\times\lambda^l(A)\right)=\Delta\left(\lambda^l(A)\right)*\Delta\left(\lambda^l(A)\right).$$

Proof. Since $c_0 = c_0 * c_0$, we assume that $\lambda^l(A)$ is Schwartz. Moreover, by Proposition 3.1.3, it is enough to prove $\Delta\left(\lambda^l(A)\times\lambda^l(A)\right)\subseteq\Delta\left(\lambda^l(A)\right)*\Delta\left(\lambda^l(A)\right)$. So, by Proposition 3.1.13, we just have to show that $\Delta\left(\lambda^l(A*A)\right)\subseteq\Delta\left(\lambda^l(A)\right)*\Delta\left(\lambda^l(A)\right)$.

Let $\xi\in\Delta\left(\lambda^l(A*A)\right)$ and $\alpha\in A$ be given. By Theorem 1.3.10, we know that there exists $\beta\in A$ such that $\alpha_n\le\beta_n$ for every $n\in\mathbb{N}_0$, $\alpha/\beta\in c_0$ and

$$\left(\xi_n\ \pi_n\left(\frac{\alpha*\alpha}{\beta*\beta}\right)\right)_{n\in\mathbb{N}_0}\in c_0.$$

But we have $\pi_{2n}\left(\dfrac{\alpha*\alpha}{\beta*\beta}\right)=\pi_{2n+1}\left(\dfrac{\alpha*\alpha}{\beta*\beta}\right)=\pi_n\left(\dfrac{\alpha}{\beta}\right)$ for every $n\in\mathbb{N}_0$. As a consequence, we have

$$\left(\xi_{2n}\ \pi_n\left(\frac{\alpha}{\beta}\right)\right)_{n\in\mathbb{N}_0}\in c_0\qquad\text{and}\qquad\left(\xi_{2n+1}\ \pi_n\left(\frac{\alpha}{\beta}\right)\right)_{n\in\mathbb{N}_0}\in c_0.$$

Therefore, $\xi\in\Delta\left(\lambda^l(A)\right)*\Delta\left(\lambda^l(A)\right)$. □

Consequently, if $\lambda^l(A)$ is stable, it is also Δ-stable. But, for smooth sequence spaces (cf. Section 1.4), we can say even more: for them, the notions of Δ-stability and stability coincide:

Proposition 3.1.15. *If $\lambda^l(A)$ is a smooth sequence space, then it is Δ-stable if and only if it is stable.*

Proof. By assumption, A is a Köthe matrix and we write $A=(a_k)_{k\in\mathbb{N}_0}$. In that case, it is clear we have

$$\lambda^l(A*A)=\lambda^l(\mathcal{A}),$$

where $\mathcal{A}$ is the Köthe matrix $(a_k*a_k)_{k\in\mathbb{N}_0}$. But, then, it is easy to see that $\lambda^l(\mathcal{A})$ is itself a smooth sequence space of the same type as $\lambda^l(A)$. Therefore, by Propositions 3.1.13 and 3.1.14, $\lambda^l(A)$ is Δ-stable if and only if

$$\Delta(\lambda^l(A))=\Delta(\lambda^l(\mathcal{A})),$$

which is equivalent to the fact that $\lambda^l(A)$ and $\lambda^l(\mathcal{A})$ are isomorphic, by Theorem 1.4.5. The conclusion follows from Proposition 3.1.13. □

Consequently, in order to find examples or counterexamples of Δ-stable smooth sequence spaces, we just have to characterize stability in their context. It is the purpose of the following result, which was originally proved in [32] for nuclear smooth sequence spaces associated to the admissible space l_1:

Proposition 3.1.16.

(1) If $\lambda^l(A)$ is a G_1-space, then it is stable if and only if, for every $m \in \mathbb{N}_0$, there exists $k \in \mathbb{N}_0$ with $(a_m(n)/a_k(2n+1))_{n\in\mathbb{N}_0} \in l_\infty$.

(2) If $\lambda^l(A)$ is a G_∞-space, then it is stable if and only if, for every $m \in \mathbb{N}_0$, there exists $k \in \mathbb{N}_0$ such that $(a_m(2n+1)/a_k(n))_{n\in\mathbb{N}_0} \in l_\infty$.

Proof. We assume that $\lambda^l(A)$ is a smooth sequence space. Once again, we define the Köthe matrix $\mathcal{A} := (a_k * a_k)_{k\in\mathbb{N}_0}$. Then, by Proposition 3.1.13, $\lambda^l(A)$ is stable if and only if $\lambda^l(A)$ and $\lambda^l(\mathcal{A})$ are isomorphic. Besides, $\lambda^l(\mathcal{A})$ is a smooth sequence space of the same type as $\lambda^l(A)$.

Therefore, by Theorem 1.4.5, $\lambda^l(A)$ is stable if and only if $\lambda^l(A) = \lambda^l(\mathcal{A})$ algebraically and topologically. This is equivalent to the fact that, for every $m \in \mathbb{N}_0$, there exist $k \geq m$ and $C > 0$ with $a_m(n) \leq C(a_k * a_k)(n)$ and $(a_m * a_m)(n) \leq Ca_k(n)$ for every n (cf. Proposition A.2.1).

Then, we split the argument according to the type of $\lambda^l(A)$.

(1) We suppose that $\lambda^l(A)$ is a G_1-space. Since, for any $m \in \mathbb{N}_0$, a_m is a decreasing sequence, we have $a_m(n) \leq (a_m * a_m)(n)$ for every $n \in \mathbb{N}_0$. Thus $\lambda^l(A)$ is stable if and only if, for every $m \in \mathbb{N}_0$, there exist $k \geq m$ and $C > 0$ for which $(a_m * a_m)(n) \leq Ca_k(n)$ for each $n \in \mathbb{N}_0$. But having $(a_m * a_m)(n) \leq Ca_k(n)$ for any n is equivalent to

$$a_m(n) \leq Ca_k(2n) \quad \text{and} \quad a_m(n) \leq Ca_k(2n+1)$$

for all $n \in \mathbb{N}_0$ and we conclude because $a_k(2n+1) \leq a_k(2n)$.

(2) Now, we suppose that $\lambda^l(A)$ is a G_∞-space. Because the sequence a_m is increasing if $m \in \mathbb{N}_0$, it means that $(a_m * a_m)_n \leq a_m(n)$ for every $n \in \mathbb{N}_0$. Consequently, $\lambda^l(A)$ is stable if and only if, for every $m \in \mathbb{N}_0$, there exist $k \geq m$ and $C > 0$ for which $a_m(n) \leq C(a_k * a_k)(n)$ for each $n \in \mathbb{N}_0$. But this last inequality is equivalent to

$$a_m(2n) \leq Ca_k(n) \quad \text{and} \quad a_m(2n+1) \leq Ca_k(n)$$

for all $n \in \mathbb{N}_0$ and, once more, we conclude since $a_m(2n) \leq a_m(2n+1)$.

Hence the conclusion. □

From the last result, it is very easy to deduce the characterization of stable power series spaces, which was for instance already pointed out in [24] for $l = l_1$:

Proposition 3.1.17. *Let α be an unbounded increasing sequence of $[0, \infty)$ and $r \in \{0, \infty\}$. Then the power series space $\Lambda^l_r(\alpha)$ is stable if and only if*

$$(\alpha_{2n+1}/\alpha_n)_{n\in\mathbb{N}_0} \in l_\infty.$$

Proof. (1) The space $\Lambda_0^l(\alpha)$ is a G_1-space, so, by the previous result, it is stable if and only if

$$\begin{aligned}
&\forall m \in \mathbb{N}, \exists k \geq m, C > 0 : e^{-\alpha_n/m} \leq Ce^{-\alpha_{2n+1}/k} \quad \forall n \in \mathbb{N}_0\\
\Leftrightarrow\ &\forall m \in \mathbb{N}, \exists k \geq m, c \in \mathbb{R} : \frac{\alpha_{2n+1}}{k} \leq c + \frac{\alpha_n}{m} \quad \forall n \in \mathbb{N}_0\\
\Leftrightarrow\ &\forall m \in \mathbb{N}, \exists k \geq m, c \in \mathbb{R} : \frac{\alpha_{2n+1}}{\alpha_n} \leq \frac{ck}{\alpha_n} + \frac{k}{m} \quad \forall n \in \mathbb{N}_0 .
\end{aligned}$$

Therefore, if $\Lambda_0^l(\alpha)$ is stable, taking $m = 1$ above, the sequence $(\alpha_{2n+1}/\alpha_n)_{n\in\mathbb{N}_0}$ is upper-bounded by $ck/\alpha_0 + k$. Conversely, if this sequence is upper-bounded by $J \in \mathbb{N}$ and if $m \in \mathbb{N}$ is given, it is enough to take $c = 0$ and $k = Jm$ in the condition above.

(2) As for the space $\Lambda_\infty^l(\alpha)$, it is a G_∞-space, thus it is stable if and only if

$$\begin{aligned}
&\forall m \in \mathbb{N}_0, \exists k \geq m, C > 0 : e^{m\alpha_{2n+1}} \leq Ce^{k\alpha_n} \quad \forall n \in \mathbb{N}_0\\
\Leftrightarrow\ &\forall m \in \mathbb{N}_0, \exists k \geq m, c \in \mathbb{R} : m\alpha_{2n+1} \leq c + k\alpha_n \quad \forall n \in \mathbb{N}_0\\
\Leftrightarrow\ &\forall m \in \mathbb{N}, \exists k \geq m, c \in \mathbb{R} : \frac{\alpha_{2n+1}}{\alpha_n} \leq \frac{c}{m\alpha_n} + \frac{k}{m} \quad \forall n \in \mathbb{N}_0 .
\end{aligned}$$

It is enough to proceed exactly in the same way as in (1).

Hence the conclusion. □

Because of this result, we will say that the sequence α is *stable* if it verifies the condition $(\alpha_{2n+1}/\alpha_n)_{n\in\mathbb{N}_0} \in l_\infty$.

Examples 3.1.18. If $\alpha = (\log(n + 1))_{n\in\mathbb{N}_0}$, then $s := \Lambda_\infty(\alpha)$ is called the *space of rapidly decreasing sequences*. It is a stable Köthe space and so a Δ-stable space.
However, if $\beta = (e^n)_{n\in\mathbb{N}_0}$, the power series space $\Lambda_0(\beta)$ is not stable and so not Δ-stable. But, as a Köthe space, it verifies $\Delta(\Lambda_0(\beta)) = \Delta^\infty(\Lambda_0(\beta))$.

Hence, Δ-stability is just a *sufficient* condition to have the equality between Δ and Δ^∞.

3.2 Prominent bounded sets and property $(\overline{\Omega})$

The notion of prominent bounded sets originates from the idea that, in some cases, one single bounded set can generate both diametral dimensions Δ and Δ_b ([34]). Moreover, some links between the existence of such prominent sets and the property $(\overline{\Omega})$ were recently pointed out ([6, 13]).

First of all, let us clarify what Terzioğlu's notion of prominent bounded sets exactly means:

Definition 3.2.1. Let E be a metrizable locally convex space and $(U_k)_{k\in\mathbb{N}_0}$ be a basis of absolutely convex 0-neighbourhoods in E. A bounded set B in E is *prominent* if

$$\Delta(E) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall k \in \mathbb{N}_0, (\xi_n \delta_n(B, U_k))_{n\in\mathbb{N}_0} \in c_0\right\}.$$

Of course, the links between prominent bounded sets and diametral dimensions is clear:

Proposition 3.2.2. *If the metrizable locally convex space E has a prominent set, then*

$$\Delta(E) = \Delta_b(E).$$

For example, in [34], Terzioğlu proves that smooth sequence spaces of finite type have prominent bounded sets; we will see that it is in fact the consequence of another property of these spaces (cf. Proposition 3.2.13).

Now, we will explicit the definition of these prominent sets thanks to a characterization due to Terzioğlu ([34]). However, an argument in the associated proof is not always correct (in fact, it is not when the considered space verifies $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$). In order to take it into account, we make this preliminary remark:

Remark 3.2.3. If the metrizable space E is such that $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$, then every bounded set in E is prominent.

Proof. Let $(U_k)_{k\in\mathbb{N}_0}$ be a basis of absolutely convex 0-neighbourhoods in E and B be a bounded set in E. Then, we have

$$\Delta(E) \subseteq \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall k \in \mathbb{N}_0, (\xi_n \delta_n(B, U_k))_{n\in\mathbb{N}_0} \in c_0\right\} \subseteq \mathbb{C}^{\mathbb{N}_0} = \Delta(E).$$

This implies that B is prominent. □

Remark 3.2.3 means that we just have to consider spaces E for which $\Delta(E) \neq \mathbb{C}^{\mathbb{N}_0}$. In Section 4.2, we will see that, in metrizable spaces, the equality $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ is actually equivalent to the fact that E is isomorphic to a subspace of ω (or isomorphic to ω if E is itself Fréchet and infinite-dimensional).

With the assumption $\Delta(E) \neq \mathbb{C}^{\mathbb{N}_0}$, the following characterization of prominent sets can be proved ([34]):

Proposition 3.2.4. *Let E be a metrizable locally convex space such that $\Delta(E) \neq \mathbb{C}^{\mathbb{N}_0}$ and $(U_k)_{k\in\mathbb{N}_0}$ be a decreasing basis of absolutely convex 0-neighbourhoods in E. Then, a bounded set B of E is prominent if and only if, for all $m \in \mathbb{N}_0$, there exist $k \geq m$ and $C > 0$ such that*

$$\delta_n(U_k, U_m) \leq C\delta_n(B, U_k)$$

for every $n \in \mathbb{N}_0$.

Remark 3.2.5. The condition in this proposition is in fact sufficient even when $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$.

Proof. We just have to prove that the condition is necessary. For this, we fix a prominent bounded set B. So, we have

$$\Delta(E) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall k \in \mathbb{N}_0, (\xi_n \delta_n(B, U_k))_{n\in\mathbb{N}_0} \in c_0 \right\}.$$

Actually, the set $\mathcal{B} := \left\{ (\delta_n(B, U_k))_{n\in\mathbb{N}_0} : k \in \mathbb{N}_0 \right\}$ defines a Köthe set. Indeed, we obviously have $\delta_n(B, U_k) \leq \delta_n(B, U_{k+1})$ for all $k, n \in \mathbb{N}_0$ and, for every $n \in \mathbb{N}_0$, there exists $k \in \mathbb{N}_0$ with $\delta_n(B, U_k) > 0$. Otherwise, there is $n_0 \in \mathbb{N}_0$ such that, for each $k \in \mathbb{N}_0$, $\delta_{n_0}(B, U_k) = 0$: this implies that $\delta_n(B, U_k) = 0$ for all $k \in \mathbb{N}_0$ and $n \geq n_0$, so $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$.

Thus, we have

$$\Delta(E) = \lambda_0(\mathcal{B})$$

and $\Delta(E)$ is itself a Fréchet space. But, for every $m \in \mathbb{N}_0$, we also have

$$\Delta(E) \subseteq \bigcup_{k\geq m} \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : (\xi_n \delta_n(U_k, U_m))_{n\in\mathbb{N}_0} \in c_0 \right\}.$$

Then, we have two possibilities:

1. There exist k_0, n_0 such that $\delta_{n_0}(U_{k_0}, U_m) = 0$ and so $\delta_n(U_k, U_m) = 0$ for all $k \geq k_0$, $n \geq n_0$. As a consequence, if we choose $k \geq k_0$ such that $\delta_{n_0}(B, U_k) > 0$, then we can find $C > 0$ for which $\delta_n(U_k, U_m) \leq C\delta_n(B, U_k)$ for all $n \in \mathbb{N}_0$.

2. We have $\delta_n(U_k, U_m) > 0$ for all k, n. Therefore, the space

   $$\left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall k \in \mathbb{N}_0, (\xi_n \delta_n(U_k, U_m))_{n\in\mathbb{N}_0} \in c_0 \right\}$$

   is a Banach space. Applying Grothendieck's Factorization Theorem (cf. Corollary A.1.5) to the inclusion above, we can find $k_0 \geq m$ such that

   $$\lambda_0(\mathcal{B}) \subseteq \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : (\xi_n \delta_n(U_{k_0}, U_m))_{n\in\mathbb{N}_0} \in c_0 \right\}.$$

   Now, using properties of inclusions between Köthe spaces (cf. Corollary A.2.2), we know there exist $k \geq k_0$ and $C > 0$ with

   $$\delta_n(U_{k_0}, U_m) \leq C\delta_n(B, U_k)$$

   for every n. Since this inequality remains true if we replace k_0 by k, we conclude.

$\square$

With this characterization, we can directly highlight some links with the property of large bounded sets:

Proposition 3.2.6. *If E is metrizable space with a prominent bounded set such that $\Delta(E) \neq \mathbb{C}^{\mathbb{N}_0}$, then it has the property of large bounded sets.*

Proof. Let B be a prominent set in E. We fix $m \in \mathbb{N}_0$ and a sequence $(r_k)_{k\geq m}$ in $(0,\infty)$. By the previous characterization, we find $k_0 \geq m$ and $C > 0$ such that

$$\delta_n(U_{k_0}, U_m) \leq C\delta_n(B, U_{k_0})$$

for all $n \in \mathbb{N}_0$. Thus, we obtain

$$\delta_n(Cr_{k_0}B, U_{k_0}) \geq r_{k_0}\delta_n(U_{k_0}, U_m) \geq \inf_{k\geq m} \left(r_k\delta_n(U_k, U_m)\right).$$

Hence the conclusion. □

Besides, we can weaken a little bit more the condition in Proposition 3.2.4, thanks to a classic property of metrizable spaces:

Proposition 3.2.7. *Let E be a metrizable locally convex space such that $\Delta(E) \neq \mathbb{C}^{\mathbb{N}_0}$ and $(U_k)_{k\in\mathbb{N}_0}$ be a decreasing basis of absolutely convex 0-neighbourhoods in E. Then, E has a prominent bounded set if and only if, for all $m \in \mathbb{N}_0$, there exist $k \geq m$ and a bounded set B_m such that*

$$\delta_n(U_k, U_m) \leq \delta_n(B_m, U_k)$$

for every $n \in \mathbb{N}_0$.

Proof. If B is a prominent set in E, Proposition 3.2.4 implies that, for every $m \in \mathbb{N}_0$, there exist $k_m \geq m$ and $C_m > 0$ with $\delta_n(U_{k_m}, U_m) \leq C_m\delta_n(B, U_{k_m})$ for each n. Therefore, it is enough to take $B_m = C_mB$ for every $m \in \mathbb{N}_0$.

Now, assume that, for every m, there exist $k_m \geq m$ and a bounded set B_m such that $\delta_n(U_{k_m}, U_m) \leq \delta_n(B_m, U_{k_m})$ for every $n \in \mathbb{N}_0$. Since E is a metrizable space, it is well known that we can find a sequence $(\mu_m)_{m\in\mathbb{N}_0}$ in $(0,\infty)$ such that

$$B := \bigcup_{m\in\mathbb{N}_0} \mu_m B_m$$

is bounded. Then, we have $\delta_n(U_{k_m}, U_m) \leq \frac{1}{\mu_m}\delta_n(B, U_{k_m})$ for all $m, n \in \mathbb{N}_0$. Thus B is a prominent set. □

With these characterizations, we are now ready to study prominent bounded sets more precisely. In a first time, we focus on Köthe spaces.

Using Proposition 3.2.4, Terzioğlu shows in [34] that a Schwartz G_∞-space associated to l_1 has no prominent set. However, we propose here a different proof, based on the arguments in the proof of Proposition 3.2.4, which is valid for any admissible space l:

Proposition 3.2.8. *If $\lambda^l(A)$ is a Schwartz G_∞-space, then it has no prominent set.*

Proof. As usual, we write $A = (a_k)_{k\in\mathbb{N}_0}$. We assume that B is a prominent set in $\lambda^l(A)$. Then, using the notations from the proof of Proposition 3.2.4, we put $\mathcal{B} := \left\{(\delta_n(B, U_k))_{n\in\mathbb{N}_0} : k \in \mathbb{N}_0\right\}$ and we obtain

$$\lambda_0(\mathcal{B}) = \Delta\left(\lambda^l(A)\right) = \bigcup_{k\in\mathbb{N}_0} \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \xi/a_k \in c_0\right\}$$

by Proposition 1.4.3. Using Grothendieck's Factorization Theorem (cf. Corollary A.1.5), we find $k_0 \in \mathbb{N}_0$ such that

$$\Delta\left(\lambda^l(A)\right) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \xi/a_{k_0} \in c_0\right\}.$$

But, since $\lambda^l(A)$ is a Schwartz G_∞-space, $a_{k_0} \in \Delta\left(\lambda^l(A)\right)$, so $a_{k_0}/a_{k_0} \in c_0$, which is of course impossible. $\square$

This result particularly shows that the existence of prominent bounded sets is not only a non-necessary condition to have the equality of Δ and Δ_b, but also a non-necessary condition to have the property of large bounded sets.

