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Algebraic thinking, pattern activities and knowledge for teaching at the transition between primary and secondary school

Isabelle Demonty[1,2] • Joëlle Vlassis[2] • Annick Fagnant[1]

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Abstract Research focusing on algebra from primary to early secondary school level has made several major advances over the past decades. Students' difficulties have been identified and supportive teaching and learning environments have been set up (Cai & Knuth, 2011; Kieran, 2007; Radford, 2008, *Mathematics Education Research Journal, 26*, 257–277, 2014). The effectiveness of these environments relies on the teachers' ability to pay careful attention to students' thinking, which then guides their instructional decisions. This inevitably raises the crucial question of the teachers' knowledge for managing these types of situations in the classroom. In this context, this paper focuses on the mathematical knowledge for teaching figural pattern activities of 100 teachers at the primary and early secondary school levels. The results show that many primary teachers lack the essential knowledge for teaching these types of activities: they do not have a clear idea of their goal, they do not consider non-standard algebraic generalisations to be correct and they generally seem unable to help students to improve their arithmetical generalisation. Secondary school teachers also seem unable to give adequate feedback to improve students' arithmetical generalisation. Although they seem to recognise that these activities are intended to improve algebraic thinking, they do not have a clear perception of the role that primary school learning can play in the development of this algebraic thinking.

Keywords Algebraic thinking · Pattern activities · Knowledge for teaching · Primary mathematics · Secondary mathematics

✉ Isabelle Demonty
Isabelle.demonty@uliege.be

[1] University of Liège, Quartier Agora, Place des Orateurs, 2 - Bat. B32, 4000 Liège, Belgium

[2] Maison des Sciences Humaines, University of Luxembourg, 11, Porte des Sciences, L-4366 Esch-sur-Alzette, Luxembourg

1 Introduction

This article focuses on knowledge in the area of algebraic thinking in teachers responsible for the mathematical education of students aged 10 to 14 years. Despite the large number of studies of algebraic thinking carried out in recent years (Cai & Knuth, 2011; Kieran, 2007; Warren, Trigueros, & Ursini, 2016), it is clear that at the beginning of secondary education, students still experience significant difficulties: they continue to make many mistakes that reflect a lack of understanding of algebraic concepts, largely due to problems in making the transition from arithmetic to algebra (Kieran, 2007). Hitherto, despite extensive investigation of students' approaches and the design of supportive environments in the research literature, there has been little exploration of the mathematical knowledge of teachers in this area.

Traditionally, school curricula separate the study of arithmetic, which is reserved for primary school, from that of algebra, which is treated as a secondary school subject. This approach is based on the premise that a solid foundation in arithmetical knowledge is a prerequisite for a successful start in algebra, teaching of which begins in secondary school (Cai, Wang, Moyer, Wang, & Nie, 2011; Carraher & Schliemann, 2007; Radford, 2014). This separation of content has been criticised by research focusing on 'early algebra' (Carraher & Schliemann, 2007; Kieran, 2007; Warren et al., 2016), which by contrast advocates the development of algebraic thinking from primary school onwards. Such thinking, it is argued, is not about the use of formal algebraic symbolism; rather, it is a particular type of reasoning which requires operations to be carried out involving unknown quantities (Radford, 2014). Algebraic thinking can therefore develop in the context of numerical learning at both primary and secondary school (Cai et al., 2011; Radford, 2008, 2014; Warren et al., 2016), particularly in the context of figural linear patterns. These require students to analyse a sequence of numbers in order to discover the general rule for creating the next element (Cooper & Warren, 2011; Moss & McNab, 2011; Radford, 2008).

Despite these advances in research, their spread to the classroom has been far from optimal and the performance of 14–15 year olds in algebra remains worrying (Kieran, 2007; Warren et al., 2016). One of the reasons put forward to explain this problem relates to the decisive role of teachers in managing these activities: for such methods to be effective, teachers must in particular be able to analyse students' problem-solving approaches in detail in order to encourage proper algebraic thinking (Cai et al., 2011; Kieran, 2007). This capacity brings us back to the question of what knowledge teachers must have in order to support students in their learning. It is in this context that this article aims to investigate the knowledge of teachers working in the final years of primary school and the early years of secondary school that enables them to support algebraic thinking. The investigation focuses on a particularly stimulating learning environment: pattern activities.