After such results in smooth sequence spaces, we can wonder what happens for general Köthe-Schwartz echelon spaces. For this, we fix an admissible space l and a Köthe matrix $A = (a_k)_{k\in\mathbb{N}_0}$ such that $a_k/a_{k+1} \in c_0$. Using Propositions 1.3.9, 2.2.12, and 3.2.7, we see that the space $\lambda^l(A)$ has a prominent bounded set if, for every $m \in \mathbb{N}_0$, we can find $k_m > m$ and a sequence $\left(r_j^{(m)}\right)_{j\geq k_m}$ of $(0,\infty)$ such that

$$\pi_n\left(a_m/a_{k_m}\right) \leq \pi_n\left(\inf_{j\geq k_m}\left(r_j^{(m)}\left(a_{k_m}/a_j\right)\right)\right) \tag{3.1}$$

for each $n \in \mathbb{N}_0$. To obtain a condition which does not depend on the decreasing-reorganization map, we will use the following property:

Lemma 3.2.9. *Let $x, y \in c_0 \cap [0,\infty)^{\mathbb{N}_0}$ be such that $x_n \leq y_n$ for every $n \in \mathbb{N}_0$. Then we have*

$$\pi_n(x) \leq \pi_n(y)$$

for all $n \in \mathbb{N}_0$.

Proof. When $n = 0$, we have

$$\pi_0(x) = \sup\{x_k : k \in \mathbb{N}_0\} \leq \sup\{y_k : k \in \mathbb{N}_0\} = \pi_0(y).$$

Now, we take $n \in \mathbb{N}_0$. Then, we have two possibilities.

1. If there exists $j \in \{0, ..., n\}$ such that $\varphi(x,j) \notin \{\varphi(y,0), ..., \varphi(y,n)\}$, then

$$\pi_{n+1}(x) \leq \pi_j(x) = x_{\varphi(x,j)} \leq y_{\varphi(x,j)} \leq \sup\{y_k : k \notin \{\varphi(y,0), ..., \varphi(y,n)\}\} = \pi_{n+1}(y).$$

2. Assume that $\{\varphi(x,0),...,\varphi(x,n)\} = \{\varphi(y,0),...,\varphi(y,n)\}$. In this case,

$$\begin{aligned}\pi_{n+1}(x) &= \sup\{x_k : k \notin \{\varphi(x,0),...,\varphi(x,n)\}\} \\ &= \sup\{x_k : k \notin \{\varphi(y,0),...,\varphi(y,n)\}\} \\ &\leq \sup\{y_k : k \notin \{\varphi(y,0),...,\varphi(y,n)\}\} \\ &= \pi_{n+1}(y).\end{aligned}$$

Hence the conclusion. □

Consequently, the inequality (3.1) is verified if

$$\frac{a_m(n)}{a_{k_m}(n)} \leq \inf_{j\geq k_m} \left(r_j^{(m)} \frac{a_{k_m}(n)}{a_j(n)} \right)$$

for all $n \in \mathbb{N}_0$, or, equivalently, if

$$\frac{a_m(n)}{a_{k_m}(n)} \leq r_j^{(m)} \frac{a_{k_m}(n)}{a_j(n)}$$

for all $j \geq k_m$ and $n \in \mathbb{N}_0$. Therefore, the space $\lambda^l(A)$ has a prominent bounded set if, for every $m \in \mathbb{N}_0$, there exists $k_m \geq m$ such that, for all $j \geq k_m$, there exists $C_j^{(m)} > 0$ with

$$a_{k_m}^2(n) \geq C_j^{(m)} a_m(n) a_j(n)$$

for all $n \in \mathbb{N}_0$. But such a condition on the weights of A is already known as a characterization of an important property in Functional Analysis, namely the *property* $(\overline{\Omega})$ of Vogt and Wagner.

Definition 3.2.10 ([24])**.** Let E be a Fréchet space and $(\|.\|_k)_{k\in\mathbb{N}_0}$ be a fundamental system of seminorms of E.

(1) The *dual norm* of $\|.\|_k$ is defined by the map

$$\|.\|_k^* : x' \in E' \mapsto \sup\{|x'(x)| : \|x\|_k \leq 1\}.$$

(2) The space E has the *property* $(\overline{\Omega})$ if, for every $m \in \mathbb{N}_0$, there exists $k \in \mathbb{N}_0$ such that, for every $j \in \mathbb{N}_0$, there is $C > 0$ with

$$\left(\|x'\|_k^*\right)^2 \leq C\|x'\|_m^*\|x'\|_j^*$$

for each $x' \in E'$.

This property is used in the theory of nuclear Fréchet spaces (e.g. to describe such spaces in terms of quotients of s) and in the theory of splitting short exact sequences (see [24, 35, 37] for more details).

The property $(\overline{\Omega})$ is itself a topological invariant (if two Fréchet spaces are isomorphic, either both have the property $(\overline{\Omega})$ or no one verifies it) and is inherited by quotient spaces (more details are for instance available in [11, 24]).

In the context of Köthe spaces, let us mention the following characterization of the property $(\overline{\Omega})$ for "classic" admissible spaces (see for instance [11, 24] for the proof and for more details):

Theorem 3.2.11. *If $p \in \{0\} \cup [1,\infty) \cup \{\infty\}$, then the space $\lambda_p(A)$ has the property $(\overline{\Omega})$ if and only if, for every $m \in \mathbb{N}_0$, there exists $k \in \mathbb{N}_0$ such that, for all $j \in \mathbb{N}_0$, there exists $C > 0$ with*

$$a_k^2(n) \geq C a_m(n) a_j(n)$$

for all $n \in \mathbb{N}_0$.

Unfortunately, we do not know whether this characterization remains true when we consider a general admissible space. This is the reason why we will call the property in the previous theorem the *property $(\overline{\Omega})$ for weights*.

Then, our previous developments directly give the following result ([6]):

Proposition 3.2.12. *If the space $\lambda^l(A)$ has the property $(\overline{\Omega})$ for weights, then it has a prominent bounded set.*

With this result, we can for instance prove that G_1-spaces have prominent sets, as already mentioned by Terzioğlu in [34]:

Proposition 3.2.13. *If $\lambda^l(A)$ is a G_1-space, then it has the property $(\overline{\Omega})$ for weights. In particular, it has a prominent set and so the property of large bounded sets.*

Proof. We fix $m \in \mathbb{N}_0$. By definition of G_1-spaces, there exist $k \geq m$ and $C > 0$ with

$$a_m(n) \leq C a_k^2(n)$$

for every n. Now, if $j \geq k$ is given, we have $a_j(n) \leq a_j(0)$ for all $n \in \mathbb{N}_0$, so

$$a_k^2(n) \geq \frac{1}{C a_j(0)} a_m(n) a_j(n).$$

Hence the conclusion. □

In fact, Terzioğlu's result in [34] is more precise: he shows that the unit ball of l is a prominent set in $\lambda^l(A)$ if it is a G_1-space.

The "natural" calculations above, which led to the property $(\overline{\Omega})$ for weights, were, in a way, restricted by the decreasing-reorganization map. But, if we rather consider regular spaces – for which we do no need such a map –, then we can be more precise about the links between prominent sets and property $(\overline{\Omega})$ ([13]):

Proposition 3.2.14. *If $\lambda^l(A)$ is regular, then it has a prominent bounded set if and only if it verifies the property $(\overline{\Omega})$ for weights.*

Proof. Assume that B is a prominent set in $\lambda^l(A)$ and fix $m \in \mathbb{N}_0$. We know there exists a sequence $(r_k)_{k\in\mathbb{N}_0} \in (0,\infty)^{\mathbb{N}_0}$ with $B \subseteq \bigcap_{k\in\mathbb{N}_0} r_k U_k^l$.

Then, by Proposition 3.2.4, there exist $k \geq m$ and $C > 0$ with

$$\delta_n\left(U_k^l, U_m^l\right) \leq C\, \delta_n\left(B, U_k^l\right)$$

for every $n \in \mathbb{N}_0$. Now, we fix $j \geq k$. By Proposition 1.3.14, we obtain

$$\frac{a_m(n)}{a_k(n)} = \delta_n\left(U_k^l, U_m^l\right) \leq C\,\delta_n\left(B, U_k^l\right) \leq Cr_j\,\delta_n\left(U_j^l, U_k^l\right) = Cr_j\frac{a_k(n)}{a_j(n)}.$$

So $a_k^2(n) \geq \frac{1}{Cr_j}a_m(n)a_j(n)$ for any $n \in \mathbb{N}_0$. Hence the conclusion. □

In particular, this means that the property $(\overline{\Omega})$ (for weights) is closely linked to the existence of prominent bounded sets in (regular) Köthe spaces. Therefore, one can naturally wonder whether it is still the case in general Fréchet spaces. To see this, we first quote a useful characterization of the property $(\overline{\Omega})$ (cf. [24], Lemmata 29.13 and 29.16):

Proposition 3.2.15. *Let E be a Fréchet space and $(U_k)_{k\in\mathbb{N}_0}$ be a basis of absolutely convex 0-neighbourhoods in E. The space E has the property $(\overline{\Omega})$ if and only if there exists a Banach disk B such that, for all $m \in \mathbb{N}_0$ and $\theta \in (0,1)$, there exist $k \in \mathbb{N}_0$ and $C > 0$ for which*

$$U_k \subseteq rU_m + Cr^{1-\frac{1}{\theta}}B$$

for every $r > 0$.

With this, we can prove the following result ([13]):

Theorem 3.2.16. *Let E be a Fréchet space. If E verifies the property $(\overline{\Omega})$, then it has a prominent bounded set. In particular, we have $\Delta(E) = \Delta_b(E)$.*

Proof. Let $(U_k)_{k\in\mathbb{N}_0}$ be a decreasing basis of absolutely convex 0-neighbourhoods in E. By the previous proposition, there exists a bounded set B in E such that, for all $m \in \mathbb{N}_0$, there exist $k \geq m$ and $C > 0$ with

$$U_k \subseteq rU_m + \frac{C}{r}B$$

for every $r > 0$. Now, we fix $m \in \mathbb{N}_0$ and we choose $k \geq m$ and $C > 0$ as above. Let $\delta > 0$ and $L \in \mathcal{L}_n(E)$ be such that

$$B \subseteq \delta U_k + L.$$

Taking $r = 2C\delta$, we have

$$U_k \subseteq rU_m + \frac{C}{r}B \subseteq rU_m + \frac{C\delta}{r}U_k + L = 2C\delta U_m + \frac{1}{2}U_k + L.$$

But, here, we can insert the previous inclusion in its right hand-side, which gives

$$U_k \subseteq 2C\delta\left(1 + \frac{1}{2}\right)U_m + \frac{1}{4}U_k + L.$$

If we repeat this argument, we obtain for every $j \in \mathbb{N}$

$$U_k \subseteq 2C\delta\left(1+\frac{1}{2}+...+\frac{1}{2^{j-1}}\right)U_m + \frac{1}{2^j}U_k + L \subseteq 4C\delta U_m + \frac{1}{2^j}U_k + L.$$

If we choose j such that $\frac{1}{2^j}U_k \subseteq C\delta U_m$, we deduce from this

$$U_k \subseteq 5C\delta U_m + L.$$

Hence $\delta_n(U_k, U_m) \leq 5C\delta$, which implies $\delta_n(U_k, U_m) \leq 5C\delta_n(B, U_k)$. By Proposition 3.2.4, it means that B is prominent. □

So, even in general Fréchet spaces, the property $(\overline{\Omega})$ implies the existence of prominent sets. A natural question follows from this: is the existence of prominent bounded sets equivalent to the property $(\overline{\Omega})$? Actually, it is not, as explained by the next property ([13]):

Proposition 3.2.17. *Let α be an increasing unbounded sequence of $(0,\infty)$. If it is stable, then the space $\Lambda_0^l(\alpha) \times \Lambda_\infty^l(\alpha)$ has a prominent bounded set, but does not verify the property $(\overline{\Omega})$.*

Proof. We know that $\Lambda_\infty^l(\alpha)$, as a G_∞-space, has no prominent set and so does not satisfy $(\overline{\Omega})$. Consequently, the space $\Lambda_0^l(\alpha) \times \Lambda_\infty^l(\alpha)$ itself does not have the property $(\overline{\Omega})$, since this property is inherited by quotients.

However, the space $\Lambda_0^l(\alpha)$ is a G_1-space and so has a prominent set B. Actually, we will show that the bounded set $B \times \{0\}$ is prominent in $\Lambda_0^l(\alpha) \times \Lambda_\infty^l(\alpha)$, which will provide the conclusion.

For this, we will respectively denote by U_k^0 and U_k^∞ the canonical 0-neighbourhoods in $\Lambda_0^l(\alpha)$ and $\Lambda_\infty^l(\alpha)$. Then the idea is to estimate Kolmogorov's diameters in $\Lambda_0^l(\alpha)\times\Lambda_\infty^l(\alpha)$ thanks to Lemma 3.1.1 and the particular form of Kolmogorov's diameters in (stable) power series spaces.

First, we prove that, for all $m \in \mathbb{N}$, there exists $k \geq m$ such that, for all $j \geq k$,

$$\delta_n\left(U_j^0, U_m^0\right) \leq \delta_{2n+1}\left(U_j^0, U_k^0\right)$$

for every $n \in \mathbb{N}_0$. Indeed, there exists $c > 0$ with $\alpha_{2n+1} \leq c\alpha_n$ for each n. For a given m, we take $k \geq (c+1)m$. Then, if $j > k$, we have

$$\frac{\alpha_{2n+1}}{\alpha_n} \leq c \leq \frac{k}{m} - 1 \leq \frac{k}{m} - \frac{k}{j} = \frac{1/m - 1/j}{1/k} \leq \frac{1/m - 1/j}{1/k - 1/j}$$

for each $n \in \mathbb{N}_0$. Thus, we have[7] $(1/k - 1/j)\alpha_{2n+1} \leq (1/m - 1/j)\alpha_n$ and, by Proposition 1.3.14,

$$\delta_n\left(U_j^0, U_m^0\right) = \exp((1/j - 1/m)\alpha_n) \leq \exp((1/j - 1/k)\alpha_{2n+1}) = \delta_{2n+1}\left(U_j^0, U_k^0\right)$$

[7] Even when $j = k$.

for all $n \in \mathbb{N}_0$.

Next, we fix $m \in \mathbb{N}_0$ and we choose $k \geq m$ as above. Since B is prominent in $\Lambda_0^l(\alpha)$, Proposition 3.2.4 gives $j \geq k$ and $C > 0$ for which

$$\delta_n\left(U_j^0, U_k^0\right) \leq C\delta_n\left(B, U_j^0\right)$$

for every $n \in \mathbb{N}_0$. This particularly implies that

$$\delta_n\left(U_j^0, U_m^0\right) \leq C\delta_{2n+1}\left(B, U_j^0\right).$$

Furthermore, as $\alpha_{n+1} \leq \alpha_{2n+1} \leq c\alpha_n$ for every $n \in \mathbb{N}_0$, we can find $t \geq m$ such that

$$\delta_n\left(U_t^\infty, U_m^\infty\right) = \exp((m-t)\alpha_n) \leq \exp((1/j-1/m)\alpha_{n+1}) = \delta_{n+1}\left(U_j^0, U_m^0\right) \leq \delta_n\left(U_j^0, U_m^0\right)$$

for each $n \in \mathbb{N}_0$. Then, by the choices of j and t and by Lemma 3.1.1, we obtain

$$\begin{aligned}
\delta_{2n}\left(U_j^0 \times U_t^\infty, U_m^0 \times U_m^\infty\right) &\leq \sup\left\{\delta_n\left(U_j^0, U_m^0\right), \delta_n\left(U_t^\infty, U_m^\infty\right)\right\} \\
&= \delta_n\left(U_j^0, U_m^0\right) \\
&\leq C\delta_{2n+1}\left(B, U_j^0\right) \\
&\leq C\delta_{2n}\left(B, U_j^0\right) \\
&= C\delta_{2n}\left(B \times \{0\}, U_j^0 \times U_t^\infty\right)
\end{aligned}$$

for every $n \in \mathbb{N}_0$ (the last equality directly follows from the definition of Kolmogorov's diameters). Likewise, we have

$$\begin{aligned}
\delta_{2n+1}\left(U_j^0 \times U_t^\infty, U_m^0 \times U_m^\infty\right) &\leq \sup\left\{\delta_{n+1}\left(U_j^0, U_m^0\right), \delta_n\left(U_t^\infty, U_m^\infty\right)\right\} \\
&= \delta_{n+1}\left(U_j^0, U_m^0\right) \\
&\leq \delta_n\left(U_j^0, U_m^0\right) \\
&\leq C\delta_{2n+1}\left(B, U_j^0\right) \\
&= C\delta_{2n+1}\left(B \times \{0\}, U_j^0 \times U_t^\infty\right)
\end{aligned}$$

for all $n \in \mathbb{N}_0$. Therefore, we have

$$\delta_n\left(U_j^0 \times U_t^\infty, U_m^0 \times U_m^\infty\right) \leq C\delta_n\left(B \times \{0\}, U_j^0 \times U_t^\infty\right),$$

which implies by Proposition 3.2.4 that $B \times \{0\}$ is a prominent set in $\Lambda_0^l(\alpha) \times \Lambda_\infty^l(\alpha)$. □

In particular, this result implies that the property of having prominent sets is not inherited by quotients, contrary to the property $(\overline{\Omega})$. Moreover, it can be used, for instance, to construct Köthe spaces which are not regular and even not isomorphic to any regular space:

Corollary 3.2.18. *Let α be an increasing, unbounded, and stable sequence in $(0, \infty)$ and l be one of the three following admissible spaces: l_p (for $p \geq 1$), l_∞, and c_0. Then, the space $\Lambda_0^l(\alpha) \times \Lambda_\infty^l(\alpha)$ is isomorphic to a Köthe echelon space, but it is not isomorphic to any regular Köthe space.*

Proof. It is obvious by the previous property and by Propositions 3.1.13 and 3.2.14. □

In conclusion, we have just seen that the property $(\overline{\Omega})$ is a sufficient but non-necessary condition to have some prominent sets.

The existence of prominent bounded sets constitutes the last property assuring the equality of Δ and Δ_b in metrizable spaces that we present in this work. In this context, since we only obtained positive partial answers to our problem, we can wonder whether it is possible to do the same in non-metrizable spaces. Nevertheless, it appeared that there exist some Schwartz – or even nuclear – non-metrizable counterexamples to our open question. The construction of these counterexamples is presented in the next chapter.

Chapter 4

Construction of counterexamples

In this chapter, we construct (non-metrizable) locally convex spaces E such that $\Delta(E) \neq \Delta_b(E)$. We also explain why we cannot adapt this construction in metrizable spaces.

4.1 Counterexamples in non-metrizable spaces

In the previous chapters, we obtained some properties which assure the equality of Δ and Δ_b in metrizable spaces. On the other hand, the fact that Köthe-Schwartz echelon spaces verify this equality makes the research of potential counterexamples harder.

In that context, another way to obtain counterexamples would be to consider some appropriate topological properties which "force" the two diametral dimensions to be different. We already know some topological properties which bring some information about the diametral dimensions, such as the fact of being Montel, Schwartz, and nuclear. Now, our idea is the following one: can we find a condition on a Schwartz locally convex space E to have $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$? And, if so, can we simultaneously have $\Delta(E) \subsetneq \mathbb{C}^{\mathbb{N}_0}$?

For this, we consider the following definition:

Definition 4.1.1. A bounded set B of a locally convex space E is *finite-dimensional* if $\mathrm{span}(B)$ is finite-dimensional.

Of course, a natural condition to have $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$ is to ask that each bounded set in E is finite-dimensional. Indeed, we have

$$\delta_n(B, U) = 0$$

if $n \geq \dim(\mathrm{span}(B))$, when B is a finite-dimensional bounded set and U is a 0-neighbourhood. Unfortunately, such a property cannot be found in infinite-dimensional metrizable spaces:

Proposition 4.1.2. *If E is a metrizable locally convex space for which all the bounded sets are finite-dimensional, then E is itself finite-dimensional.*

Proof. Assume there exists a sequence $(x_n)_{n\in\mathbb{N}_0}$ of linearly independent elements of E. Since each singleton $\{x_n\}$ is bounded, we can find a sequence $(\mu_n)_{n\in\mathbb{N}_0}$ of $(0,\infty)$ such that

$$B := \{\mu_n x_n : n \in \mathbb{N}_0\}$$

is bounded in E. In particular, the elements of B are linearly independent. But, by assumption on the bounded sets, B is also finite-dimensional. Hence a contradiction. $\square$

This implies that the property of having only finite-dimensional bounded sets is a sufficient but non-necessary condition to have $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$ (because, for instance, the space ω is an infinite-dimensional metrizable space which verifies $\Delta_b(\omega) = \mathbb{C}^{\mathbb{N}_0}$, see below).

This is why the counterexamples we will present are not metrizable. But, to construct them, we first have to characterize spaces with only finite-dimensional bounded sets. To do this, we need the following notion:

Definition 4.1.3. A linear map $T : E \to F$ between two locally convex spaces is *locally bounded* if, for every bounded set B of E, the set $T(B)$ is bounded in F.

For example, every continuous map is locally bounded and, conversely, a locally bounded map on a bornological space is continuous (cf. [24] for more details).

Moreover, we will consider the following notations: if E is a locally convex space, we denote by E^* the *algebraic dual* of E and by E^b the set of all locally bounded linear maps from E into $\mathbb{C}$.

With these notions, we consider the next property, which is quoted in [39] for Hausdorff spaces:

Proposition 4.1.4. *Let E be a locally convex space. If $E^b = E^*$, then every bounded set of E is finite-dimensional. Conversely, if E is Hausdorff and if every bounded set in E is finite-dimensional, then $E^b = E^*$.*

Proof. Firstly, we assume that $E^b = E^*$ and we fix a bounded set B in E. If B is not finite-dimensional, we can find a sequence $(x_n)_{n\in\mathbb{N}_0}$ of linearly independent elements of B. Then, we can define a linear map $x^* \in E^*$ such that $x^*(x_n) = n$ for every $n \in \mathbb{N}_0$. But, by assumption, x^* is locally bounded, so $x^*(B)$ is bounded in $\mathbb{C}$, which is of course impossible.

Secondly, we suppose that E is a Hausdorff space for which each bounded set is finite-dimensional. We fix $x^* \in E^*$ and a bounded set B in E and we have to prove that $x^*(B)$ is bounded in $\mathbb{C}$.

By assumption, there exist $x_1, ..., x_N \in E$ ($N \in \mathbb{N}$) with $B \subseteq F := \mathrm{span}(\{x_1, ..., x_N\})$. Because E is Hausdorff, F is isomorphic to $\mathbb{C}^N$ (as locally convex spaces). As a consequence, for every $n \in \{1, ..., N\}$, the canonical projection $\Phi_n : F \to \mathbb{C}$ associated to x_n is continuous. In particular, there exists $C_n > 0$ with $|\Phi_n(b)| \leq C_n$ for each $b \in B$. Then we conclude, since we have

$$|x^*(b)| \leq \sum_{n=1}^{N} C_n |x^*(x_n)|$$

for every $b \in B$. $\square$

Remark 4.1.5. The assumption "Hausdorff" is essential in the previous result. Indeed, if we consider the space $E := \mathbb{C}$, endowed with the trivial topology, then E is bounded in itself, but the linear map

$$\mathrm{id} : E \to \mathbb{C}$$

is not locally bounded because $\mathrm{id}(E) = \mathbb{C}$ is not bounded in $\mathbb{C}$.

With this characterization, it is easy to find topologies with only finite-dimensional bounded sets. For this purpose, we recall the notion of weak topologies:

Definition 4.1.6. Let E be a vector space and F be a vector subspace of E^* (we say that the couple (E, F) is a *dual pair*). Then, we define on E the *weak topology* $\sigma(E, F)$ by the family of seminorms

$$p_M : x \in E \mapsto \sup_{x' \in M} |x'(x)|,$$

where M is a finite subset of F.

It is well-known that $\sigma(E, F)$ is the coarsest locally convex topology $\mathcal{T}$ on E such that $(E, \mathcal{T})' = F$ (cf. [24]).

This property clearly implies that the weak topology $\sigma(E, E^*)$ has only finite-dimensional bounded sets (by Proposition 4.1.4), so

$$\Delta_b(E, \sigma(E, E^*)) = \mathbb{C}^{\mathbb{N}_0}.$$

Thus, we would like to have $\Delta(E, \sigma(E, E^*)) \subsetneq \mathbb{C}^{\mathbb{N}_0}$. Unfortunately, it is not the case, as explained by the next property.

Proposition 4.1.7. *If E is a locally convex space endowed with a weak topology, then we have $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$. In particular, we have $\Delta(E) = \Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$.*

Proof. Let M be a finite subset of E^*. We consider the seminorm

$$p_M : x \in E \mapsto \sup_{x' \in M} |x'(x)|$$

and its closed unit ball U. By properties of linear functionals, we know that $\ker(p_M)$ has a finite codimension N. As a consequence, there exists an N-dimensional vector subspace L of E with

$$E = \ker(p_M) \oplus L.$$

Therefore, for every $\delta > 0$, we have $E \subseteq \delta U + L$ and so

$$\delta_N(E, U) = 0.$$

Hence the conclusion. □

In particular, this result directly implies that every weak topology is nuclear by Theorem 1.2.10. But it also means that weak topologies are not counterexamples to our open question. Consequently, we have to work a little bit more to find such counterexamples.

Nevertheless, this last property provides an equality already claimed several times in the present thesis:

Example 4.1.8. The space ω verifies $\Delta(\omega) = \Delta_b(\omega) = \mathbb{C}^{\mathbb{N}_0}$.

Now, our idea is to compare the diametral dimension of two topologies on a same vector space. Since a linear and surjective map $T : E \to F$ between two locally convex spaces E and F, with F barrelled, is nearly open, Proposition 1.2.3 implies the following result:

Proposition 4.1.9. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two locally convex topologies on the same vector space E. If $\mathcal{T}_2$ is finer that $\mathcal{T}_1$ and if $\mathcal{T}_1$ is barrelled, then*

$$\Delta(E, \mathcal{T}_2) \subseteq \Delta(E, \mathcal{T}_1).$$

Proof. Indeed, the map

$$\mathrm{id} : (E, \mathcal{T}_2) \to (E, \mathcal{T}_1)$$

is then linear, continuous, and nearly open. □

Consequently, with barrelledness, this proposition means that the diametral dimension is decreasing for the topology. Then, our idea is simple: we will consider a space E with $\Delta(E) \neq \mathbb{C}^{\mathbb{N}_0}$ and we will "strengthen" its topology (and so make its diametral dimension decrease) until it only remains finite-dimensional bounded sets.

More precisely, we fix a vector space E and a barrelled topology $\mathcal{T}_1$ on E such that

$$\Delta(E, \mathcal{T}_1) \subsetneq \mathbb{C}^{\mathbb{N}_0}.$$

Remark that there are many spaces with such properties, like regular Köthe spaces or smooth sequence spaces, and which can be chosen to be Schwartz or nuclear. More generally, we will see in Section 4.2 that a metrizable space F (resp. an infinite-dimensional Fréchet space F) is isomorphic to a subspace of ω (resp. to ω) if and only if it verifies $\Delta(F) = \mathbb{C}^{\mathbb{N}_0}$. Therefore, it is for instance enough to choose an infinite-dimensional Fréchet space $(E, \mathcal{T}_1)$ which is not isomorphic to ω.

With these assumptions, we obtain a family of counterexamples ([6]):

Theorem 4.1.10. *Let $\mathcal{T}_2$ be a locally convex topology on E for which each bounded set is finite-dimensional (for instance, $\mathcal{T}_2 = \sigma(E, E^*)$) and let $\mathcal{T}$ be the topology on E whose 0-neighbourhoods are the intersections of those of $\mathcal{T}_1$ and $\mathcal{T}_2$. Then, we have*

$$\Delta(E, \mathcal{T}) \subsetneq \Delta_b(E, \mathcal{T}).$$

In particular, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are Schwartz (resp. nuclear), then $\mathcal{T}$ is itself Schwartz (resp. nuclear).

Proof. Since $\mathcal{T}$ is finer than $\mathcal{T}_2$, every bounded set in $(E, \mathcal{T})$ is finite-dimensional. Then, by the previous proposition, we obtain

$$\Delta(E, \mathcal{T}) \subseteq \Delta(E, \mathcal{T}_1) \subsetneq \mathbb{C}^{\mathbb{N}_0} = \Delta_b(E, \mathcal{T}).$$

Hence the conclusion. □

Since $\mathcal{T}_2 := \sigma(E, E^*)$ is nuclear, we just have to take a nuclear barrelled topology $\mathcal{T}_1$ with $\Delta(E, \mathcal{T}_1) \neq \mathbb{C}^{\mathbb{N}_0}$ to obtain a nuclear non-metrizable counterexample in the previous construction. Therefore, contrary to the situation in metrizable spaces, the nuclearity is in general non-sufficient to have the equality of Δ and Δ_b. Consequently, Hilbertizability and the equality $\Delta = \Delta^\infty$ are also non-sufficient to have the equality of Δ and Δ_b.

In particular, we have just seen that our open question is in general false for (non-metrizable) Schwartz locally convex spaces, while it remains open for Schwartz metrizable spaces.

In this context, we note the importance of the assumption of metrizability for our results. But, given the previous developments, we can wonder whether we could find some other topological properties in metrizable spaces which give

$$\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}\,.$$

Such properties, combined with arguments similar to Theorem 4.1.10, could maybe bring some metrizable counterexamples. Unfortunately, as we will see in the next section, metrizable spaces with such a particular diametral dimension Δ_b cannot be counterexamples.

4.2 Metrizable spaces with $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ or $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$

The purpose of this section is to characterize (infinite-dimensional) metrizable spaces E which verify

$$\Delta(E) = \mathbb{C}^{\mathbb{N}_0} \qquad \text{and/or} \qquad \Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}\,.$$

As explained in the previous section, the Fréchet space ω verifies these two equalities. But it is also the case for the space of finite sequences

$$\varphi := \bigoplus_{n \in \mathbb{N}_0} \mathbb{C},$$

seen as a topological subspace of ω, because it is then also endowed with a weak topology. However, it is well known that this space is not complete[8], so that it cannot be isomorphic to ω, even though it has the same diametral dimensions.

More generally, every subspace E of ω verifies $\Delta(E) = \Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$. Naturally, we can wonder whether there exist some other metrizable spaces with such diametral dimensions and we will see that it is not the case.

[8]Indeed, φ is not closed in ω, since the sequence $\left(e^{(m)}\right)_{m \in \mathbb{N}_0}$ of φ, with $e^{(m)} := \sum_{k=0}^{m} e_k$, converges pointwise to $e := (1)_{k \in \mathbb{N}_0} \notin \varphi$.

To show this, we need a topological characterization of ω and its subspaces, which follows from properties of projective limits. Therefore, we first make some recalls about this notion.

Definition 4.2.1 ([24]). Let E be a complex vector space and, for every $\alpha \in A$, let E_α be a locally convex space and $\pi_\alpha : E \to E_\alpha$ be a linear map. The *projective limit associated to the projective system* $(E_\alpha, \pi_\alpha)_{\alpha \in A}$ is the space E endowed with the coarsest locally convex topology for which all the maps π_α are continuous.

It is denoted by

$$\operatorname*{proj}_{\alpha \in A}(E_\alpha, \pi_\alpha).$$

The topology of this space is defined by the seminorms

$$\sup_{\alpha \in M} (p_\alpha \circ \pi_\alpha),$$

where M is a finite subset of A and p_α is a continuous seminorm on E_α for all $\alpha \in M$.

Of course, Cartesian products (like ω itself) are projective limits for the system of canonical projections. But, in fact, we can say more: in some sense, projective limits are subspaces of products ([24]).

Proposition 4.2.2. *Let* $E := \operatorname*{proj}_{\alpha \in A}(E_\alpha, \pi_\alpha)$ *be given and suppose that* $\cap_{\alpha \in A}\{x \in E : \pi_\alpha(x) = 0\} = \{0\}$ *(it is the case if E is Hausdorff). Then the linear map*

$$\Phi : E \to \prod_{\alpha \in A} E_\alpha : x \mapsto (\pi_\alpha(x))_{\alpha \in A}$$

is an isomorphism between E and $\Phi(E)$, endowed with the induced topology.

Proof. The injectivity of Φ follows from the fact that $\cap_{\alpha \in A}\{x \in E : \pi_\alpha(x) = 0\} = \{0\}$. Besides, it is clear that Φ is linear and continuous and that $\Phi^{-1} : \Phi(E) \to E$ is also continuous. □

In fact, every locally convex space carries a projective limit topology induced by local Banach spaces ([24]) (cf. Definition 2.3.7 for the notations):

Proposition 4.2.3. *Let E be a locally convex space. If $\mathcal{P}$ is a fundamental system of seminorms of E, then*

$$E = \operatorname*{proj}_{p \in \mathcal{P}}(E_p, \iota^p).$$

In particular, if E is Hausdorff, it is (isomorphic to) a subspace of $\prod_{p \in \mathcal{P}} E_p$.

Proof. Since $\iota^p : E \to E_p$ is continuous for every $p \in \mathcal{P}$, it means that the topology of E is finer than the topology of the projective limit. Moreover, we have $p(x) = \|\iota_p(x)\|_p$ for any $x \in E$ and $p \in \mathcal{P}$, thus the topology of E is coarser than the topology of the projective limit.

As for the particular case, it directly follows from the last proposition. □

Applying these results to ω, we obtain a characterization of ω and its subspaces thanks to their local Banach spaces:

Proposition 4.2.4. *Let E be a metrizable locally convex space. Then, E is (isomorphic to) a subspace of ω if and only if all its local Banach spaces are finite-dimensional.*

Proof. We know that the topology of ω is defined by the seminorms

$$p_k : (x_n)_{n\in\mathbb{N}_0} \in \omega \mapsto \sup\{|x_0|, ..., |x_k|\},$$

where $k \in \mathbb{N}_0$. But it is clear that $\omega/\ker(p_k)$ is isomorphic to $\mathbb{C}^{k+1}$. Therefore, the local Banach space associated to p_k in (any subspace of) ω is finite-dimensional. Since this property is true for a fundamental system of seminorms of ω, it is also true for any continuous seminorm of ω.

Conversely, let $(p_m)_{m\in\mathbb{N}_0}$ be a fundamental system of seminorms of E and assume that E_{p_m} is finite-dimensional for every $m \in \mathbb{N}_0$. Then, by the last proposition, we know that E is a subspace of $\prod_{m\in\mathbb{N}_0} E_{p_m}$, which is isomorphic to ω. □

Remark 4.2.5. If E is an infinite-dimensional Fréchet space, we can be more precise: E is isomorphic to ω if and only if all its local Banach spaces are finite-dimensional.

Indeed, using the notion of *minimal Hausdorff spaces*, it is shown in [10] that every infinite-dimensional closed subspace of ω is itself isomorphic to ω. Consequently, if E is Fréchet and is an infinite-dimensional subspace of ω, then it is in particular closed in ω (as a complete subspace of a Hausdorff space) and so isomorphic to ω.

Thanks to Proposition 4.2.4, we will characterize metrizable spaces E with $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ or $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$. But, first of all, we need to explicit these equalities:

Proposition 4.2.6. *Let E be a metrizable locally convex space and $(U_k)_{k\in\mathbb{N}_0}$ be a decreasing basis of 0-neighbourhoods in E.*

(1) We have $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ if and only if, for every $m \in \mathbb{N}_0$, there exist $k \geq m$ and $n \in \mathbb{N}_0$ with $\delta_n(U_k, U_m) = 0$.

(2) We have $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$ if and only if, for all bounded set B and $m \in \mathbb{N}_0$, there exists $n \in \mathbb{N}_0$ with $\delta_n(B, U_m) = 0$.

Proof. It is clear that both conditions are sufficient, so we just have to prove that they are necessary.

(1) We assume that $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ and that there exists $m \in \mathbb{N}_0$ such that, for all $k \geq m$ and $n \in \mathbb{N}_0$, $\delta_n(U_k, U_m) > 0$. Then, we consider a sequence ξ such that

$$\xi_n = \sup\left\{\frac{1}{\delta_n(U_m, U_m)}, ..., \frac{1}{\delta_n(U_n, U_m)}\right\}$$

for every $n \geq m$. By construction, for every $k \geq m$, we have $\xi_n \delta_n(U_k, U_m) \geq 1$ when $n \geq k$, which is impossible since $\xi \in \mathbb{C}^{\mathbb{N}_0} = \Delta(E)$.

(2) Now, we assume that $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$ and we fix a bounded set B in E and $m \in \mathbb{N}_0$. Since we have

$$\Delta_b(E) \subseteq \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : (\xi_n \delta_n(B, U_m))_{n\in\mathbb{N}_0} \in c_0\right\},$$

this particularly means that

$$\mathbb{C}^{\mathbb{N}_0} = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : (\xi_n \delta_n(B, U_m))_{n\in\mathbb{N}_0} \in c_0\right\}.$$

Therefore, if $\delta_n(B, U_m) > 0$ for each $n \in \mathbb{N}_0$, the sequence $\xi := (1/\delta_n(B, U_m))_{n\in\mathbb{N}_0}$ is such that $(\xi_n \delta_n(B, U_m))_{n\in\mathbb{N}_0} \in c_0$, which is impossible.

Hence the conclusion. □

Thus, we see that the equalities $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ and $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$ are linked to Kolmogorov's diameters equal to 0. Consequently, we have to determine what such an equality to 0 exactly means. Actually, it has already been characterized in [26] for normed spaces:

Proposition 4.2.7. *Let E be a normed space and U be its closed unit ball. If B is a bounded set of E for which there exists $n \in \mathbb{N}_0$ such that*

$$\delta_n(B, U) = 0,$$

then $\dim(\mathrm{span}(B)) \leq n$.

Proof. Assume there exist $n+1$ linearly independent vectors in B, denoted by $b_1, ..., b_{n+1}$. Then, using Hahn-Banach Theorem, we find $x'_1, ..., x'_{n+1} \in E'$ such that $x'_j(b_k) = \delta_{j,k}$[9] for any $j, k \in \{1, ..., n+1\}$.

Since we have $\det\left((\delta_{j,k})_{1\leq j,k\leq n+1}\right) = 1$, there exists $\varepsilon > 0$ for which

$$|a_{j,k}| \leq \varepsilon \quad \forall j,k \in \{1, ..., n+1\} \quad \Longrightarrow \quad \det\left((\delta_{j,k} - a_{j,k})_{1\leq j,k\leq n+1}\right) \neq 0.$$

Then, we define

$$\delta := \frac{\varepsilon}{\sup\left\{\|x'_j\|^* : j \in \{1, ..., n+1\}\right\}},$$

where $\|.\|^*$ is the dual norm of the norm of E. By assumption, there exists an at most n-dimensional subspace L of E such that $B \subseteq \delta U + L$. As a consequence, for every $k \in \{1, ..., n+1\}$, we can find $u_k \in U$, $l_k \in L$ such that $b_k = \delta u_k + l_k$. But we have $|x'_j(\delta u_k)| \leq \delta \|x'_j\|^* \leq \varepsilon$ for all j, k, so

$$\det\left((x'_j(l_k))_{1\leq j,k\leq n+1}\right) = \det\left((x'_j(b_k) - x'_j(\delta u_k))_{1\leq j,k\leq n+1}\right) \neq 0.$$

Nevertheless, because $l_1, ..., l_{n+1}$ are linearly dependent in L, we also have

$$\det\left((x'_j(l_k))_{1\leq j,k\leq n+1}\right) = 0.$$

Hence a contradiction. □

[9] $\delta_{j,k}$ is the Kronecker delta, i.e. $\delta_{j,j} = 1$ and $\delta_{j,k} = 0$ if $j \neq k$.