2 Theoretical framework

Attempts to define the nature of the knowledge required for teaching have preoccupied researchers for many years, across all disciplines. In the late 1980s, Shulman's work (1987) profoundly shifted the focus by drawing attention, through the concept of pedagogical content knowledge, to the connections between content knowledge and pedagogical knowledge in order to determine the knowledge that is essential for the teaching profession (Depaepe, Verschaffel, & Kelchtermans, 2013).

In mathematics, three large-scale studies have established an empirical basis for studying the knowledge required for teaching: (1) MKT (Mathematical Knowledge for Teaching), (2) COACTIV (Cognitive Activation in the Mathematics Classroom) and (3) TEDS-M (Teacher Education and Development Study in Mathematics). These studies have highlighted the importance of content knowledge, and more importantly of pedagogical content knowledge, as factors in the effectiveness of teaching practices (Depaepe et al., 2015). They have also more specifically studied the link between these two types of knowledge: not only are they positively correlated, but in addition, a command of content knowledge is a necessary but not sufficient condition for pedagogical content knowledge. In other words, teachers need adequate content knowledge in order to be able to develop their pedagogical content knowledge (Baumert et al., 2010; Depaepe et al., 2015).

Following on from this work, studies of the knowledge needed to train current and future teachers at primary and secondary level have led to the following observation: although teachers working in secondary school often have a better mastery of the knowledge content than those working in primary school, this is not necessarily accompanied by a higher level of pedagogical content knowledge (Depaepe et al., 2015; Kleickmann et al., 2013; Krauss et al., 2008; Senk et al., 2012).

However, most of these studies have carried out their investigations in the field of mathematics in general, without focusing on particular content. A number of authors have therefore pointed to the need to narrow down the scope of studies in order to analyse the situation in relation to more limited fields of mathematics (Baumert et al., 2010; Depaepe et al., 2015; Hill, Sleep, Lewis, & Ball, 2007). The research literature on teaching and learning within these restricted fields represents an essential body of reference for such purposes. This section aims to use the research literature to provide such a framework, in order to study the knowledge needed for teaching algebraic thinking to students making the transition between primary and secondary school. It is structured in two parts. The first part summarises the main findings of research into the current knowledge of teachers working with students in the transitional years of late primary and early secondary education. The second part supplements this picture by identifying the knowledge needed to teach algebraic thinking, on the basis of studies that have examined how this kind of thinking can be developed using pattern activities.

2.1 Teachers' knowledge and algebraic thinking: an under-developed research area

With regard to algebraic thinking, little is currently known about what happens in ordinary classrooms, particularly in terms of teaching (Demosthenous & Stylianides, 2014; Kieran, 2007; Warren et al., 2016). The few findings that have been made concern the use in class of activities stimulating algebraic thinking and the conceptions concerning the actual definition of algebra held by teachers working with students in late primary and early secondary education.

In terms of classroom activities, Demosthenous & Stylianides, (2014) analysed textbooks. They found that the majority of tasks that potentially encourage algebraic thinking are pattern activities and appear in sixth-grade textbooks. This type of activity is also found at the beginning of secondary education (grades 7 and 8), where it is regarded by researchers as one of the main ways of introducing algebra (Mason, 1996). However, these activities do not necessarily lead to the development of algebraic thinking by students (Cooper & Warren, 2011; Moss & McNab, 2011; Radford, 2008, 2014): for example, they may simply appear as a context for introducing the idea of a variable (Cai et al., 2011). Unfortunately, there is very little information available on how teachers actually use these tasks (Demosthenous & Stylianides, 2014).