Now, we are ready to prove the characterization claimed at the beginning of this section:

Theorem 4.2.8. *Let E be a metrizable space. If $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ or $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$, then E is (isomorphic to) a subspace of ω.*

Proof. Of course, we can assume that E is infinite-dimensional. Besides, since $\Delta(E) \subseteq \Delta_b(E)$, we just have to prove the result when $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$.

By Proposition 4.2.6, this implies that, for all bounded set B of E and all absolutely convex 0-neighbourhood U of E, there exists $n \in \mathbb{N}_0$ with $\delta_n(B, U) = 0$.

If E is not isomorphic to a subspace of ω, it means by Proposition 4.2.4 that there exists a continuous seminorm p on E for which $E/\ker p$ is infinite-dimensional. Consequently, we can find a sequence $(x_m)_{m\in\mathbb{N}_0}$ of linearly independent elements of $E/\ker p$.

Next, we choose a sequence $(y_m)_{m\in\mathbb{N}_0}$ in E such that $x_m = \Phi_p(y_m) = y_m + \ker p$ for each $m \in \mathbb{N}_0$. Moreover, we take a sequence $(\lambda_m)_{m\in\mathbb{N}_0}$ in $(0, \infty)$ for which

$$B := \{\lambda_m y_m : m \in \mathbb{N}_0\}$$

is bounded in E. By assumption, there exists $n \in \mathbb{N}_0$ with $\delta_n(B, U) = 0$, where $U = \{x \in E : p(x) \leq 1\}$. Then, by Proposition 1.1.7, we have

$$0 \leq \delta_n(\Phi_p(B), \Phi_p(U)) \leq \delta_n(B, U) = 0.$$

By the previous proposition, this implies that $\operatorname{span}(\Phi_p(B))$ is at most n-dimensional. Hence a contradiction. $\square$

Remark 4.2.9. In the same way as in Remark 4.2.5, we can precise the previous result if E is Fréchet. Indeed, if $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$ or $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$, then E is a closed subspace of ω: either it is finite-dimensional, or it is isomorphic to ω (cf. [10]).

In summary, the only metrizable spaces E for which $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$ are exactly – up to isomorphism – the subspaces of ω, and they all verify $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$. And, more precisely, ω is the unique – up to isomorphism – infinite-dimensional Fréchet space E verifying $\Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}$ (and for which we also have $\Delta(E) = \mathbb{C}^{\mathbb{N}_0}$).

In particular, this means that it is *impossible* to find a metrizable locally convex space E such that

$$\Delta(E) \subsetneq \mathbb{C}^{\mathbb{N}_0} \qquad \text{and} \qquad \Delta_b(E) = \mathbb{C}^{\mathbb{N}_0}.$$

This explains why we cannot imitate the arguments of Theorem 4.1.10 in metrizable spaces. For this reason, we do not know whether it is possible to construct metrizable counterexamples to our open question. Consequently, this problem remains open for such spaces.

These last results conclude the first part of the present thesis, dedicated to the diametral dimensions Δ and Δ_b and their equality. In the next part, we will use some notions presented in the previous sections (such as the diametral dimensions and the property $(\overline{\Omega})$) to pursue the topological study of the so-called spaces S^ν.

Part II

Applications to spaces S^ν

Chapter 5

Definition and main properties of spaces S^ν

As explained previously, spaces S^ν were defined in the context of multifractal analysis in order to determine (or at least approximate) the spectrum of singularities of a given signal. But some notions of topology were then needed to certify that "most of" the signals (for instance, in the sense of prevalence) belonging to S^ν have a spectrum of singularities equal to (a variation of) ν (cf. [3] for more details and explanations). In this work, we will focus on the functional analysis aspects of spaces S^ν.

Although spaces S^ν were introduced to study functions, Jaffard defined them thanks to their wavelet coefficients in a wavelet basis ([17]). In fact, the required conditions in the definition of spaces S^ν are independent of the choice of the wavelet basis, so that spaces S^ν can be seen as sequence spaces.

In this chapter, we provide the definition of sequence spaces S^ν and their main topological properties, which are needed to study the diametral dimensions and the property $(\overline{\Omega})$ in the context of such spaces.

5.1 General definition and topology of spaces S^ν

First of all, we present some notations used to introduce spaces S^ν. In fact, coefficients in wavelet basis are not indexed by "traditional" natural numbers, but by a binary tree. More precisely, we will consider sequences indexed by the set

$$\Lambda := \left\{(j,k) \in \mathbb{N}_0^2 : k \leq 2^j - 1\right\}.$$

Moreover, in the following, we will need an order on Λ: we endow Λ with the lexicographical order, i.e.

$$(j_1,k_1) \leq (j_2,k_2) \qquad \Longleftrightarrow \qquad (j_1 < j_2) \quad \text{or} \quad (j_1 = j_2 \quad \text{and} \quad k_1 \leq k_2).$$

This order naturally leads to a canonical identification of Λ with $\mathbb{N}_0$, which will be used several times when we will study the diametral dimension of spaces S^ν.

Then, we consider the set of all complex sequences indexed by Λ:

$$\Omega := \mathbb{C}^{\Lambda}.$$

In order to distinguish elements of Ω from usual sequences indexed by $\mathbb{N}_0$, we will denote them by symbols with an arrow, such as $\vec{c}$.

The definition of spaces S^ν is based on the number of coefficients greater than a precise lower-bound, which is formalized by the cardinality of the following sets:

$$E_j(C,\alpha)(\vec{c}) := \left\{k \in \{0,...,2^j-1\} : |c_{j,k}| \geq C2^{-\alpha j}\right\},$$

where $j \in \mathbb{N}_0$, $C > 0$, $\alpha \in \mathbb{R}$, and $\vec{c} \in \Omega$.

Finally, let us explicit what the symbol ν means:

Definition 5.1.1. An *admissible profile* is a map $\nu : \mathbb{R} \to [0,1] \cup \{-\infty\}$ which is increasing, right-continuous, and for which $\alpha_{\min} := \inf\{\alpha \in \mathbb{R} : \nu(\alpha) \geq 0\}$ is a finite real number[10].

From now on, we fix an admissible profile ν. Besides, we introduce the following notation:

$$\alpha_{\max} := \inf\{\alpha \in \mathbb{R} : \nu(\alpha) = 1\} \in \mathbb{R} \cup \{\infty\}.$$

In particular, this means that we have

$$\begin{cases} \nu(\alpha) = -\infty \quad \text{if} \quad \alpha < \alpha_{\min}; \\ \nu(\alpha) \in [0,1) \quad \text{if} \quad \alpha_{\min} \leq \alpha < \alpha_{\max}; \\ \nu(\alpha) = 1 \quad \text{if} \quad \alpha \geq \alpha_{\max}. \end{cases}$$

Furthermore, we will also use the conventions $2^{-\infty} := 0$ and $2^{\infty} := \infty$. Now, we are ready to define the space S^ν ([17]):

Definition 5.1.2. The *space* S^ν is the set of all complex sequences $\vec{c} \in \Omega$ such that

$$\forall \alpha \in \mathbb{R}, \forall \varepsilon > 0, \forall C > 0, \exists J \in \mathbb{N}_0 : \forall j \geq J, \# E_j(C,\alpha)(\vec{c}) \leq 2^{(\nu(\alpha)+\varepsilon)j}.$$

Intuitively, a sequence $\vec{c}$ is in S^ν if, asymptotically, the number of k such that $|c_{j,k}| \geq 2^{-\alpha j}$ is smaller than $2^{\nu(\alpha)j}$. Another way to describe spaces S^ν is to use the *wavelet profile* $\nu_{\vec{c}}$ of a sequence $\vec{c} \in \Omega$, defined by

$$\nu_{\vec{c}}(\alpha) := \lim_{\varepsilon \to 0^+} \left(\limsup_{j \to \infty} \left(\frac{\log(\# E_j(1, \alpha+\varepsilon)(\vec{c}))}{\log(2^j)} \right) \right).$$

More precisely, we can quote the following property (cf. [4]):

Proposition 5.1.3. *A sequence $\vec{c} \in \Omega$ belongs to S^ν if and only $\nu_{\vec{c}}(\alpha) \leq \nu(\alpha)$ for every $\alpha \in \mathbb{R}$.*

[10] Here, we use the conventions that $\inf \emptyset = \infty$ and $\inf(\mathbb{R}) = -\infty$.

Let us also mention that the set S^ν is a vector subspace of Ω. As we will see below, S^ν can be endowed with a natural topology which makes it a separable, complete, metrizable, Schwartz, non-nuclear, topological vector space. For this, we need to describe the space S^ν as a countable intersection of metrizable spaces, called *ancillary spaces* ([4]).

Definition 5.1.4. If $\alpha \in \mathbb{R}$ and $\beta \in [0,\infty) \cup \{-\infty\}$, the *ancillary space* $E(\alpha,\beta)$ is the set

$$E(\alpha,\beta) = \left\{ \vec{c} \in \Omega : \exists C, C' \geq 0 \text{ such that } \# E_j(C,\alpha)(\vec{c}) \leq C' 2^{\beta j} \ \forall j \in \mathbb{N}_0 \right\}.$$

In this situation, the map

$$d_{\alpha,\beta} : (\vec{c}, \vec{d}) \in E(\alpha,\beta)^2 \mapsto \inf \left\{ C + C' : C, C' \geq 0, \ \# E_j(C,\alpha)(\vec{c} - \vec{d}) \leq C' 2^{\beta j} \ \forall j \in \mathbb{N}_0 \right\}$$

defines a metric on $E(\alpha,\beta)$. Besides, it can be proved that $(E(\alpha,\beta), d_{\alpha,\beta})$ is a complete metric space, the topology of which is stronger than the topology of pointwise convergence (cf. [4]). Moreover, the link between the space S^ν and the ancillary spaces is the following one ([4]):

Theorem 5.1.5. *We have*

$$S^\nu = \bigcap_{\varepsilon > 0} \bigcap_{\alpha \in \mathbb{R}} E(\alpha, \nu(\alpha) + \varepsilon) = \bigcap_{m \in \mathbb{N}_0} \bigcap_{n \in \mathbb{N}_0} E(\alpha_n, \nu(\alpha_n) + \varepsilon_m),$$

where $(\varepsilon_m)_{m \in \mathbb{N}_0}$ is a null sequence of $(0,\infty)$ and $(\alpha_n)_{n \in \mathbb{N}_0}$ is a dense sequence in $\mathbb{R}$.

As announced above, the space S^ν can be seen as a countable intersection of metrizable spaces, so we can use a classic construction in metrizable spaces to define a metric topology on S^ν (cf. [4]):

Proposition 5.1.6. *Let $(\varepsilon_m)_{m \in \mathbb{N}_0}$ be a null sequence of $(0,\infty)$ and $(\alpha_n)_{n \in \mathbb{N}_0}$ be a dense sequence in $\mathbb{R}$. Then, the map*

$$\delta := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^{-m-n} \frac{d_{\alpha_n, \nu(\alpha_n) + \varepsilon_m}}{1 + d_{\alpha_n, \nu(\alpha_n) + \varepsilon_m}}$$

defines a translation-invariant metric on S^ν. What is more, the space (S^ν, δ) is a separable, complete, topological vector space such that

(1) its topology is the coarsest one for which the inclusions $S^\nu \hookrightarrow E(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$ are continuous for all m and n;

(2) a sequence is convergent (resp. Cauchy) in (S^ν, δ) if and only if it is convergent (resp. Cauchy) in $E(\alpha_n, \nu(\alpha_n) + \varepsilon_m)$ for all m and n.

In this situation, we can wonder whether this topology depends on the choice of the sequences $(\varepsilon_m)_{m \in \mathbb{N}_0}$ and $(\alpha_n)_{n \in \mathbb{N}_0}$. In fact, it does not and, more generally, Closed Graph Theorem for complete metrizable topological vector spaces provides the following property:

Proposition 5.1.7. *All the complete metrizable topologies on S^ν which are finer than the topology of pointwise convergence are equivalent.*

Therefore, from now on, we endow the space S^ν with the topology defined in Proposition 5.1.6.

5.2 Concave profiles

When the profile ν is concave, the description of the space S^ν – and its topology – is more simple. To understand this, we just recall the general notion of pseudonorms:

Definition 5.2.1. Let E be a $\mathbb{C}$-vector space and let $p \in (0,1]$ be given. A map $q : E \to [0,\infty)$ is a *p-seminorm* if

(1) for all $x \in E$ and $\lambda \in \mathbb{C}$, we have $q(\lambda x) = |\lambda| q(x)$;

(2) for all $x, y \in E$, $q(x+y)^p \leq q(x)^p + q(y)^p$.

If, furthermore, $q(x) = 0 \Rightarrow x = 0$, then q is a *p-norm*. If $p = 1$, then q is a (semi)norm. In general, a *pseudo(-semi)norm* is a p-(semi)norm for some $p \in (0,1]$.

For instance, if X is a countable set and $p > 0$, the space $l_p(X)$ is a complete $\min(1,p)$-normed space (more details, especially when $p < 1$, can be found in [19, 20]). Besides, we can mention the following property ([16, 24, 28]), which is quite important for the following:

Proposition 5.2.2. *If $p, q > 0$ are such that $p \leq q$ and if X is a countable set, then*

$$l_p(X) \subseteq l_q(X)$$

and $\|.\|_{l_q(X)} \leq \|.\|_{l_p(X)}$ on $l_p(X)$.

Pseudonorms lead to the general notion of locally pseudoconvex spaces, in the same way as seminorms define locally convex topologies:

Definition 5.2.3. Let $p \in (0,1]$ be given. A topological vector space E is *locally p-convex* if its topology can be defined by a fundamental system of p-seminorms. It is *locally pseudoconvex* if its topology can be defined by a fundamental system of pseudo-seminorms[11].

We will see later that the space S^ν is always locally pseudoconvex and, sometimes, locally p-convex for a particular p.

Now, we can define Besov spaces:

[11] In particular, two pseudo-seminorms of the topology of E are not necessarily p-seminorms for the same p.

Definition 5.2.4. Let $p > 0$ and $s \in \mathbb{R}$ be given. The Besov space $b^s_{p,\infty}$ is the vector space

$$b^s_{p,\infty} := \left\{ \vec{c} \in \Omega : \|\vec{c}\|_{b^s_{p,\infty}} := \sup_{j \in \mathbb{N}_0} \left[2^{\left(s - \frac{1}{p}\right) j} \left(\sum_{k=0}^{2^j - 1} |c_{j,k}|^p \right)^{1/p} \right] < \infty \right\},$$

endowed with the $\min(1,p)$-norm $\|.\|_{b^s_{p,\infty}}$. Moreover, we also define the space

$$b^s_{\infty,\infty} := \left\{ \vec{c} \in \Omega : \|\vec{c}\|_{b^s_{\infty,\infty}} := \sup_{j \in \mathbb{N}_0} \sup_{0 \le k \le 2^j - 1} \left(2^{sj} |c_{j,k}| \right) < \infty \right\},$$

endowed with the norm $\|.\|_{b^s_{\infty,\infty}}$.

Some inclusions between Besov spaces are already known ([1, 4]):

Proposition 5.2.5. *Let $p, q > 0$ and $r, s \in \mathbb{R}$ be given.*

(1) If $p \le q$ and $s - \frac{1}{p} \ge r - \frac{1}{q}$, then $b^s_{p,\infty} \subseteq b^r_{q,\infty}$ and

$$\|\cdot\|_{b^r_{q,\infty}} \le \|\cdot\|_{b^s_{p,\infty}}$$

on $b^s_{p,\infty}$.

(2) If $p \le q$, then $b^s_{q,\infty} \subseteq b^s_{p,\infty}$ and

$$\|\cdot\|_{b^s_{p,\infty}} \le \|\cdot\|_{b^s_{q,\infty}}$$

on $b^s_{q,\infty}$.

When ν is concave, S^ν can be described as a countable intersection of Besov spaces, with parameters linked to the *concave conjugate* of ν (which actually corresponds to the Legendre transform of ν):

Definition 5.2.6. The *concave conjugate* of ν is the map

$$\eta : p > 0 \mapsto \inf_{\alpha \ge \alpha_{\min}} \left(\alpha p - \nu(\alpha) + 1 \right).$$

The main properties of the concave conjugate η of ν are the following ones ([4]):

Proposition 5.2.7.

(1) The map $p > 0 \mapsto \frac{\eta(p)}{p}$ is decreasing.

(2) The map $p > 0 \mapsto \frac{\eta(p)}{p} - \frac{1}{p}$ is increasing.

(3) The profile ν is concave if and only if

$$\nu(\alpha) = \inf_{p > 0} \{ \alpha p - \eta(p) + 1 \}$$

for each $\alpha \ge \alpha_{\min}$.

Now, we can consider the following useful result ([4]):

Theorem 5.2.8. *Let $(p_n)_{n\in\mathbb{N}_0}$ be a dense sequence in $(0,\infty)$ and $(\varepsilon_m)_{m\in\mathbb{N}_0}$ be a null sequence of $(0,\infty)$. Then*

$$S^\nu \subseteq \bigcap_{p>0}\bigcap_{\varepsilon>0} b^{\eta(p)/p-\varepsilon}_{p,\infty} = \bigcap_{n\in\mathbb{N}_0}\bigcap_{m\in\mathbb{N}_0} b^{\eta(p_n)/p_n-\varepsilon_m}_{p_n,\infty}$$

and this inclusion becomes an equality if and only if ν is concave.

As a consequence, when we consider a concave profile ν, we just have to study spaces of type

$$\bigcap_{n\in\mathbb{N}_0}\bigcap_{m\in\mathbb{N}_0} b^{\eta(p_n)/p_n-\varepsilon_m}_{p_n,\infty}.$$

In Chapter 6.1, we will provide some new techniques to determine the diametral dimension of such intersection spaces. We will also show how to use these methods to study the property $(\overline{\Omega})$ in the context of this particular type of spaces S^ν.

5.3 Local pseudoconvexity

When ν is not concave, the description of the space S^ν is not so "simple", even if its topology can be also described thanks to some Besov spaces (see below). Actually, this description will prove that such a space is always locally pseudoconvex and we will see under which conditions it is even locally p-convex.

In fact, these results originate from a property of the concave conjugate η of ν. Indeed, in [1], Aubry and Bastin define the *convexity index* of S^ν by

$$p_0 := \inf\left\{1, \inf_{\alpha_{\min}\leq\alpha<\alpha_{\max}} \underline{\partial}^+\nu(\alpha)\right\},$$

where $\underline{\partial}^+\nu(\alpha) = \liminf_{h\to 0^+}\frac{\nu(\alpha+h)-\nu(\alpha)}{h}$ is the *right-inf derivative* of ν. Then, if $p_0>0$, they show that

$$\frac{\eta(p)}{p} = \frac{\eta(p_0)}{p_0} = \alpha_{\max}$$

if $0<p\leq p_0$[12]. Using Proposition 5.2.5 (2) for a concave profile ν, this implies that

$$S^\nu = \bigcap_{p\geq p_0}\bigcap_{\varepsilon>0} b^{\eta(p)/p-\varepsilon}_{p,\infty},$$

so that S^ν is at least locally p_0-convex. This consideration can be generalized to non-concave profiles ([1]):

Theorem 5.3.1. *The space S^ν is not p-normed for any $p\in(0,1]$. Moreover,*

[12] In particular, $\alpha_{\max}$ is always finite in that situation; this fact will be important in the following.

(1) if $p_0 > 0$, then S^ν is locally p_0-convex;

(2) if $p_0 < 1$, then S^ν is not locally p-convex for any $p \in (p_0, 1]$.

In particular, S^ν is a Fréchet space if and only if $p_0 = 1$.

When $p_0 > 0$, the topology of S^ν can be described by sums of Besov spaces. To explain this, we use the same notations as in [2]. If $\alpha, s \in \mathbb{R}$, we define on $b^\alpha_{\infty,\infty} + b^s_{p_0,\infty}$ the p_0-norm

$$\|\vec{c}\|_{\alpha,s} := \inf\left\{\|\vec{c'}\|_{b^\alpha_{\infty,\infty}} + \|\vec{c''}\|_{b^s_{p_0,\infty}} : \vec{c} = \vec{c'} + \vec{c''}\right\}.$$

Next, we consider the set

$$\mathbb{U} := \{(A, \varepsilon) : A := \{\alpha_1 \leq ... \leq \alpha_L\} \subseteq (-\infty, \alpha_{\max}),\ \varepsilon > 0\}$$

and we define an associated p_0-norm

$$|||\vec{c}|||_{A,\varepsilon} := \sup_{1 \leq l \leq L} \|\vec{c}\|_{\alpha_l - \varepsilon, \alpha_l - \varepsilon + (1 - \nu(\alpha_l))/p_0}.$$

In the following, we will denote by $B_{A,\varepsilon}$ the closed unit ball associated to $|||.|||_{A,\varepsilon}$.