By contrast, we know somewhat more about teachers' conceptions regarding algebra. Primary school teachers tend to view this subject as directly related to symbols and their use (Stephens, 2008). Moreover, they do not feel comfortable with algebraic content defined as such (Jacobs, Franke, Carpenter, Levi, & Battey, 2007). Few studies of secondary school teachers have been carried out in recent years (Warren et al., 2016). Among older studies, Bishop and Stump (2000, cited by Kieran, 2007) showed that many future secondary school teachers also understand algebra in terms of procedural aspects to do with the use of symbols, even when they analyse a pattern activity; they do not perceive the generalisation activity needed to produce the algebraic expression as being algebraic in nature. This trend seems less marked among practising teachers. For example, a study by Menzel and Clarke (1998, cited by Kieran, 2007) found that teachers were divided about whether a verbal description of a mathematical relationship can be seen as algebraic or not.

2.2 Types of knowledge needed to teach algebraic thinking in the context of pattern activities

The studies outlined above suggest that although pattern activities are present in classrooms, teachers may well fail to fully appreciate their algebraic aspect, given the weaknesses identified in their understanding of the characteristics of algebra (Bishop & Stump, 2000 cited by Kieran, 2007; Jacobs et al., 2007; Menzel & Clarke, 1998 cited by Kieran, 2007; Stephens, 2008). The activities used in the classroom do not necessarily have to be changed in order to develop algebraic thinking: the problems which are conventionally used in arithmetic and algebra can constitute rich learning environments. What does need to change in order to encourage the development of algebraic thinking is the way in which these activities are understood (Carraher & Schliemann, 2007; Warren et al., 2016). Clearly, then, it is necessary to throw light on the right way to view such activities, which are fully integrated in students' gradated arithmetical or algebraic learning. This is the aim of the two following points, which investigate the notions of algebraic generalisation and relational thinking as they appear in activities involving figural patterns.

2.2.1 Algebraic generalisation and figural pattern activities

To consider pattern activities in algebraic terms, it is first of all necessary to understand them in terms of the generalisation activity that such problems can give rise to (Radford, 2008). However, not all generalisation is algebraic: in many spontaneous strategies used by students, generalisations of an arithmetical nature are also found (Radford, 2008). What are the points in common and the differences between these two types of thinking?

Algebraic generalisation comes about through identifying a regular pattern based on observation of the terms known. This approach is called abduction. Deduction is then necessary in order to use the observed regularity to produce an expression for any term in the sequence, whether this is close to or distant from a known term. According to Radford (2014), it is during this last stage that the student develops reasoning involving indeterminate quantities which, it should be remembered, is the very essence of algebra. Rivera (2013) argues for the idea that the process of algebraic generalisation implies a continuous interplay between thinking and analysis of the pattern: abduction and induction are, in his view, at the heart of algebraic generalisation. For example, when a student is interested in determining more distant terms (for example, term 10 or term 100), he finds himself reconsidering his initial generalisation in order to make adjustments in light of the numerical tests he conducts. On the basis of these two authors' work, it therefore appears that abduction, induction and deduction are the three key elements needed to establish an algebraic generalisation.

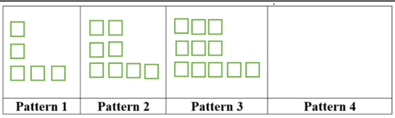
Arithmetical generalisation is not unrelated to *algebraic generalisation*: in both forms of reasoning, observation of known terms leads to an understanding, via abduction, of a regular pattern between the terms of the sequence. In the case of *arithmetical generalisation*, the regular pattern takes the form of a constant increase between two consecutive terms of the sequence. According to Radford (2008), the major difference between arithmetical and algebraic generalisation lies in how the pattern is used: in *arithmetical generalisation*, it is used to determine terms close to the known terms, but cannot be used to deduce more distant terms, unlike *algebraic generalisation*. When the generalisation is arithmetical, the student therefore does not carry out operations involving indeterminate quantities. It is in this sense that the approach is not algebraic, even though it too involves a process of abduction and deduction.