Then, we have the following result ([2]):

Theorem 5.3.2. *If $p_0 > 0$, a fundamental system of p_0-norms of S^ν is given by the family of p_0-norms $|||.|||_{A,\varepsilon}$, where $(A, \varepsilon) \in \mathbb{U}$.*

We will use this description of the topology of S^ν in Chapter 7 to determine its diametral dimension and to prove that it has the property $(\overline{\Omega})$.

When $p_0 = 0$, then S^ν is locally pseudoconvex, but not locally p-convex for any p ([2]). In this situation, the topology of S^ν is described by the same kind of pseudonorms, except they do not have a fixed convexity index ([2]):

Theorem 5.3.3. *Let $(\alpha_n)_{n\in\mathbb{N}_0}$ be a dense sequence of $(\alpha_{\min} - 1/2, \infty)$ and $(\varepsilon_m)_{m\in\mathbb{N}_0}$ be a null sequence of $(0, \infty)$ and assume that $p_0 = 0$. Then, the topology of S^ν is defined by the family of pseudonorms*

$$\|\vec{c}\|_{\mathfrak{b}_{m,n}} := \inf\left\{\|\vec{c'}\|_{b^{\alpha_n}_{\infty,\infty}} + \|\vec{c''}\|_{b^{s_{m,n}}_{p_{m,n},\infty}} : \vec{c} = \vec{c'} + \vec{c''}\right\},$$

where $m, n \in \mathbb{N}_0$, $p_{m,n} := \frac{\varepsilon_m}{2(\alpha_n - \alpha_{\min} + 1)}$ and $s_{m,n} := \alpha_n + \frac{1 - \nu(\alpha_n) - \varepsilon_m}{p_{m,n}}$.

5.4 Diametral dimension and spaces S^ν

As explained in the introduction, we do not know whether spaces S^ν are isomorphic. Of course, the convexity index can be used to distinguish such spaces, but we do not know what happens for two spaces S^ν with the same convexity index.

This is the reason why the study of some topological invariants in the context of spaces S^ν could be interesting. Actually, in [2], Aubry and Bastin determine the formula of the diametral dimension of locally p-convex spaces S^ν: if $p_0 > 0$, we have

$$\Delta\left(S^\nu\right) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0, \left(\xi_n (n+1)^{-s}\right)_{n\in\mathbb{N}_0} \in c_0\right\}.$$

This result can be used to prove that such spaces are Schwartz and non-nuclear, but it is already known that every space S^ν (even locally pseudoconvex) is Schwartz (cf. [2]). For the non-nuclearity, Theorem 1.2.10 brings a proof when $p_0 = 1$. Nevertheless, when $p_0 < 1$, the situation is more complex. Indeed, there is no clear definition of nuclearity for non-locally convex spaces (cf. [2, 22])[13], so that we can consider S^ν to be non-nuclear by default in that case.

The reader can also remark that the diametral dimension above is the same for all the locally p-convex spaces S^ν, which implies that it is impossible to topologically distinguish them thanks to this invariant. Another idea is to use the diametral dimension Δ_b. However, it is easy to see that

$$\left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0, \left(\xi_n (n+1)^{-s}\right)_{n\in\mathbb{N}_0} \in c_0\right\} = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0, \left(\xi_n (n+1)^{-s}\right)_{n\in\mathbb{N}_0} \in l_\infty\right\},$$

which implies that

$$\Delta\left(S^\nu\right) = \Delta^\infty\left(S^\nu\right).$$

Therefore, using a simple adaptation of Theorem 2.3.4, we obtain:

Theorem 5.4.1. *If $p_0 > 0$, we have*

$$\Delta\left(S^\nu\right) = \Delta_b\left(S^\nu\right) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0, \left(\xi_n (n+1)^{-s}\right)_{n\in\mathbb{N}_0} \in c_0\right\}.$$

This unfortunately means that Δ_b is itself unhelpful to distinguish spaces S^ν. Nonetheless, given its links with the diametral dimensions (cf. Proposition 3.2.16), we decided to study the property $(\overline{\Omega})$ in the context of spaces S^ν. This pushed us into considering two different situations ([12]).

1. When the profile ν is concave, spaces S^ν take a particular form, which is easier to manipulate. With this assumption, we will obtain an extension of the formula of the diametral dimension of spaces S^ν to some non-locally p-convex ones. Moreover, it appears that the techniques used in that context can also be used to prove that these spaces have the property $(\overline{\Omega})$.

2. When $p_0 > 0$, the techniques developed in the previous point can be adapted to provide another proof for the formula of $\Delta\left(S^\nu\right)$ and to show that spaces S^ν verify the property $(\overline{\Omega})$.

[13] In these papers, it is explained that taking the locally convex definition of nuclearity or the characterization from Theorem 1.2.10 as a definition of nuclearity in locally pseudoconvex spaces actually implies the local convexity.

These different developments will be presented in the next two chapters. However, we already point out the fact that $(\overline{\Omega})$ cannot distinguish the studied spaces S^{ν} either, since they all verify it.

Chapter 6

The concave case

From now on, we assume that the admissible profile ν is concave. Therefore, using Theorem 5.2.8, we know that

$$S^\nu = \bigcap_{n\in\mathbb{N}_0}\bigcap_{m\in\mathbb{N}_0} b^{\eta(p_n)/p_n-\varepsilon_m}_{p_n,\infty},$$

if $(p_n)_{n\in\mathbb{N}_0}$ is a dense sequence in $(0,\infty)$ and $(\varepsilon_m)_{m\in\mathbb{N}_0}$ is a null sequence of $(0,\infty)$. Without loss of generality, we will assume that

$$\varepsilon_m := \frac{1}{m+1} \qquad \text{and} \qquad \{p_n : n\in\mathbb{N}_0\} = \mathbb{Q}\cap(0,\infty) =: \mathbb{Q}^+ .$$

To simplify the notations, we also put

$$p'_n = \frac{\eta(p_n)}{p_n}.$$

As far as the topology of S^ν is concerned, Theorem 5.1.7 implies that it is the coarsest one for which the inclusions

$$S^\nu \hookrightarrow b^{p'_n-\varepsilon_m}_{p_n,\infty}$$

are continuous. Therefore, the topology of S^ν is defined by the family of pseudonorms

$$P^{(I)}_m(\vec{c}) := \sup_{i\in I} \|\vec{c}\|_{b^{p'_i-\varepsilon_m}_{p_i,\infty}} = \sup_{i\in I}\sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_i-\frac{1}{p_i}-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_i}\right)^{1/p_i}\right],$$

where $m\in\mathbb{N}_0$ and I is a finite subset of $\mathbb{N}_0$. The closed unit ball associated to $P^{(I)}_m$ will be denoted by

$$B_{P^{(I)}_m}.$$

Our purpose is to determine the diametral dimension of S^ν (independently of developments from [2]) and to check if it verifies the property $(\overline{\Omega})$ or not. Unless otherwise specified, the results presented in this chapter come from [12].

6.1 Diametral dimension

In order to calculate the diametral dimension of S^ν, we will follow some developments presented in Köthe spaces (cf. Proposition 1.3.5) to approximate Kolmogorov's diameters. Indeed, some weights of the form

$$(2^{\alpha j})_{(j,k)\in\Lambda},$$

with $\alpha \in \mathbb{R}$, explicitly appear in the pseudonorms above. However, unlike Köthe spaces, these weights are not indexed by natural numbers, but by the binary tree Λ. Consequently, we will use the lexicographical order of Λ to define an enumeration of the weights.

More precisely, the idea is to define indexes on Λ: we decide that $(0,0)$ is the couple of index 0 in Λ and, after, we index the other couples of Λ by following its order. For instance, the couples $(1,0)$, $(1,1)$, $(2,0)$, and $(2,1)$ will be respectively indexed by 1, 2, 3, and 4.

Then, we define the natural number $j(n)$ as the first component j of the couple (j,k) of index $n \in \mathbb{N}_0$ in Λ. In other words, $j(n)$ is the first component of the $(n+1)$-th couple (j,k) of Λ, according to the order of Λ and beginning with $(0,0)$. It is illustrated in the next figure:

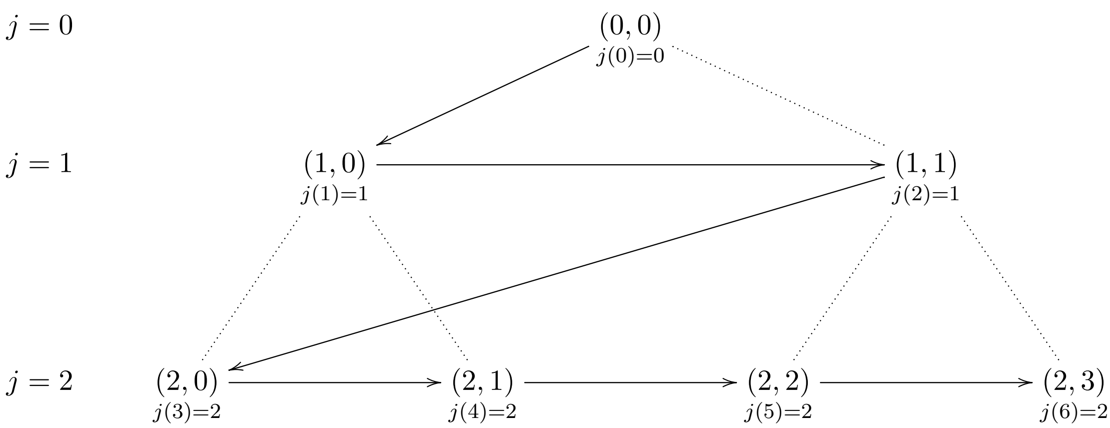

Figure 6.1: The first couples of Λ, crossed according to the order of Λ, and the associated scales $j(n)$.

Hence, the $(n+1)$-th component of the sequence $(2^{\alpha j})_{(j,k)\in\Lambda}$ is $2^{\alpha j(n)}$. In order to determine the precise value of $j(n)$, we consider the following result ([11]):

Lemma 6.1.1. *If $n \in \mathbb{N}_0$, then $j(n)$ is the unique natural number verifying*

$$2^{j(n)} - 1 \leq n \leq 2^{j(n)+1} - 2.$$

Proof. If (j,k) is the couple of index $n \in \mathbb{N}_0$ in Λ, then we know that $j(n) = j$. More generally, all the couples of the form $(j,k') \in \Lambda$ are associated to an index $n' \in \mathbb{N}_0$ such that $j(n') = j$. So, to conclude, we just have to count the number of couples of the form (j,k), for a given $j \in \mathbb{N}_0$, and to determine the corresponding indices.

For this, let us have a look at the first couples of Λ (cf. Figure 6.1).

1. If $j = 0$, there is only one corresponding couple, namely $(0,0)$, with an index equal to 0. It verifies $j(0) = 0$.

2. If $j = 1$, there are two associated couples: $(1,0)$ and $(1,1)$. Besides, their indices are respectively 1 and 2, so $j(1) = j(2) = 2$.

3. If $j = 2$, we have four couples, $(2,0)$, $(2,1)$, $(2,2)$, and $(2,3)$, with respective indices 3, 4, 5, and 6. Consequently, we have $j(3) = j(4) = j(5) = j(6) = 2$.

By induction, if $j \in \mathbb{N}$ is given, the couples of Λ of the form (j,k) are indexed from $1 + 2 + ... + 2^{j-1}$ to $2 + ... + 2^j$, i.e. from

$$\frac{2^j - 1}{2 - 1} = 2^j - 1 \qquad \text{to} \qquad 2\frac{2^j - 1}{2 - 1} = 2^{j+1} - 2.$$

Also remark that this remains true if $j = 0$. So, this means that for the indices $n \in \mathbb{N}_0$ such that $2^j - 1 \leq n \leq 2^{j+1} - 2$, we have $j(n) = j$. Hence the conclusion. □

In the following, we will also consider the *unit sequences* $\overrightarrow{e_{j,k}} \in \Omega$, with $(j,k) \in \Lambda$, the components of which are equal to 0, except the component (j,k) which is equal to 1.

Now, we are ready to study the diametral dimension of S^ν with, first, an upper-bound for some Kolmogorov's diameters:

Proposition 6.1.2. *Let $m, k_0 \in \mathbb{N}_0$, with $k_0 \geq m$, $n \in \mathbb{N}_0$, and a finite subset I of $\mathbb{N}_0$ be given. Then, we have*

$$\delta_n\left(B_{P_{k_0}^{(I)}}, B_{P_m^{(I)}}\right) \leq 2^{(\varepsilon_{k_0} - \varepsilon_m) j(n)}.$$

Proof. Let $\vec{c} \in B_{P_{k_0}^{(I)}}$. We consider the projection $P_n : S^\nu \to S^\nu$ onto the linear span of the first n vectors $\overrightarrow{e_{j,k}}$ (according to the order of Λ and beginning with $\overrightarrow{e_{0,0}}$). Then, since we have

$$\left(\vec{c} - P_n(\vec{c})\right)_{j,k} = 0$$

if $j < j(n)$, we obtain

$$\begin{aligned}
P_m^{(I)}\left(\vec{c} - P_n(\vec{c})\right) &= \sup_{i\in I}\sup_{j\in\mathbb{N}_0}\left[2^{\left(p_i' - \frac{1}{p_i} - \varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}\left|\left(\vec{c} - P_n(\vec{c})\right)_{j,k}\right|^{p_i}\right)^{1/p_i}\right] \\
&\leq \sup_{i\in I}\sup_{j\geq j(n)}\left[2^{\left(p_i' - \frac{1}{p_i} - \varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_i}\right)^{1/p_i}\right] \\
&= \sup_{i\in I}\sup_{j\geq j(n)}\left[2^{(\varepsilon_{k_0}-\varepsilon_m)j}2^{\left(p_i' - \frac{1}{p_i} - \varepsilon_{k_0}\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_i}\right)^{1/p_i}\right] \\
&\leq 2^{(\varepsilon_{k_0}-\varepsilon_m)j(n)}P_{k_0}^{(I)}(\vec{c}) \\
&\leq 2^{(\varepsilon_{k_0}-\varepsilon_m)j(n)}.
\end{aligned}$$

Therefore, we have

$$\vec{c} = \left(\vec{c} - P_n(\vec{c})\right) + P_n(\vec{c}) \in 2^{(\varepsilon_{k_0}-\varepsilon_m)j(n)}B_{P_m^{(I)}} + P_n(S^\nu)$$

and we conclude because $\dim\left(P_n(S^\nu)\right) = n$. □

This first result implies an inclusion for the description of the diametral dimension of S^ν.

Corollary 6.1.3. *If ν is concave, we have*

$$\left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0,\ \left(\xi_n(n+1)^{-s}\right)_{n\in\mathbb{N}_0} \in c_0\right\} \subseteq \Delta(S^\nu).$$

Proof. We fix $m \in \mathbb{N}_0$ and a finite subset I of $\mathbb{N}_0$ and we take $k_0 > m$. By Lemma 6.1.1, we have $(n+2)/2 \leq 2^{j(n)}$ for any $n \in \mathbb{N}_0$. Consequently, the previous proposition implies that

$$\delta_n\left(B_{P_{k_0}^{(I)}}, B_{P_m^{(I)}}\right) \leq 2^{(\varepsilon_{k_0}-\varepsilon_m)j(n)} \leq \left(\frac{n+2}{2}\right)^{\varepsilon_{k_0}-\varepsilon_m} \leq 2^{\varepsilon_m-\varepsilon_{k_0}}(n+1)^{\varepsilon_{k_0}-\varepsilon_m}.$$

Hence the conclusion. □

However, the other inclusion is not so easily obtained. Indeed, for this purpose, we have to find some finite index sets $I \subseteq \mathbb{N}_0$ and some $m \in \mathbb{N}_0$ for which we have a "suitable" lower-bound for diameters of type

$$\delta_n\left(B_{P_{k_0}^{(J)}}, B_{P_m^{(I)}}\right),$$

where $k_0 \geq m$ and J is a finite subset of $\mathbb{N}_0$ with $J \supseteq I$. But, in that situation, we have to compare (pseudo)norms of type l_p with different p, which was not the case in Köthe spaces.

In that context, we will use the following general result ([26]), which brings a lower-bound for Kolmogorov's diameters and which is called Tikhomirov's Theorem. Remark this can be easily generalized to p-normed spaces.

Proposition 6.1.4. *Let $(E, \|.\|)$ be a normed space, U be the closed unit ball of E, and B be a bounded set of E. If there exist $\delta > 0$ and a projection $P : E \to E$ with $\|P\| \leq 1$, $\dim P(E) = n + 1$, and*

$$\delta U \cap P(E) \subseteq B,$$

then $\delta_n(B, U) \geq \delta$.

Proof. Assume that $\delta_n(B, U) < \delta$. Then, we take $\delta_0 > 0$, with $\delta_n(B, U) < \delta_0 < \delta$, and $L \in \mathcal{L}_n(E)$ such that

$$B \subseteq \delta_0 U + L.$$

Since E is Hausdorff, $P(L)$ is a closed proper subspace of $P(E)$. Therefore, there exists $x \in P(E) \setminus P(L)$ and $\varepsilon > 0$ such that

$$\{y \in P(E) : \|x - y\| \leq \varepsilon\} \subseteq P(E) \setminus P(L).$$

In particular,

$$\lambda := \inf \{\|x - z\| : z \in P(L)\} \geq \varepsilon > 0.$$

Next, we choose $z_0 \in P(L)$ such that

$$\lambda_0 := \|x - z_0\| < \frac{\lambda\delta}{\delta_0}.$$

Using the facts that $P(U) \subseteq U$ by assumption and that P is a projection, we obtain

$$U \cap P(E) = P(U \cap P(E)) \subseteq \frac{1}{\delta} P(B) \subseteq \frac{\delta_0}{\delta} P(U) + P(L) \subseteq \frac{\delta_0}{\delta} U + P(L).$$

Therefore, there exist $u \in U$ and $z \in P(L)$ such that

$$\frac{x - z_0}{\lambda_0} = \frac{\delta_0}{\delta} u + z.$$

This implies that $\|x - z_0 - \lambda_0 z\| \leq \lambda_0(\delta_0/\delta) < \lambda$. But, by definition of λ and by the fact that $z_0 + \lambda_0 z \in P(L)$, we also have $\|x - z_0 - \lambda_0 z\| \geq \lambda$, which leads to a contradiction. □

Now, we have to construct some index sets – on the basis of the previous proposition – which will lead us to the formula of $\Delta(S^\nu)$, as explained above. Nonetheless, the construction we will present needs an assumption on the asymptotic behaviour of the map $p > 0 \mapsto \eta(p)/p$ around 0. For this, we consider the following lemma:

Lemma 6.1.5. *We have*

$$\lim_{p \to 0^+} \frac{\eta(p)}{p} = \alpha_{\max}.$$

Proof. Let L be the limit $\lim_{p\to 0^+} \eta(p)/p$, which exists since the map $p > 0 \mapsto \eta(p)/p$ is decreasing. Now, we distinguish two different situations:

1. If $\alpha_{\max}$ is finite, we have

$$\eta(p) = \inf_{\alpha\geq\alpha_{\min}} (\alpha p - \nu(\alpha) + 1) \leq \alpha_{\max} p - \nu(\alpha_{\max}) + 1 = \alpha_{\max} p,$$

so $L \leq \alpha_{\max} < \infty$. Moreover, by Proposition 5.2.7, we have

$$\nu(\alpha) = \inf_{p>0} \{\alpha p - \eta(p) + 1\} = \inf_{p>0} \{p(\alpha - \eta(p)/p) + 1\} = 1$$

if $\alpha \geq L$, because $\eta(p)/p \leq L$ for any $p > 0$. Thus $\alpha_{\max} \leq L$ and $L = \alpha_{\max}$.

2. If $\alpha_{\max}$ is infinite, then L is also infinite. Otherwise, we apply Proposition 5.2.7 in the same way as in the previous paragraph and we obtain $\alpha_{\max} \leq L < \infty$ again, which is impossible.

Hence the conclusion. □

In what follows, our construction of some finite index sets will need the map $p > 0 \mapsto \eta(p)/p$ to be bounded around 0. According to the previous result, this means that we will assume that $\alpha_{\max}$ is finite. As explained in Section 5.3, this is particularly the case when S^ν is locally p-convex.

Nevertheless, there also exist some concave profiles ν such that $\alpha_{\max}$ is finite and S^ν is only locally pseudoconvex. It is for instance the case for

$$\nu_0 : \alpha \in \mathbb{R} \mapsto \begin{cases} -\infty \quad \text{if} \quad \alpha < 0, \\ \sqrt{1-(\alpha-1)^2} \quad \text{if} \quad 0 \leq \alpha \leq 1, \\ 1 \quad \text{if} \quad \alpha > 1, \end{cases}$$

since it is easily verified that p_0 is then equal to 0. In particular, this explains why the next developments will bring an extension of the formula of $\Delta(S^\nu)$ to some non-locally p-convex spaces.

However, the assumption that $\alpha_{\max}$ is finite is not always verified, as illustrated by the following concave profile:

$$\nu_1 : \alpha \in \mathbb{R} \mapsto \begin{cases} -\infty & \text{if} \quad \alpha < 0, \\ 1 - e^{-\alpha} & \text{if} \quad \alpha \geq 0. \end{cases}$$

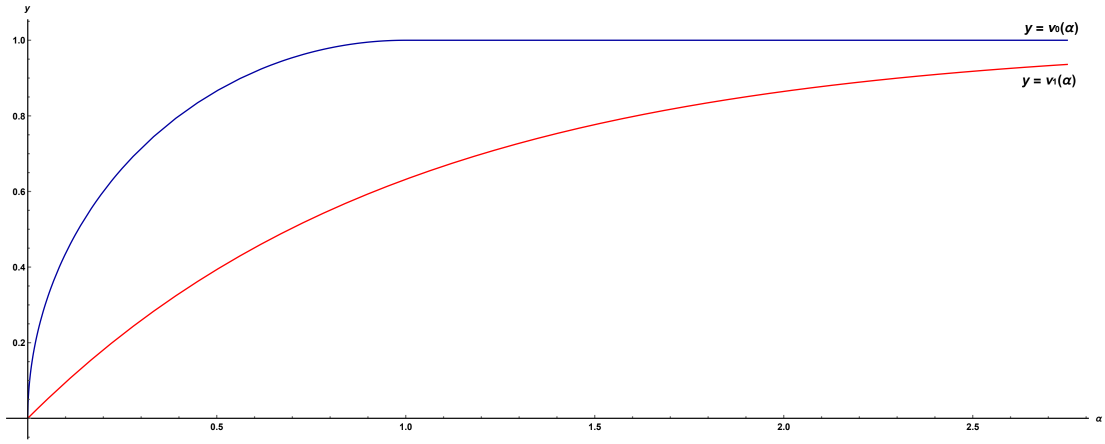

Figure 6.2: Representation of the profiles ν_0 (blue) and ν_1 (red)

Therefore, it will remain some concave profiles ν for which we do not know the exact expression of $\Delta(S^\nu)$.

Now, we are ready to present the announced construction of finite index sets.

Construction 6.1.6. We assume that $\alpha_{\max} < \infty$ and we fix $\varepsilon \in \mathbb{Q}^+$. Then, we construct the index set I_ε by the following procedure:

1. Since the map $p > 0 \mapsto \eta(p)/p$ is decreasing and admits a finite limit for $p \to 0^+$ and since the sequence $(p_n)_{n\in\mathbb{N}_0}$ is dense in $(0,\infty)$, we can choose $i_0 \in \mathbb{N}_0$ such that
$$0 < p < p_{i_0} \Rightarrow \frac{\eta(p)}{p} - \frac{\eta(p_{i_0})}{p_{i_0}} \leq \varepsilon.$$
2. There exists $l \in \mathbb{N}_0$ such that $l\varepsilon < \dfrac{1}{p_{i_0}} \leq (l+1)\varepsilon$. If $l > 0$, we define $i_k \in \mathbb{N}_0$ for $k \in \{1, ..., l\}$ by
$$\frac{1}{p_{i_k}} = \frac{1}{p_{i_0}} - k\varepsilon.$$
3. We put $I_\varepsilon := \{i_0, ..., i_l\}$.

The introduction of such index sets I_ε is in fact justified by the following proposition, which corresponds to the inclusion needed in Tikhomirov's Theorem (Proposition 6.1.4) to obtain a lower-bound for Kolmogorov's diameters:

Proposition 6.1.7. *If $\alpha_{\max}$ is finite and if $n \in \mathbb{N}_0$ and $\varepsilon \in \mathbb{Q}^+$ are given, then there exists $i \in I_\varepsilon$ such that, for all $m \in \mathbb{N}_0$ and $\vec{c} \in S^\nu$,*
$$\|\vec{c}\|_{b^{p'_n-\varepsilon_m}_{p_n,\infty}} \leq \sup_{j\in\mathbb{N}_0} \left[2^{\left(p'_i - \frac{1}{p_i} + \varepsilon - \varepsilon_m\right)j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right],$$
where the right-hand side takes its values in $[0,\infty]$.

Proof. Referring to the definition of the set I_ε, we consider three different cases.

1. If $p_n \leq p_{i_0}$, Proposition 5.2.5 gives

$$
\begin{aligned}
\|\vec{c}\|_{b^{p'n-\varepsilon_m}_{p_n,\infty}} \leq \|\vec{c}\|_{b^{p'n-\varepsilon_m}_{p_{i_0},\infty}} &= \sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_n-\frac{1}{p_{i_0}}-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_0}}\right)^{1/p_{i_0}}\right] \\
&= \sup_{j\in\mathbb{N}_0}\left[2^{\left(\left(p'_n-p'_{i_0}\right)+p'_{i_0}-\frac{1}{p_{i_0}}-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_0}}\right)^{1/p_{i_0}}\right] \\
&\leq \sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_{i_0}-\frac{1}{p_{i_0}}+\varepsilon-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_0}}\right)^{1/p_{i_0}}\right],
\end{aligned}
$$

   by the first point in Construction 6.1.6.

2. If $p_{i_0} < p_n < p_{i_l}$, there exists $t \in \{0, ..., l-1\}$ with $p_{i_t} \leq p_n \leq p_{i_{t+1}}$. In the same way as in the previous paragraph, we obtain

$$
\begin{aligned}
\|\vec{c}\|_{b^{p'n-\varepsilon_m}_{p_n,\infty}} &\leq \sup_{j\in\mathbb{N}_0}\left[2^{\left(\left(p'_n-p'_{i_{t+1}}\right)+p'_{i_{t+1}}-\frac{1}{p_{i_{t+1}}}-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_{t+1}}}\right)^{1/p_{i_{t+1}}}\right] \\
&\leq \sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_{i_{t+1}}-\frac{1}{p_{i_{t+1}}}+\varepsilon-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_{t+1}}}\right)^{1/p_{i_{t+1}}}\right],
\end{aligned}
$$

   since Proposition 5.2.7 gives

$$
\begin{aligned}
p'_n - p'_{i_{t+1}} &\leq p'_{i_t} - p'_{i_{t+1}} \\
&= \left(p'_{i_t} - \frac{1}{p_{i_t}}\right) - \left(p'_{i_{t+1}} - \frac{1}{p_{i_{t+1}}}\right) + \frac{1}{p_{i_t}} - \frac{1}{p_{i_{t+1}}} \\
&\leq \frac{1}{p_{i_t}} - \frac{1}{p_{i_{t+1}}} \\
&= \varepsilon.
\end{aligned}
$$

3. If now $p_n \geq p_{i_l}$, Propositions 5.2.2 and 5.2.7 lead to

$$\begin{aligned}
\|\vec{c}\|_{b^{p'n-\varepsilon_m}_{p_n,\infty}} &\leq \sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_n-\frac{1}{p_n}-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_l}}\right)^{1/p_{i_l}}\right] \\
&\leq \sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_{i_l}-\frac{1}{p_n}-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_l}}\right)^{1/p_{i_l}}\right] \\
&\leq \sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_{i_l}-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_l}}\right)^{1/p_{i_l}}\right] \\
&\leq \sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_{i_l}-\frac{1}{p_{i_l}}+\frac{1}{p_{i_l}}-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_l}}\right)^{1/p_{i_l}}\right] \\
&\leq \sup_{j\in\mathbb{N}_0}\left[2^{\left(p'_{i_l}-\frac{1}{p_{i_l}}+\varepsilon-\varepsilon_m\right)j}\left(\sum_{k=0}^{2^j-1}|c_{j,k}|^{p_{i_l}}\right)^{1/p_{i_l}}\right]
\end{aligned}$$

since, by Construction 6.1.6, we have $1/p_{i_l} \leq \varepsilon$.

Hence the conclusion. □

Combining Propositions 6.1.4 and 6.1.7, we are now ready to find a lower-bound for some Kolmogorov's diameters in S^ν:

Proposition 6.1.8. *Let $m, k_0 \in \mathbb{N}_0$, with $k_0 \geq m$, and $\varepsilon \in \mathbb{Q}^+$ be given. If $\alpha_{\max}$ is finite and if J is a finite subset of $\mathbb{N}_0$ such that $J \supseteq I_\varepsilon$, then we have*

$$\delta_n\left(B_{P^{(J)}_{k_0}}, B_{P^{(I_\varepsilon)}_m}\right) \geq 2^{(\varepsilon_{k_0}-\varepsilon_m-\varepsilon)j(n)}$$

for every $n \in \mathbb{N}_0$.

Proof. Let $P_{n+1} : S^\nu \to S^\nu$ be the projection onto the linear span of the first $n+1$ vectors $\overrightarrow{e_{j,k}}$ (beginning with $\overrightarrow{e_{0,0}}$).

Of course, $B_{P^{(J)}_{k_0}}$ is a bounded set of the (pseudo)normed space $\left(S^\nu, P^{(I_\varepsilon)}_m\right)$ and we have

$$P^{(I_\varepsilon)}_m(P_{n+1}(\vec{c})) \leq P^{(I_\varepsilon)}_m(\vec{c})$$

for each $\vec{c} \in S^\nu$. Therefore, by Proposition 6.1.4, we just have to show that

$$2^{(\varepsilon_{k_0}-\varepsilon_m-\varepsilon)j(n)}B_{P^{(I_\varepsilon)}_m} \cap P_{n+1}\left(S^\nu\right) \subseteq B_{P^{(J)}_{k_0}}.$$

Equivalently, we will get the conclusion if we prove the following inequality on $P_{n+1}(S^\nu)$:

$$P_{k_0}^{(J)} \leq 2^{(\varepsilon_m+\varepsilon-\varepsilon_{k_0})j(n)} P_m^{(I_\varepsilon)}.$$

But, for any $\vec{c} \in P_{n+1}(S^\nu)$, Proposition 6.1.7 gives

$$\begin{aligned}
P_{k_0}^{(J)}(\vec{c}) &\leq \sup_{i\in I_\varepsilon} \sup_{j\in\mathbb{N}_0} \left[2^{\left(p_i'-\frac{1}{p_i}+\varepsilon-\varepsilon_{k_0}\right)j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&= \sup_{i\in I_\varepsilon} \sup_{j\leq j(n)} \left[2^{(\varepsilon_m+\varepsilon-\varepsilon_{k_0})j} 2^{\left(p_i'-\frac{1}{p_i}-\varepsilon_m\right)j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&\leq 2^{(\varepsilon_m+\varepsilon-\varepsilon_{k_0})j(n)} \sup_{i\in I_\varepsilon} \sup_{j\leq j(n)} \left[2^{\left(p_i'-\frac{1}{p_i}-\varepsilon_m\right)j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&= 2^{(\varepsilon_m+\varepsilon-\varepsilon_{k_0})j(n)} P_m^{(I_\varepsilon)}(\vec{c})
\end{aligned}$$

and we are done. □

This last result provides the other inclusion for the diametral dimension of S^ν.

Theorem 6.1.9. *If ν is concave and if $\alpha_{\max}$ is finite, then*

$$\Delta(S^\nu) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0,\ \left(\xi_n (n+1)^{-s}\right)_{n\in\mathbb{N}_0} \in c_0 \right\}.$$

Proof. Let $\xi \in \Delta(S^\nu)$ and $s > 0$ be given. We choose $m \in \mathbb{N}_0$ such that $\varepsilon_m \leq s/2$ and we consider the index set I_ε with $\varepsilon := \varepsilon_m$.

By definition of the diametral dimension, there exist $k_0 \geq m$ and a finite subset J of $\mathbb{N}_0$, with $J \supseteq I_\varepsilon$, such that

$$\left(\xi_n \delta_n \left(B_{P_{k_0}^{(J)}}, B_{P_m^{(I_\varepsilon)}} \right) \right)_{n\in\mathbb{N}_0} \in c_0.$$

Moreover, the previous result gives

$$\begin{aligned}
\delta_n \left(B_{P_{k_0}^{(J)}}, B_{P_m^{(I_\varepsilon)}} \right) &\geq 2^{(\varepsilon_{k_0}-\varepsilon_m-\varepsilon)j(n)} \\
&= 2^{(\varepsilon_{k_0}-2\varepsilon_m)j(n)} \\
&\geq 2^{-2\varepsilon_m j(n)} \\
&\geq (n+1)^{-2\varepsilon_m} \\
&\geq (n+1)^{-s},
\end{aligned}$$

because $2^{j(n)} \leq n+1$ by Lemma 6.1.1. Therefore, $(\xi_n(n+1)^{-s})_{n\in\mathbb{N}_0} \in c_0$. We conclude by Corollary 6.1.3. □

Consequently, as explained before, we have just extended the formula of $\Delta(S^\nu)$ given in [2] to some locally pseudoconvex spaces S^ν (in fact, to the spaces S^ν with ν concave, $\alpha_{\max} < \infty$, and $p_0 = 0$).

However, it appeared that the index sets I_ε can be also used to prove that the associated spaces S^ν verify the property $(\overline{\Omega})$, as we will see in the next section. This explains why we will adapt Construction 6.1.6 in the context of locally p-convex spaces S^ν in Chapter 7. Indeed, we will see there that these adapted arguments not only provide another technique to obtain the formula of $\Delta(S^\nu)$ (different from the proof in [2]), but they can also be used to show that such spaces have the property $(\overline{\Omega})$.

6.2 Property $(\overline{\Omega})$

As we saw in Section 3.2, property $(\overline{\Omega})$ is defined by means of dual norms. Unfortunately, this implies that its definition is in general not easy to handle, especially when we consider some pseudonorms as in the case of spaces S^ν.

This is the reason why we decided to use a characterization of this property, based on some inclusions between 0-neighbourhoods (cf. [24]):

Theorem 6.2.1. *Let E be a Fréchet space and $(U_k)_{k\in\mathbb{N}_0}$ be a basis of 0-neighbourhoods in E. Then the space E verifies $(\overline{\Omega})$ if and only if*

$$\forall m \in \mathbb{N}_0 \ \exists k \in \mathbb{N}_0 \ \forall j \in \mathbb{N}_0 \ \exists C > 0 : U_k \subseteq rU_j + \frac{C}{r} U_m \ \forall r > 0. \tag{6.1}$$

Nonetheless, this characterization is obtained thanks to Bipolar Theorem, so that we do not know whether (6.1) remains equivalent to $(\overline{\Omega})$ or not when we consider non-locally convex spaces. Therefore, in order to distinguish (6.1) from $(\overline{\Omega})$, we will denote the property (6.1) by (Ω_{id}), according to the notations used in [36].

Now, we will use index sets I_ε – and more precisely Proposition 6.1.7 – to prove that the considered spaces S^ν have the property (Ω_{id}).

Theorem 6.2.2. *If ν is concave and if $\alpha_{\max}$ is finite, then S^ν verifies the property (Ω_{id}). In particular, if ν is concave and S^ν is locally convex, then it has the property $(\overline{\Omega})$.*

Proof. Let $m \in \mathbb{N}_0$ and a finite subset I_m of $\mathbb{N}_0$ be given. Then, we choose $k_0 \geq m$ such that $\varepsilon_{k_0} < \varepsilon_m/2$ and we define $I_{k_0} := I_\varepsilon \cup I_m$, with $\varepsilon := \varepsilon_m/2 - \varepsilon_{k_0}$.

In that context, we fix $j_0 \in \mathbb{N}_0$, a finite subset I_{j_0} of $\mathbb{N}_0$, and $r > 0$ and we prove the inclusion

$$B_{P_{k_0}^{(I_{k_0})}} \subseteq rB_{P_{j_0}^{(I_{j_0})}} + \frac{1}{r} B_{P_m^{(I_m)}}.$$

For this, we consider two different situations.

1. If $r \leq 1$, we clearly have

$$B_{P_{k_0}^{(I_{k_0})}} \subseteq B_{P_m^{(I_m)}} \subseteq \frac{1}{r} B_{P_m^{(I_m)}} \subseteq rB_{P_{j_0}^{(I_{j_0})}} + \frac{1}{r} B_{P_m^{(I_m)}}.$$

2. If $r \geq 1$, we choose $J \in \mathbb{N}_0$ such that $2^{\frac{\varepsilon_m}{2}J} \leq r \leq 2^{\frac{\varepsilon_m}{2}(J+1)}$. For a given $\vec{c} \in B_{P_{k_0}^{(I_{k_0})}}$, we put

$$\vec{c_1} := \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} c_{j,k}\, \overrightarrow{e_{j,k}} \qquad \text{and} \qquad \vec{c_2} := \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{j,k}\, \overrightarrow{e_{j,k}}.$$

Therefore, since $I_\varepsilon \subseteq I_{k_0}$ and $\varepsilon + \varepsilon_{k_0} - \varepsilon_{j_0} \leq \varepsilon_m/2$, Proposition 6.1.7 implies that

$$\begin{aligned}
P_{j_0}^{(I_{j_0})}(\vec{c_1}) &\leq \sup_{i \in I_\varepsilon} \sup_{j \leq J} \left[2^{\left(p_i' - \frac{1}{p_i} + \varepsilon - \varepsilon_{j_0}\right)j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&\leq \sup_{i \in I_{k_0}} \sup_{j \leq J} \left[2^{(\varepsilon + \varepsilon_{k_0} - \varepsilon_{j_0})j} 2^{(p_i' - \frac{1}{p_i} - \varepsilon_{k_0})j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&\leq 2^{\frac{\varepsilon_m}{2}J} \sup_{i \in I_{k_0}} \sup_{j \leq J} \left[2^{(p_i' - \frac{1}{p_i} - \varepsilon_{k_0})j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&\leq 2^{\frac{\varepsilon_m}{2}J} P_{k_0}^{(I_{k_0})}(\vec{c}) \\
&\leq r.
\end{aligned}$$

Thus we have $\vec{c_1} \in rB_{P_{j_0}^{(I_{j_0})}}$. Besides, because $I_m \subseteq I_{k_0}$ and $\varepsilon_{k_0} < \varepsilon_m/2$, we also have

$$\begin{aligned}
P_m^{(I_m)}(\vec{c_2}) &= \sup_{i \in I_m} \sup_{j \geq J+1} \left[2^{(p_i' - \frac{1}{p_i} - \varepsilon_m)j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&\leq \sup_{i \in I_{k_0}} \sup_{j \geq J+1} \left[2^{(\varepsilon_{k_0} - \varepsilon_m)j} 2^{(p_i' - \frac{1}{p_i} - \varepsilon_{k_0})j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&\leq 2^{\frac{-\varepsilon_m}{2}(J+1)} \sup_{i \in I_{k_0}} \sup_{j \geq J+1} \left[2^{(p_i' - \frac{1}{p_i} - \varepsilon_{k_0})j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_i} \right)^{1/p_i} \right] \\
&\leq 2^{\frac{-\varepsilon_m}{2}(J+1)} P_{k_0}^{(I_{k_0})}(\vec{c}) \\
&\leq \frac{1}{r},
\end{aligned}$$

so $\vec{c_2} \in \frac{1}{r} B_{P_m^{(I_m)}}$. Subsequently,

$$\vec{c} = \vec{c_1} + \vec{c_2} \in rB_{P_{j_0}^{(I_{j_0})}} + \frac{1}{r} B_{P_m^{(I_m)}}.$$

Hence the conclusion. $\square$

In particular, we have just shown that locally convex spaces S^ν, with ν concave, verify the property $(\overline{\Omega})$. Therefore, we can wonder what happens for locally convex spaces S^ν without the assumption that ν is a concave profile.