Helping students to generalise pattern activities is far from simple: given the variety of possible approaches to patterns and the difficulty of anticipating them all, there is a high risk that teachers will focus on their own adult perspective when they use these activities. However, this perspective may be incompatible with the ways in which students think about these patterns (Rivera, 2013). What teaching approaches should therefore be emphasised to help students generalise patterns? Studies agree that aids which include verbal, figural and numerical representations of patterns, and which highlight the connections between these representations, may help students to generalise (Radford, 2008; Rivera, 2013). Warren et al. (2016) consider two types of approach that are particularly helpful for making such connections: looking at the invariant relations between the pictorial clues provided by visual arrangements accompanying the pattern, and inviting students to express but also justify their generalisations. According to Rivera (2013), encouraging students to engage in multiplicative thinking can also help them to generalise linear patterns. This approach then makes it possible to move from arithmetical generalisation to algebraic generalisation, by allowing the iteration of a constant increase between two terms to be generalised. The exchange between a teacher and a student presented in Fig. 1 illustrates this idea.

In this excerpt, the teacher supports the student in his reasoning so that he moves from additive reasoning (adding 3 every time) to multiplicative reasoning (adding 3 a certain number of times, with this number corresponding to the difference between the number of the unknown pattern and 3—this last number being the number of the pattern taken as the starting point).

More generally, encouraging students to understand the pattern using multiplication can help them to develop a rule that will work for any term in the pattern being sought (Rivera, 2013).

These considerations suggest that pattern activities are stimulating environments for developing algebraic thinking when the emphasis is on generalising the pattern. Although arithmetical generalisation is not algebraic in nature, it can still lead to such an approach, provided that teachers can support students' thinking by means of a detailed analysis of how students spontaneously conceptualise the pattern. This attention to the student's own method of solving the problem then enables them to offer help as the student develops his or her thinking. The different ways of presenting the material (visual and numerical) and the multiplicative component that these patterns convey are essential elements on which the teacher can base his or her support. The ability to support the student's current thought processes in this way is a matter of pedagogical content knowledge.



Note : E stays for student and P for teacher.

E: I can do it if I'm trying to find a pattern close to one I already know, but not if I'm looking for one that is further ahead.
P : Can you explain to me what you have already discovered?
E First, I saw that 3 squares are added every time, so from pattern 3, I can easily find pattern 4, then 5 and so on. But if we have to find pattern 100, it will take a lot of time... Or I will need to know what pattern 99 is.
P : So to go from pattern 3 to pattern 4 you add 3. Is that right?
E : Yes. That gives 11 + 3, which makes 14.
P : If you then want to find pattern 5, how many times do you have to add 3 to 11?
E I have to add 3 again, so in total, 2 times 3.
P : And if you wanted to find pattern 7?
E : Add 3 again to find pattern 6, and then 3 again to get pattern 7.
P : So starting from pattern 3, how many times do you have to add 3 to get to pattern 7?
E : In total… 4 times 3…. So in total that makes 11+12, so 23 squares.
P : And to find pattern 8?
E : 5 times 3
P : So, to find pattern 4, you add 1 times 3, to find pattern 5 you add 2 times 3…
E : Yes, and for pattern 6, you add 3 times 3, then 4 times 3 for pattern 7…
P : What about if you want to find pattern 15?
E : Um… 12 times 3?
P : How did you find that?
E : Each time you have to take away 3 from the number of the pattern you are looking for.

Fig. 1 Exchange between a teacher and a student showing the support given to help the student move from arithmetical generalisation to algebraic generalisation (Demonty, 2015)

2.2.2 Relational thinking and pattern activities

In addition to their usefulness in encouraging generalisation, research has shown that the problem-solving approaches generated by patterns involving visual aids can provide a solid grounding in the understanding of algebraic expressions and techniques related to their transformation (Cai et al., 2011; Geraniou, Mavrikis, Hoyles, & Noss, 2011; Warren et al., 2016). This is because the way in which visual aids are perceived leads to a greater variety of approaches than when the analysis is based only on numbers: Rivera & Becker (2011) specify that while some generalisations are based on reason and are close to the classical formula $y = mx + p$, others stem from a particular perception of the visual aids that accompany them. The latter, described as non-standard, leads to formulas equivalent to the standard formula but involving additive and multiplicative operations that depend on how the student visualises the materials accompanying the sequences of numbers.

These varied algebraic generalisations can themselves be symbolised in various ways, thus opening the way to comparisons of numerical or algebraic expressions (İmre & Akkoç, 2012; Lannin, 2005; Radford, 2014; Rivera, 2013; Warren & Cooper, 2009) and hence the development of a structural understanding of the expressions and of equality (Kieran, 2007). This ability to approach numbers and operations not by seeking to determine the result of operations, but by understanding them in their entirety, makes use of relational thinking (Jacobs et al., 2007).

Engaging students in relational thinking from primary school onwards is essential in order not only to lay the foundations of algebra (Carraher & Schliemann, 2007) but to get students at primary school to improve their calculating abilities—something of direct relevance to arithmetical learning

(Jacobs et al., 2007). It is therefore an approach that primary school teachers might value even if they are unaware of the algebraic implications of pattern activities (Stephens, 2008).

However, pattern activities can easily be used in a way that overlooks their potential for engaging relational thinking in both primary and early secondary education (Warren et al., 2016). For example, if the emphasis is put on constructing tables of values arising from pattern sequences, this can lead to the development of a single formula which students cannot easily relate to the visual aids accompanying the sequence. This approach is detrimental to the ability of students to identify a variety of equivalent expressions that can arise from the concrete situation—a skill that is essential in developing relational thinking.

In conclusion, the development of relational thinking in the context of pattern activities is bound to be encouraged if teachers can at least recognise the variety of correct problem-solving approaches to which these number sequences with accompanying visual representation naturally give rise.

3 Research questions

While pattern activities are potentially useful for encouraging algebraic thinking, they must be focused in a particular direction, in terms of both generalisation (Radford, 2014; Rivera, 2013) and relational thinking (Warren et al., 2016). To impart such a direction, teachers will often need to step back from their own problem-solving approach and analyse in detail how students reason, so that they can offer relevant support (Rivera, 2013). This requires a mastery on the teacher's part of the necessary teaching-related knowledge, in particular the knowledge relating to identification of why each activity is important, the analysis of students' problem-solving approaches and the types of intervention likely to assist their algebraic thinking. Such knowledge is partly content knowledge (when it is a matter of identifying a variety of correct approaches) and partly pedagogical content knowledge.

In terms of the development of algebraic thinking, the points identified above suggest that arithmetical generalisation is an important milestone on the path to developing algebraic generalisation, for two reasons: firstly, because arithmetical generalisation is an approach commonly used by students when analysing pattern activities (Radford, 2014), and secondly, because arithmetical generalisation and algebraic generalisation are partially related—although both involve the use of abductive and deductive reasoning (Radford, 2008), algebraic generalisation involves multiplicative thinking, which is not the case with arithmetical generalisation (Rivera, 2013). Consequently, students who are able to perceive the multiplicative component of linear patterns will be able to take their arithmetical generalisation further, to the point where it leads to the use of an indeterminate quantity and thus becomes algebraic in nature.

Currently, the information we have about teachers' knowledge is very patchy and suggests that many of them have a restricted understanding of algebra (Bishop & Stump, 2000 cited by Kieran, 2007; Menzel & Clarke, 1998 cited by Kieran, 2007; Stephens, 2008). How do these two groups of teachers ultimately think about these activities? This is the problem that we consider in this exploratory study, which more specifically aims to shed light on the following two research questions:

RQ1. What role do primary and secondary school teachers think pattern activities play?

RQ2. What is their view of arithmetical generalisation produced by students?

4 Methodology

4.1 Participants and context

One hundred French-speaking Belgian teachers responsible for instruction in mathematics for students in the final years of primary school (grades 5 and 6) ($n = 50$) and the first 2 years of secondary school (grades 7 and 8) ($n = 50$) participated in the study. They were recruited on a voluntary basis to take part in a survey examining the transition in mathematics between primary and secondary education in French-speaking Belgium. After being informed of the objectives of this survey, these teachers agreed to fill in a questionnaire. The teachers came from 29 primary schools and 19 secondary schools in French-speaking Belgium. This sample of convenience made it possible to gather information from a group of teachers interested in the problem of the transition between primary and secondary school in the area of mathematics.