In fact, as explained before, we will see in the next chapter that they still have the property $(\overline{\Omega})$ and, more generally, that every locally p-convex space S^ν has the property (Ω_{id}). Consequently, these developments will extend the previous theorem to locally p-convex spaces S^ν, but we recall the fact that Theorem 6.2.2 is also valid for some locally pseudoconvex spaces S^ν.

Chapter 7

The locally p-convex case

In this chapter, we no longer assume that ν is concave. Besides, from now on, we suppose that

$$p_0 = \inf\left\{1, \inf_{\alpha_{\min}\leq\alpha<\alpha_{\max}} \underline{\partial}^+\nu(\alpha)\right\} > 0,$$

so S^ν is locally p_0-convex. From Section 5.3, we recall that the topology of S^ν is given by the pseudonorms

$$|||\vec{c}|||_{A,\varepsilon} = \sup_{1\leq l\leq L} \|\vec{c}\|_{\alpha_l-\varepsilon,\alpha_l-\varepsilon+(1-\nu(\alpha_l))/p_0},$$

where $A=\{\alpha_1,...,\alpha_L\}$ and ε are such that $(A,\varepsilon)\in\mathbb{U}$ (i.e. $\alpha_1\leq...\leq\alpha_L<\alpha_{\max}$ and $\varepsilon>0$) and

$$\|\vec{c}\|_{\alpha_l-\varepsilon,\alpha_l-\varepsilon+(1-\nu(\alpha_l))/p_0} = \inf\left\{\|\vec{c'}\|_{b^{\alpha_l-\varepsilon}_{\infty,\infty}} + \|\vec{c''}\|_{b^{\alpha_l-\varepsilon+(1-\nu(\alpha_l))/p_0}_{p_0,\infty}} : \vec{c}=\vec{c'}+\vec{c''}\right\}.$$

Moreover, we point out the fact that we have

$$\|\vec{c'}\|_{b^{\alpha_l-\varepsilon}_{\infty,\infty}} = \sup_{j\in\mathbb{N}_0}\sup_{0\leq k\leq 2^j-1}\left(2^{(\alpha_l-\varepsilon)j}|c'_{j,k}|\right)$$

and

$$\|\vec{c''}\|_{b^{\alpha_l-\varepsilon+(1-\nu(\alpha_l))/p_0}_{p_0,\infty}} = \sup_{j\in\mathbb{N}_0}\left[2^{\left(\alpha_l-\varepsilon-\frac{\nu(\alpha_l)}{p_0}\right)j}\left(\sum_{k=0}^{2^j-1}|c''_{j,k}|^{p_0}\right)^{1/p_0}\right].$$

As explained previously, we will adapt the construction of the index sets I_ε presented in the previous chapter to prove that S^ν verifies the property (Ω_{id}). We will also show that this new construction can be used to obtain the expression of $\Delta(S^\nu)$ with other techniques than those used in [2].

7.1 Diametral dimension

In the same way as in Section 6.1, we can easily obtain a first inclusion for the diametral dimension of S^ν, which was already proved in [2]. But, before this, we make a small remark, which is straightforward but important for the use of the pseudonorms of S^ν:

Remark 7.1.1. Let $\alpha, s \in \mathbb{R}$, $J \in \mathbb{N}_0$, and $\vec{c} \in \Omega$ be given. If $c_{j,k} = 0$ if $j < J$, then

$$\|\vec{c}\|_{\alpha,s} = \inf\left\{\|\vec{c'}\|_{b^{\alpha}_{\infty,\infty}} + \|\vec{c''}\|_{b^{s}_{p_0,\infty}} : \vec{c} = \vec{c'} + \vec{c''},\ c'_{j,k} = c''_{j,k} = 0 \text{ if } j < J\right\}.$$

Proof. Indeed, if $c_{j,k} = 0$ when $j < J$ and if $\vec{c} = \vec{c'} + \vec{c''}$, then the sequences

$$\vec{c_1} := \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} c'_{j,k}\overrightarrow{e_{j,k}} \qquad \text{and} \qquad \vec{c_2} := \sum_{j=J}^{\infty} \sum_{k=0}^{2^j-1} c''_{j,k}\overrightarrow{e_{j,k}}$$

are such that $\vec{c} = \vec{c_1} + \vec{c_2}$ and

$$\|\vec{c_1}\|_{b^{\alpha}_{\infty,\infty}} + \|\vec{c_2}\|_{b^{s}_{p_0,\infty}} \leq \|\vec{c'}\|_{b^{\alpha}_{\infty,\infty}} + \|\vec{c''}\|_{b^{s}_{p_0,\infty}}.$$

□

This leads to the following lemma ([2]), which is useful to compare two pseudonorms of S^ν:

Lemma 7.1.2. *Let $(A, \varepsilon) \in \mathbb{U}$, $\varepsilon' \in (0, \varepsilon)$, $\vec{c} \in S^\nu$, and $J \in \mathbb{N}_0$ be given. If $c_{j,k} = 0$ for every $j < J$, then*

$$|||\vec{c}|||_{A,\varepsilon} \leq 2^{(\varepsilon'-\varepsilon)J} |||\vec{c}|||_{A,\varepsilon'}.$$

Proof. Assume that $c_{j,k} = 0$ if $j < J$ and that $\vec{c} = \vec{c'} + \vec{c''}$, with $c'_{j,k} = c''_{j,k} = 0$ if $j < J$. Then, fix $\alpha \in A$. We have

$$\begin{aligned}
\|\vec{c'}\|_{b^{\alpha-\varepsilon}_{\infty,\infty}} &= \sup_{j \geq J} \sup_{0 \leq k \leq 2^j-1} \left(2^{(\alpha-\varepsilon)j} |\vec{c'}_{j,k}|\right) \\
&= \sup_{j \geq J} \sup_{0 \leq k \leq 2^j-1} \left(2^{(\varepsilon'-\varepsilon)j} 2^{(\alpha-\varepsilon')j} |\vec{c'}_{j,k}|\right) \\
&\leq 2^{(\varepsilon'-\varepsilon)J} \|\vec{c'}\|_{b^{\alpha-\varepsilon'}_{\infty,\infty}}
\end{aligned}$$

and

$$\begin{aligned}
\|\vec{c''}\|_{b^{\alpha-\varepsilon+(1-\nu(\alpha))/p_0}_{p_0,\infty}} &= \sup_{j \geq J} \left[2^{\left(\alpha-\varepsilon-\frac{\nu(\alpha)}{p_0}\right)j} \left(\sum_{k=0}^{2^j-1} |c''_{j,k}|^{p_0}\right)^{1/p_0}\right] \\
&= \sup_{j \geq J} \left[2^{(\varepsilon'-\varepsilon)j} 2^{\left(\alpha-\varepsilon'-\frac{\nu(\alpha)}{p_0}\right)j} \left(\sum_{k=0}^{2^j-1} |c''_{j,k}|^{p_0}\right)^{1/p_0}\right] \\
&\leq 2^{(\varepsilon'-\varepsilon)J} \|\vec{c''}\|_{b^{\alpha-\varepsilon'+(1-\nu(\alpha))/p_0}_{p_0,\infty}}.
\end{aligned}$$

Therefore,

$$\|\vec{c'}\|_{b^{\alpha-\varepsilon}_{\infty,\infty}} + \|\vec{c''}\|_{b^{\alpha-\varepsilon+(1-\nu(\alpha))/p_0}_{p_0,\infty}} \leq 2^{(\varepsilon'-\varepsilon)J}\left(\|\vec{c'}\|_{b^{\alpha-\varepsilon'}_{\infty,\infty}} + \|\vec{c''}\|_{b^{\alpha-\varepsilon'+(1-\nu(\alpha))/p_0}_{p_0,\infty}}\right),$$

which gives the conclusion. □

Using this tool, we are now ready to obtain an upper-bound for Kolmogorov's diameters:

Proposition 7.1.3. *Let $(A,\varepsilon) \in \mathbb{U}$ and $\varepsilon' \in (0,\varepsilon)$ be given. Then, for each $n \in \mathbb{N}_0$, we have*

$$\delta_n\left(B_{A,\varepsilon'}, B_{A,\varepsilon}\right) \leq 2^{(\varepsilon'-\varepsilon)j(n)}.$$

Proof. Let $P_n : S^\nu \to S^\nu$ be the projection onto the linear span of the first n vectors $\overrightarrow{e_{j,k}}$ (beginning with $\overrightarrow{e_{0,0}}$) and let us take $\vec{c} \in B_{A,\varepsilon'}$. Then, the sequence $\vec{c} - P_n(\vec{c})$ is such that $(\vec{c} - P_n(\vec{c}))_{j,k} = 0$ if $j < j(n)$. Therefore, we get

$$|||\vec{c} - P_n(\vec{c})|||_{A,\varepsilon} \leq 2^{(\varepsilon'-\varepsilon)j(n)} |||\vec{c} - P_n(\vec{c})|||_{A,\varepsilon'}$$

by the previous lemma. Since $|||\vec{c} - P_n(\vec{c})|||_{A,\varepsilon'} \leq |||\vec{c}|||_{A,\varepsilon'} \leq 1$, we obtain

$$\vec{c} = (\vec{c} - P_n(\vec{c})) + P_n(\vec{c}) \in 2^{(\varepsilon'-\varepsilon)j(n)} B_{A,\varepsilon} + P_n(S^\nu).$$

Hence the conclusion. □

Corollary 7.1.4. *We have*

$$\left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0,\ \left(\xi_n (n+1)^{-s}\right)_{n \in \mathbb{N}_0} \in c_0\right\} \subseteq \Delta(S^\nu).$$

Proof. It is clear, because if $(A,\varepsilon) \in \mathbb{U}$ and $\varepsilon' \in (0,\varepsilon)$ are given, we get, by the previous result and by Lemma 6.1.1,

$$\delta_n\left(B_{A,\varepsilon'}, B_{A,\varepsilon}\right) \leq 2^{(\varepsilon'-\varepsilon)j(n)} \leq \left(\frac{n+2}{2}\right)^{\varepsilon'-\varepsilon} \leq 2^{\varepsilon-\varepsilon'}(n+1)^{\varepsilon'-\varepsilon}$$

for each $n \in \mathbb{N}_0$. □

For the other inclusion, the idea is the same as in Section 6.1: we will construct some suitable index sets to use Tikhomirov's Theorem ([12]).

Construction 7.1.5. Let $\varepsilon_0 > 0$ be given. We define a finite subset A_{ε_0} of $(-\infty, \alpha_{\max})$ according to the following procedure:

1. We choose $\alpha_1 \in (-\infty, \alpha_{\min})$.
2. There exists $L \in \mathbb{N}$ such that $\alpha_1 + (L-1)\varepsilon_0 < \alpha_{\max} \leq \alpha_1 + L\varepsilon_0$. For $l \in \{1, ..., L\}$, we put $\alpha_l := \alpha_1 + (l-1)\varepsilon_0$.

3. We define $A_{\varepsilon_0} := \{\alpha_1, ..., \alpha_L\}$.

In the same way as in Proposition 6.1.7, we have the following inequalities:

Proposition 7.1.6. *Let $\alpha \in (-\infty, \alpha_{\max})$ and $\varepsilon', \varepsilon_0 > 0$ be given. Then, there exists $\alpha' \in A_{\varepsilon_0}$ such that, for every $\vec{c} \in \Omega$, we have*

1. $\|\vec{c}\|_{b^{\alpha-\varepsilon'+(1-\nu(\alpha))/p_0}_{p_0,\infty}} \leq \sup_{j\in\mathbb{N}_0} \left[2^{(\varepsilon_0-\varepsilon')j} 2^{\left(\alpha' - \frac{\nu(\alpha')}{p_0}\right)j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_0} \right)^{1/p_0} \right]$,

2. $\|\vec{c}\|_{b^{\alpha-\varepsilon'}_{\infty,\infty}} \leq \sup_{j\in\mathbb{N}_0} \left[2^{(\varepsilon_0-\varepsilon')j} 2^{\alpha' j} \left(\sup_{0\leq k\leq 2^j-1} |c_{j,k}| \right) \right]$,

where the right-hand sides both take their values in $[0, \infty]$.

Proof. We consider three different situations.

- If $\alpha \leq \alpha_1 < \alpha_{\min}$, then $\nu(\alpha) = \nu(\alpha_1) = -\infty$ and the first inequality is verified when $\alpha' = \alpha_1$. Indeed, if $\vec{c} = \vec{0}$, then this inequality becomes $0 \leq 0$ and if $\vec{c} \neq \vec{0}$, it becomes $\infty \leq \infty$.

  Besides, since we have $\alpha \leq \alpha_1 + \varepsilon_0$, the second inequality is also true.

- If $\alpha \geq \alpha_L$, we have $\alpha_L \leq \alpha < \alpha_{\max} \leq \alpha_L + \varepsilon_0$ by construction, so that the second inequality is correct for $\alpha' = \alpha_L$. Moreover, since ν is increasing, we obtain

$$\alpha - \varepsilon' - \frac{\nu(\alpha)}{p_0} \leq \alpha_L + \varepsilon_0 - \varepsilon' - \frac{\nu(\alpha)}{p_0} \leq \alpha_L + \varepsilon_0 - \varepsilon' - \frac{\nu(\alpha_L)}{p_0},$$

  which implies

$$\|\vec{c}\|_{b^{\alpha-\varepsilon'+(1-\nu(\alpha))/p_0}_{p_0,\infty}} \leq \sup_{j\in\mathbb{N}_0} \left[2^{(\varepsilon_0-\varepsilon')j} 2^{\left(\alpha_L - \frac{\nu(\alpha_L)}{p_0}\right)j} \left(\sum_{k=0}^{2^j-1} |c_{j,k}|^{p_0} \right)^{1/p_0} \right].$$

- If $\alpha_1 < \alpha < \alpha_L$, then there exists $l \in \{1, ..., L-1\}$ with $\alpha_l \leq \alpha \leq \alpha_{l+1}$. Therefore, we get $\alpha \leq \alpha_l + \varepsilon_0$ (by construction of A_{ε_0}) and $-\nu(\alpha)/p_0 \leq -\nu(\alpha_l)/p_0$. In the same way as in the previous point, this gives the two claimed inequalities for $\alpha' = \alpha_l$.

This leads to the conclusion. □

Thanks to these two inequalities, we obtain an upper-bound for the pseudonorms of S^ν of the same kind as in Proposition 6.1.7:

Corollary 7.1.7. *Let $\alpha \in (-\infty, \alpha_{\max})$, $\varepsilon, \varepsilon', \varepsilon_0 > 0$, and $J \in \mathbb{N}_0$ be given. Then, if $\vec{c} \in \Omega$ is such that $c_{j,k} = 0$ $\forall j > J$, we have*

$$\|\vec{c}\|_{\alpha-\varepsilon', \alpha-\varepsilon'+(1-\nu(\alpha))/p_0} \leq 2^{(\varepsilon_0+\varepsilon)J} |||\vec{c}|||_{A_{\varepsilon_0}, \varepsilon}.$$

Proof. We take the parameter $\alpha' \in A_{\varepsilon_0}$ provided by the previous property and two sequences $\vec{c'}$ and $\vec{c''}$ such that $\vec{c} = \vec{c'} + \vec{c''}$ and $c'_{j,k} = c''_{j,k} = 0$ if $j > J$. Then, we obtain

$$\begin{aligned}
\|\vec{c'}\|_{b^{\alpha-\varepsilon'}_{\infty,\infty}} &\leq \sup_{j\in\mathbb{N}_0}\left[2^{(\varepsilon_0-\varepsilon')j}2^{\alpha' j}\left(\sup_{0\leq k\leq 2^j-1}|c'_{j,k}|\right)\right] \\
&= \sup_{j\leq J}\left[2^{(\varepsilon_0+\varepsilon-\varepsilon')j}2^{(\alpha'-\varepsilon)j}\left(\sup_{0\leq k\leq 2^j-1}|c'_{j,k}|\right)\right] \\
&\leq \sup_{j\leq J}\left[2^{(\varepsilon_0+\varepsilon)j}2^{(\alpha'-\varepsilon)j}\left(\sup_{0\leq k\leq 2^j-1}|c'_{j,k}|\right)\right] \\
&\leq 2^{(\varepsilon_0+\varepsilon)J}\|\vec{c'}\|_{b^{\alpha'-\varepsilon}_{\infty,\infty}}.
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
\|\vec{c''}\|_{b^{\alpha-\varepsilon'+(1-\nu(\alpha))/p_0}_{p_0,\infty}} &\leq \sup_{j\in\mathbb{N}_0}\left[2^{(\varepsilon_0-\varepsilon')j}2^{\left(\alpha'-\frac{\nu(\alpha')}{p_0}\right)j}\left(\sum_{k=0}^{2^j-1}|c''_{j,k}|^{p_0}\right)^{1/p_0}\right] \\
&= \sup_{j\leq J}\left[2^{(\varepsilon_0+\varepsilon-\varepsilon')j}2^{\left(\alpha'-\frac{\nu(\alpha')}{p_0}-\varepsilon\right)j}\left(\sum_{k=0}^{2^j-1}|c''_{j,k}|^{p_0}\right)^{1/p_0}\right] \\
&\leq \sup_{j\leq J}\left[2^{(\varepsilon_0+\varepsilon)j}2^{\left(\alpha'-\frac{\nu(\alpha')}{p_0}-\varepsilon\right)j}\left(\sum_{k=0}^{2^j-1}|c''_{j,k}|^{p_0}\right)^{1/p_0}\right] \\
&\leq 2^{(\varepsilon_0+\varepsilon)J}\|\vec{c''}\|_{b^{\alpha'-\varepsilon+(1-\nu(\alpha'))/p_0}_{p_0,\infty}}.
\end{aligned}$$

Consequently, we have

$$\|\vec{c'}\|_{b^{\alpha-\varepsilon'}_{\infty,\infty}} + \|\vec{c''}\|_{b^{\alpha-\varepsilon'+(1-\nu(\alpha))/p_0}_{p_0,\infty}} \leq 2^{(\varepsilon_0+\varepsilon)J}\left(\|\vec{c'}\|_{b^{\alpha'-\varepsilon}_{\infty,\infty}} + \|\vec{c''}\|_{b^{\alpha'-\varepsilon+(1-\nu(\alpha'))/p_0}_{p_0,\infty}}\right),$$

which proves

$$\|\vec{c}\|_{\alpha-\varepsilon',\alpha-\varepsilon'+(1-\nu(\alpha))/p_0} \leq 2^{(\varepsilon_0+\varepsilon)J}\|\vec{c}\|_{\alpha'-\varepsilon,\alpha'-\varepsilon+(1-\nu(\alpha'))/p_0} \leq 2^{(\varepsilon_0+\varepsilon)J}|||\vec{c}|||_{A_{\varepsilon_0},\varepsilon}.$$

□

Now, we can use this property of the sets A_{ε_0} to obtain a new prove providing the formula of $\Delta(S^\nu)$. For this, we will use the following approximation of Kolmogorov's diameters:

Proposition 7.1.8. *Let $(A,\varepsilon) \in \mathbb{U}$, $\varepsilon' \in (0,\varepsilon)$, and $\varepsilon_0 >$ be given, with $A_{\varepsilon_0} \subseteq A$. Then we have*

$$\delta_n(B_{A,\varepsilon'}, B_{A_{\varepsilon_0},\varepsilon}) \geq 2^{-(\varepsilon_0+\varepsilon)j(n)}$$

for any $n \in \mathbb{N}_0$.