In French-speaking Belgium, the mathematics curriculum retains a fairly clear break between arithmetic at primary school and algebra at the start of secondary education. However, a competency entitled ‘the ability to enumerate quantities’ must be worked on continuously between primary and secondary school. Figural linear patterns are frequently included in national certificative tests in order to evaluate this competency. At 12 years, the legal requirements specify that students must be able to perform this enumeration by means of a calculation. At age 14, they must be able to replace the calculation with a formula. Little explanation is given in the curriculum regarding the aim of these generalisation problems. The legal requirements likewise say nothing about the possibility of using these teaching aids to develop relational thinking.

4.2 Data collection

The data were collected using a paper-and-pencil questionnaire posted to teachers who had expressed an interest in participating in the survey. The questionnaire included six case studies relating to mathematics teaching for 10- to 14-year-olds. Three of these related to numerical learning and three to geometrical learning. In the context of this article, we will only consider the case studies relating specifically to figural linear patterns.

Once the first draft of the case studies had been completed, an attempt was made to anticipate the types of answers that might be provided by teachers from primary and secondary schools. Three experts in mathematics education at both target levels reviewed this work.

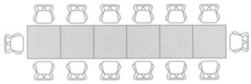
In a pilot test, the questionnaire was given to 23 teachers in order to compare expectations with teachers’ actual answers and refine the questions accordingly, as well as estimate the time taken to complete the questionnaire. This pilot test was supplemented with interviews with four teachers from two different schools, in which they were encouraged to share their views on how realistic the case studies were, and explain their answers. The *retrospective think aloud* method (Alavi, 2005) was used to throw light on the cognitive processes used by teachers in answering the questions.

4.3 Analysis of the case relating to the linear patterns

This problem was chosen because it is representative of the problems used by teachers to work on figural linear patterns. It consists of two questions, as shown in Fig. 2: the first question concerns

Here is a problem:

Jules' parents are organising a birthday party for him. They contact Monsieur Boulèfrite, the caterer. He has some small square tables. He suggests setting them out side by side to make one long table at which all the guest will sit, as shown below:



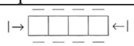
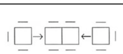
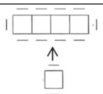
Write a rule that can be used to find the number of chairs for any number of tables that you have.

1) In your view, what is the relevance of using problems of this kind?

a. At primary level:

b. At secondary level:

2) Here are some examples of rules produced by some students:

	Visual presentation	Rule suggested
Student 1		Number of chairs = (number of tables x 2) + 2
Student 2		Number of chairs = (number of tables - 2) x 2 + 6
Student 3		Each time you add a table, you have 2 more chairs Number of chairs = Number of tables + 2

What comment would you make to each student?

RQ1: What role do primary and secondary school teachers think pattern activities play?

RQ2: What is their view of arithmetic generalisation?

Fig. 2 Analysis of the case study submitted to participants

the usefulness of this type of activity for primary students and for secondary students, while in the second, respondents are asked to describe what they would say to three students who offer different generalisations, two of which are algebraic in nature and one of which is arithmetical.

The analysis of answers given by teachers to the first question helped to identify whether they were aware of the algebraic element potentially involved in pattern activities in primary and secondary school, which may take the form either of generalisation or of relational thinking (RQ1). A closer look was taken at this openness to relational thinking through analysis of the comments made with regard to the first two students' approaches put forward in the second question, which presents two types of algebraic generalisation: the first answer represents reasoning close to the standard $y = mx + p$ formula, while the second is of a non-standard nature since it stems from a particular perception regarding the arrangement of tables (Rivera & Becker, 2011). Incidentally, the decision to ask the teachers about the relevance of these activities in primary and secondary school was particularly justified for primary school teachers, who might not see the relevance to algebra at their level despite being able to do so for secondary education (Stephens, 2008). For teachers who were well aware of the algebraic nature of these problems, the design of the questions, focusing in this way on the two levels of schooling, also made it possible to determine their view on the expected progression in the development of algebraic thinking between the end of primary and the start of secondary education.