Proof. Let $P_{n+1} : S^\nu \to S^\nu$ be the projection onto the linear span of the first $n+1$ vectors $\overrightarrow{e_{j,k}}$ (beginning with $\overrightarrow{e_{0,0}}$). If $\vec{c} \in P_{n+1}(S^\nu)$, then $c_{j,k} = 0$ if $j > j(n)$ and, by Corollary 7.1.7, this implies that

$$|||\vec{c}|||_{A,\varepsilon'} \leq 2^{(\varepsilon_0+\varepsilon)j(n)} |||\vec{c}|||_{A_{\varepsilon_0},\varepsilon}.$$

Thus $2^{-(\varepsilon_0+\varepsilon)j(n)} B_{A_{\varepsilon_0},\varepsilon} \cap P_{n+1}(S^\nu) \subset B_{A,\varepsilon'}$ and we conclude by Tikhomirov's Theorem (Proposition 6.1.4). □

Finally, we obtain:

Theorem 7.1.9. *If S^ν is locally p_0-convex (i.e. $p_0 > 0$), then*

$$\Delta(S^\nu) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0, \ \left(\xi_n (n+1)^{-s}\right)_{n \in \mathbb{N}_0} \in c_0 \right\}.$$

Proof. Let $\xi \in \Delta(S^\nu)$ and $s > 0$ be given. We put $\varepsilon := \varepsilon_0 := s/2$. Then, there exists $(A, \varepsilon') \in \mathbb{U}$ such that $\varepsilon' < \varepsilon$, $A_{\varepsilon_0} \subseteq A$, and

$$\left(\xi_n \delta_n (B_{A,\varepsilon'}, B_{A_{\varepsilon_0},\varepsilon})\right)_{n \in \mathbb{N}_0} \in c_0.$$

But, by the previous result and by Lemma 6.1.1, we get

$$\delta_n(B_{A,\varepsilon'}, B_{A_{\varepsilon_0},\varepsilon}) \geq 2^{-(\varepsilon_0+\varepsilon)j(n)} \geq (n+1)^{-(\varepsilon_0+\varepsilon)} = (n+1)^{-s},$$

so $(\xi_n(n+1)^{-s})_{n\in\mathbb{N}_0} \in c_0$. Hence the conclusion by Corollary 7.1.4. □

Now, we will use the parameter sets A_{ε_0} to study the property (Ω_{id}) in the context of locally p-convex spaces S^ν.

7.2 Property $(\overline{\Omega})$

As in the concave case, the locally p-convex spaces S^ν verify the property (Ω_{id}) ([12]):

Theorem 7.2.1. *The space S^ν verifies the property (Ω_{id}). In particular, if $p_0 = 1$, then S^ν has the property $(\overline{\Omega})$.*

Proof. The argument is exactly the same as in Theorem 6.2.2. Actually, it is enough to show this property:

$$\forall (A,\varepsilon) \in \mathbb{U}, \ \exists (A',\varepsilon') \in \mathbb{U} : \ \forall (A'',\varepsilon'') \in \mathbb{U}, \ B_{A',\varepsilon'} \subset r B_{A'',\varepsilon''} + \frac{1}{r} B_{A,\varepsilon} \quad \forall r > 0.$$

In that case, we fix $(A,\varepsilon) \in \mathbb{U}$. Next, we choose $\varepsilon' > 0$ with $\varepsilon' < \varepsilon/2$ and we put $A' := A \cup A_{\varepsilon_0}$, with $\varepsilon_0 := \varepsilon/2 - \varepsilon'$. Now, we take $(A'',\varepsilon'') \in \mathbb{U}$ and $r > 0$ and we prove the inclusion above.

1. If $r \leq 1$, it is clear because
$$B_{A',\varepsilon'} \subseteq B_{A,\varepsilon} \subseteq \frac{1}{r} B_{A,\varepsilon} \subseteq r B_{A'',\varepsilon''} + \frac{1}{r} B_{A,\varepsilon}.$$

2. If $r \geq 1$, we know there exists $J \in \mathbb{N}_0$ such that $2^{\frac{\varepsilon}{2}J} \leq r \leq 2^{\frac{\varepsilon}{2}(J+1)}$. If $\vec{c} \in B_{A',\varepsilon'}$, we put
$$\vec{c_1} := \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} c_{j,k} \overrightarrow{e_{j,k}} \qquad \text{and} \qquad \vec{c_2} := \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} c_{j,k} \overrightarrow{e_{j,k}}.$$
Since $A_{\varepsilon_0} \subseteq A'$ and $\varepsilon_0 + \varepsilon' = \varepsilon/2$, Corollary 7.1.7 leads to
$$|||\vec{c_1}|||_{A'',\varepsilon''} \leq 2^{(\varepsilon_0+\varepsilon')J} |||\vec{c_1}|||_{A_{\varepsilon_0},\varepsilon'} \leq 2^{(\varepsilon_0+\varepsilon')J} |||\vec{c_1}|||_{A',\varepsilon'} \leq 2^{\frac{\varepsilon}{2}J} |||\vec{c}|||_{A',\varepsilon'} \leq 2^{\frac{\varepsilon}{2}J} \leq r.$$
Besides, by Lemma 7.1.2 and by the facts that $\varepsilon' - \varepsilon < -\varepsilon/2$ and $A \subseteq A'$, we obtain
$$|||\vec{c_2}|||_{A,\varepsilon} \leq 2^{(\varepsilon'-\varepsilon)(J+1)} |||\vec{c_2}|||_{A,\varepsilon'} \leq 2^{-\frac{\varepsilon}{2}(J+1)} |||\vec{c}|||_{A',\varepsilon'} \leq 2^{-\frac{\varepsilon}{2}(J+1)} \leq \frac{1}{r}.$$
Therefore, $\vec{c} = \vec{c_1} + \vec{c_2} \in r B_{A'',\varepsilon''} + \frac{1}{r} B_{A,\varepsilon}$.

Hence the conclusion. □

In summary, in the last chapters, we proved that the spaces S^ν, with ν concave and $\alpha_{\max} < \infty$ or with $p_0 > 0$, verify the property (Ω_{id}) and the equality
$$\Delta(S^\nu) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0,\ \left(\xi_n (n+1)^{-s}\right)_{n \in \mathbb{N}_0} \in c_0 \right\}.$$
As explained before, it is easy to verify that $\Delta(S^\nu) = \Delta^\infty(S^\nu)$ under these assumptions, so that we also have
$$\Delta_b(S^\nu) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall s > 0,\ \left(\xi_n (n+1)^{-s}\right)_{n \in \mathbb{N}_0} \in c_0 \right\}.$$
Consequently, the property (Ω_{id}) and the diametral dimensions Δ and Δ_b are unhelpful to find some potential non-isomorphisms between spaces S^ν.

Moreover, even though these last properties are valid for some "strict" locally pseudoconvex spaces S^ν (i.e. with $p_0 = 0$), there remain some spaces of this kind for which we have no result about their diametral dimension(s) and the property (Ω_{id}).

In fact, we did not manage to extend our arguments with the sets I_ε and A_{ε_0} to general locally pseudoconvex spaces, despite of a similar description of their topology (cf. Section 5.3). The main difficulty comes from the fact that the exponents in the general pseudonorms of S^ν are unbounded[14], which makes the extension of Constructions 6.1.6 and 7.1.5 to these spaces impossible.

In conclusion, in future research, it should be interesting to check if general locally pseudoconvex spaces S^ν could have a different diametral dimension from what we found or if they could not verify the property (Ω_{id}). Furthermore, the study of some other topological invariants could perhaps provide some non-isomorphisms between spaces S^ν.

[14]Actually, we assumed $\alpha_{\max} < \infty$ in Chapter 6 to avoid this problem.

Appendix A

Some important theorems and applications to Köthe spaces

In this appendix, the reader will find some important results in Fréchet spaces and in Köthe spaces which are used several times in the present thesis.

A.1 Closed Graph Theorem and Grothendieck's Factorization Theorem

In the theory of Fréchet spaces, Closed Graph Theorem is a useful result to prove the continuity of a given operator. Some other versions/generalizations of this theorem are known, such as De Wilde's Closed Graph Theorem for operators from a locally convex space with a web into an ultra-bornological space (see for instance [10, 24, 29] for more details). Nevertheless, in this work, we just need the result in Fréchet spaces:

Theorem A.1.1 (Closed Graph Theorem)**.** *Let E and F be two Fréchet spaces and $T : E \to F$ be a linear operator. If the graph of T*

$$\mathcal{G}(T) := \{(x, T(x)) : x \in E\}$$

is closed in $E \times F$, then T is continuous.

An important consequence of this result is the following one:

Theorem A.1.2 (Open Mapping Theorem)**.** *If $T : E \to F$ is a linear, continuous, and surjective operator between Fréchet spaces, then it is open.*

In practice, we usually use the following corollary of these properties:

Proposition A.1.3. *Let E be a vector space and let $\mathcal{T}$ and $\mathcal{S}$ be two Fréchet topologies on E which are both finer than a same Hausdorff topology on E. Then, $\mathcal{T}$ and $\mathcal{S}$ are equivalent.*

In particular, if $\mathcal{S}$ is finer than $\mathcal{T}$, these two topologies are equivalent.

As far as Grothendieck's Factorization Theorem is concerned, it is an important result in (LF)-spaces ([24]):

Theorem A.1.4 (Grothendieck's Factorization Theorem)**.** *Let E be a Hausdorff locally convex space and F and F_n ($n \in \mathbb{N}_0$) be Fréchet spaces. If there exist linear continuous maps $T : F \to E$ and $T_n : F_n \to E$, for all $n \in \mathbb{N}_0$, such that $T(F) \subseteq \bigcup_{n\in\mathbb{N}_0} T_n(F_n)$, then there exists $n_0 \in \mathbb{N}_0$ with*

$$T(F) \subseteq T_{n_0}(F_{n_0}).$$

This leads to the following property:

Corollary A.1.5. *Let E be a Hausdorff locally convex space and F and F_n ($n \in \mathbb{N}_0$) be Fréchet spaces included in E. If the inclusions $F \hookrightarrow E$ and $F_n \hookrightarrow E$ are continuous and if*

$$F \subseteq \bigcup_{n\in\mathbb{N}_0} F_n,$$

then there exists $n_0 \in \mathbb{N}_0$ such that $F \subseteq F_{n_0}$.

A.2 Applications to Köthe spaces

In this section, we present some results about the equality of and the inclusions between Köthe echelon spaces, already known for "classic" admissible spaces, but generalized here for any admissible space. From now on, we fix an admissible space $(l, \|.\|_l)$ and two Köthe matrices $A = (a_k)_{k\in\mathbb{N}_0}$ and $B = (b_k)_{k\in\mathbb{N}_0}$.

When we compare two Köthe spaces, we usually have to determine if they are equal and if they share the same topology. In fact, all these notions are closely related to some properties of Köthe matrices:

Proposition A.2.1. *The following are equivalent:*

(1) $\lambda^l(A) = \lambda^l(B)$ algebraically;

(2) $\lambda^l(A) = \lambda^l(B)$ algebraically and topologically;

(3) for every $m \in \mathbb{N}_0$, there exist $k \geq m$ and $C > 0$ for which $a_m(n) \leq Cb_k(n)$ and $b_m(n) \leq Ca_k(n)$ for all $n \in \mathbb{N}_0$.

Proof. If $\lambda^l(A) = \lambda^l(B)$ algebraically, they have the same topology by Closed Graph Theorem (Proposition A.1.3), since they are continuously included in ω. By Proposition 1.3.2, we actually just have to prove that (2) implies (3).

If the two spaces have the same topology, for a given $m \in \mathbb{N}_0$, we can find $k_1, k_2 \geq m$ and $C_1, C_2 > 0$ such that

$$\|a_m\xi\|_l \leq C_1\|b_{k_1}\xi\|_l \qquad \text{and} \qquad \|b_m\xi\|_l \leq C_2\|a_{k_2}\xi\|_l$$

for every $\xi \in \lambda^l(A) = \lambda^l(B)$. We conclude by taking $k := \sup\{k_1, k_2\}$ and $C := \sup\{C_1, C_2\}$ and by evaluating these inequalities at $\xi = e_n$. $\square$

Sometimes, we say that A and B are *equivalent* when they verify (3) in the previous proposition. In particular, we see that the condition to have the equality of $\lambda^l(A)$ and $\lambda^l(B)$ is independent of the choice of l.

With this result, we can also characterize the (continuous) inclusions between Köthe echelon spaces:

Corollary A.2.2. *The following are equivalent:*

(1) $\lambda^l(A) \subseteq \lambda^l(B)$*;*

(2) $\lambda^l(A) \subseteq \lambda^l(B)$ *continuously;*

(3) for every $m \in \mathbb{N}_0$*, there exist* $k \geq m$ *and* $C > 0$ *for which* $b_m(n) \leq C a_k(n)$ *for all* $n \in \mathbb{N}_0$.

Proof. Of course, (2) implies (1) and, by Proposition 1.3.2, (3) implies (2). Therefore, it remains to prove that (1) implies (3).

We define a third matrix $C := (c_k)_{k\in\mathbb{N}_0}$, with $c_k := a_k + b_k$. Thus, we have

$$\lambda^l(C) \subseteq \lambda^l(A).$$

Moreover, if $\xi \in \lambda^l(A) \subseteq \lambda^l(B)$ and $k \in \mathbb{N}_0$ are given, then $a_k\xi \in l$ and $b_k\xi \in l$, which implies that $c_k\xi = a_k\xi + b_k\xi \in l$. Therefore, $\xi \in \lambda^l(C)$ and we have

$$\lambda^l(A) = \lambda^l(C).$$

In particular, the two matrices A and C are equivalent and, for a given $m \in \mathbb{N}_0$, we can find $k \geq m$ and $C > 0$ with $c_m(n) \leq C a_k(n)$ for any n. We conclude because $b_m(n) \leq c_m(n)$. $\square$

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List of symbols

$\mathbb{C}$	The set of all complex numbers	7
c_0	The space of all null (complex) sequences, endowed with the topology induced by l_∞	11
$\Delta(E)$	The "classic" diametral dimension of E	11
$\Delta^\infty(E)$	The variation of the diametral dimension $\Delta(E)$	29
$\Delta_b(E)$	The "second" diametral dimension of E	27
$\Delta_b^\infty(E)$	The variation of the diametral dimension $\Delta_b(E)$	38
$\delta_n(T)$	If $T : E \to F$ is a continuous operator between the normed spaces E and F with respective closed unit balls V and U, then $\delta_n(T) := \delta_n(T(V), U)$	45
$\delta_n(V, U)$	The n-th Kolmogorov's diameter of V with respect to U	7
$\dim(E)$	The dimension of the vector space E	8
$\underline{\partial}^+\nu$	The right-inf derivative of ν	88
$d_{\alpha,\beta}$	The distance of the ancillary space $E(\alpha, \beta)$	85
$\overrightarrow{e_{j,k}}$	The unit sequence $\overrightarrow{e_{j,k}} \in \Omega$ the components of which are all equal to 0, except the component (j, k) which is equal to 1	95
$E(\alpha, \beta)$	The ancillary space associated to α and β	85
E^*	The algebraic dual of E	72
E^b	The set of all locally bounded linear maps from E into $\mathbb{C}$	72
E_B	If B is bounded, E_B is the normed space $(\mathrm{span}(B), p_B)$	47
$E_j(C, \alpha)(\vec{c})$	The set $\left\{k \in \{0, ..., 2^j - 1\} : \mid c_{j,k} \mid \geq C2^{-\alpha j}\right\}$	84
e_k	The unit sequence defined by $(e_k)_k = 1$ and $(e_k)_n = 0$ if $n \neq k$	14
E_p	The local Banach space for the seminorm p	40
Φ_p	The quotient map $\Phi_p : E \to E/\ker(p)$	40
φ	The space of all finite sequences	75
$\varphi(x, n)$	The smallest index k such that $\pi_n(x) = x_k$	18
$\Gamma(V)$	The absolutely convex hull of V	8
$\mathcal{G}(T)$	The graph of the operator T	115

$\inf_{j\in\mathbb{N}_0} x^{(j)}$	The sequence defined by $\left(\inf_{j\in\mathbb{N}_0} x^{(j)}\right)_n := \inf_{j\in\mathbb{N}_0} x_n^{(j)}$	36
ι_p	The imbedding $\iota^p : E \to E_p : x \mapsto x + \ker(p)$ into the local Banach space associated to p	41
ι_q^p	The imbedding $\iota_q^p : E_q \to E_p$ between local Banach spaces	41
I_ε	The index set defined in Construction 6.1.6	99
$j(n)$	The first component j of the $(n+1)$-th couple (j,k) of Λ, according to the order of Λ and beginning with $(0,0)$	94
ker	The kernel of a map	9
Λ	The set $\left\{(j,k) \in \mathbb{N}_0^2 : k \leq 2^j - 1\right\}$	83
$\lambda^l(A)$	The Köthe sequence space $\lambda^l(A) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall\alpha \in A, \alpha\xi \in l\right\}$, endowed with the topology defined by the seminorms p_α^l	15
$\lambda_0(A)$	The Köthe sequence space $\lambda^{c_0}(A)$	15
$\lambda_p(A)$	The Köthe sequence space $\lambda^{l_p}(A)$	15
$\Lambda_0(\alpha)$	The power series space of finite type associated to the sequence α	20
$\Lambda_0^l(\alpha)$	The power series space of finite type associated to the sequence α and the admissible space l	21
$\lambda_\infty(A)$	The Köthe sequence space $\lambda^{l_\infty}(A)$	15
$\Lambda_\infty(\alpha)$	The power series space of infinite type associated to the sequence α	21
$\Lambda_\infty^l(\alpha)$	The power series space of infinite type associated to the sequence α and the admissible space l	21
$\mathcal{L}_n(E)$	The class of all vector subspaces of the vector space E with a dimension at most equal to n	7
$L(E,F)$	The space of all continuous linear maps from E into F, endowed with the norm $$\Vert\,.\,\Vert_{L(E,F)} : T \in L(E,F) \mapsto \sup\{\Vert\, T(x)\,\Vert_F : x \in E, \Vert\, x\,\Vert_E \leq 1\}$$	43
l_p	The sequence space $l_p = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \Vert\,\xi\,\Vert_p := \left(\sum_{n=0}^{\infty} \mid \xi_n \mid^p\right)^{1/p} < \infty\right\}$, endowed with the topology defined by the $\min(1,p)$-norm $\Vert\,.\,\Vert_p$	14

l_∞	The space of all bounded (complex) sequences, endowed with the locally convex topology defined by the norm $\|\| \, . \, \|\|_\infty : \ \xi \in l_\infty \mapsto \sup_{n\in\mathbb{N}_0} \mid \xi_n \mid$	13
$\mathbb{N}$	The set of all strictly positive natural numbers	7
$\mathbb{N}_0$	The set of all natural numbers	7
$\nu_{\vec{c}}$	The wavelet profile of $\vec{c}$	84
Ω	The space $\mathbb{C}^\Lambda$	84
ω	The linear space $\mathbb{C}^{\mathbb{N}_0}$ endowed with the topology of pointwise convergence	12
$o(\xi_n)$	$\eta_n = o(\xi_n)$ if $\forall\varepsilon > 0, \ \exists N \in \mathbb{N}_0 : \forall n \geq N, \ \mid \eta_n \mid \leq \varepsilon \mid \xi_n \mid .$	45
$\pi(x)$	The decreasing reorganization of the sequence x	18
$\operatorname{proj}_{\alpha\in A}(E_\alpha, \pi_\alpha)$	The projective limit associated to the projective system $(E_\alpha, \pi_\alpha)_{\alpha\in A}$	76
$\wp(E)$	The class of all subsets of E	7
p_0	The convexity index of S^ν	88
p_U	The gauge of U, i.e. $p_U(x) := \inf\{\lambda > 0 : x \in \lambda U\}$	9
p^l_α	The seminorm $p^l_\alpha : \xi \in \lambda^l(A) \mapsto \|\| \alpha\xi \|\|_l$	15
p^l_k	The seminorm $p^l_{a_k}$ associated to the weight a_k	19
$P^{(I)}_m$	The pseudonorm defined by $P^{(I)}_m(\vec{c}) := \sup_{i\in I} \|\| \vec{c} \, \|\|_{b^{\eta(p_i)/p_i - \varepsilon_m}_{p_i,\infty}}$	93
$\mathbb{Q}$	The set of all rational numbers	93
$\mathbb{Q}^+$	The set of all strictly positive rational numbers	93
$\mathbb{R}$	The set of all real numbers	7
$\sigma(E, F)$	The weak topology associated to the dual pair (E, F)	73
$\operatorname{span}(P)$	The linear span of the set P	9
s	The space of rapidly decreasing sequences	59
$s_n(T)$	The n-th singular number of the operator T	43
T^*	The adjoint map of the operator T	46

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