The comments made by teachers about the last student's answer provided some information relevant to the second research question, about how teachers view arithmetical generalisation (RQ2). In this answer, the student identifies a regularity (in the sense of Radford (2008): a constant increase between two consecutive terms of the sequence), together with the justification 'whenever we add a table, we have 2 more chairs' and a diagram. However, the rule proposed only works when moving from one pattern to the next and hence for deducing a nearby term from a known term. When the student tries to turn his rule into a formula ('number of chairs = number of tables + 2'), he produces an incorrect formula (focusing on the '+ 2' that he correctly identified) which does not reflect the situation as a whole. It seems quite possible to move such an approach towards an algebraic generalisation by making use of the two modes of presentation (numerical and visual) that this problem allows (Radford, 2008; Rivera, 2013; Warren et al., 2016) and/or by encouraging a multiplicative way of thinking even more directly (Rivera, 2013; Warren et al., 2016). With support of this kind from the teacher, the student may be led to change his approach by analysing with greater accuracy the link between the number of chairs and tables in any given pattern; the purpose of this would be to develop an algebraic approach more directly. The student could also be guided towards taking his arithmetical approach further in a similar way to that presented in Fig. 1, by the teacher encouraging him to focus his attention on the number of times that 2 must be added in order to move from a known number of tables to an unknown number of tables, with this number of times itself being dependent on the difference between the two groups of tables under consideration. Analysis of the comments written by the teachers with regard to this student approach aimed to identify the type of help they proposed to use when confronted with this kind of approach.

5 Results

5.1 The role played by pattern activities in algebraic thinking (RQ1)

The comments made by teachers in response to the first question and about the first two student responses to the second question were classified into three categories, according to the degree of knowledge they revealed about teaching pattern activities. We chose to focus our attention on two aspects and defined a number of criteria for each one to assess whether these aspects had been taken into account: (1) the identification of an objective for generalisation—this criterion was met if teachers explicitly understood that generalisation and/or the production of a general formula were central to these activities—and (2) possible openness to the variety of different approaches—this criterion was met if the teachers considered at least the first two student approaches to be correct. Some examples of teachers' comments (shown in italics) illustrate each category and draw attention to the fact that comments included in the same category may in reality show a certain gradation in the level of understanding of the points constituting the minimum criteria for classification in that category.

1. *Teachers who perceived the algebraic nature of these problems* by pointing to their relevance to generalisation (for at least one of the two schooling levels) and recognising the variety of possible correct approaches.

> In primary school, these activities are irrelevant. In secondary school, they enable a general formula such as "$x = y + 2$" to be introduced. The first response is very good and the second is correct but too complicated. (Primary teacher)
> From primary school onwards, these activities lead to generalisation, and in secondary they can be used to introduce algebraic expressions. With regard to the first response, both the representation and formulation are good. The second response is equally good and can be used to work on the equivalence of expressions. (Secondary teacher)

While generalisation appears clearly in these two examples of comments, a difference can be noted in the degree of openness to the variety of different approaches. The remarks made by the secondary school teacher shows much more clearly than those of his primary school colleague that he is aware of the link between the variety of different approaches and the possibility of developing thinking about the equivalence of expressions. However, in both cases, the two approaches are considered correct by the teachers, thus meeting the minimum criterion for considering that they demonstrate possible openness to a variety of approaches.

2. *Teachers who partially perceived the algebraic nature of these problems*, either (2.1) by identifying their relevance to generalisation (again, for at least one of the two schooling levels) but without being open to the variety of possible correct approaches

> In primary school, this can be used to help students discover a general rule taking a concrete example as the starting point. In secondary school, it can be used to show that algebra is simply a representation of something concrete. In the first approach, the explanation and the drawing are correct. In the second approach, the drawing is correct but the rule is not suitable for the precise situation. (Primary teacher)

> In primary school, I can't see how this type of activity would be relevant. In secondary, it can be used to generalise a sequence. The first approach is very good. The second is incorrect: the student is going round in circles. (Secondary teacher)

or (2.2) conversely, by being open to a variety of correct approaches but showing no explicit awareness of the relevance for generalisation.

> In primary, you can use this to work on logical sequences. In secondary school, the idea of proportionality can be introduced. Well done for the first response; in the second one, the process is correct but complex. (Primary school teacher)

> In both primary and secondary, this is useful for encouraging a mathematical way of thinking. The first response is very good; the second response is correct, but try to simplify it. (Secondary school teacher)

3. *Teachers who did not perceive the algebraic nature of these problems at all*, failing to identify any explicit relevance to generalisation or to recognise the correctness of the first two student responses.

> In primary school, this can be used to find an operation to solve a logical problem. In secondary school, I really do not see the point of doing this activity. The first response is very good. In the second, your operation would be correct if you removed 4 chairs rather than 2. (Primary school teacher)

In primary school, it's a problem to do with intervals. In secondary school, there is no value in doing this kind of activity. The first response is good. In the second, why did you do a multiplication? -2 + 6 equals+4. (Primary school teacher)

Table 1 shows the distribution of comments from the two schooling levels according to these categories. Only the teachers who answered the question are included in the table.

In total, 29% of primary school teachers and 63% of secondary school teachers, to judge from their comments, saw pattern activities as compatible with the development of algebraic thinking: they were aware of their relevance to generalisation and recognised that there might be several correct approaches, thereby showing a potential openness to the variety of approaches that is an essential element for the development of relational thinking.

At the other end of the spectrum, nearly one in five of the primary school teachers surveyed (17%) and a single secondary school teacher revealed significant gaps in their knowledge of how to develop algebraic thinking in this context: they did not recognise either of the two elements, and in some cases made comments about the second student's approach that were likely to give rise to a completely erroneous formula. While this finding echoes the remarks made by Rivera (2013), who referred to the danger of teachers' assistance not being based on students' spontaneous approaches, we see here an even greater risk of 'assistance' that could even hinder the student in the development of algebraic thinking.

Finally, more than half of primary school teachers (54%) and about a third of secondary school teachers (34%) had only a limited understanding of this type of activity: often, they only recognised the variety of correct approaches. These teachers' answers regarding the value of the pattern activities were, at best, lacking in precision (for example, when they expressed the idea that they were useful for developing logical understanding) and could be completely erroneous. (Although these activities involve a constant increase, they are not problems about proportionality, as some primary school teachers claimed.)

These results lead us to conclude that in primary school, only a minority of teachers recognise the algebraic value of pattern activities. This observation is similar to the findings of Stephens (2008), who also observed a difficulty for primary school teachers in seeing algebra in anything other than its formal aspects. Nevertheless, a significant proportion of teachers at this level recognised the correctness of two algebraic generalisations frequently encountered in this context, suggesting a possible openness to relational thinking. As mentioned before, however, the category 'openness to the variety of approaches' contains a certain gradation in terms of the opportunities offered by the recognition that both approaches were correct. To put it another way, the openness to

Table 1 Perceptions regarding the role played by pattern activities in algebraic thinking, according to the level of schooling

	Primary school teachers ($N = 48$)	Secondary school teachers ($N = 46$)
1) Those who perceived the algebraic nature of these problems (generalisation and recognition of the variety of correct approaches)	29% (14)	63% (29)
2) Those who partially perceived the algebraic nature of these problems:		
2.1. Only identifying their relevance to generalisation	10% (5)	4% (2)
2.2. Only recognising the variety of possible correct approaches	44% (21)	30% (14)
3) Those who did not perceive the algebraic nature of these problems: they failed to identify their relevance to generalisation or to recognise the variety of possible correct approaches	17% (8)	17% (8)

the variety of approaches is much more definite for some teachers than for others. An in-depth analysis, not described in detail here, showed that in secondary school, teachers were in reality more likely than their primary school colleagues to identify the algebraic aspects of these problems as well as the variety of approaches that problems of this type generate. They also expressed the value of this variety of approaches with regard to relational thinking more explicitly than primary school teachers. These results are in line with those obtained by Menzel and Clark (1998, cited by Kieran, 2007).

Our diagnosis with regard to those teachers who identified the relevance of these activities to algebraic thinking (i.e., those whose comments were assigned to category 1 of the previous analysis) can be further refined in order to ascertain their views on how the learning of algebraic thinking progresses between primary and secondary school.

Some clearly expected to see such a progression and felt that it was possible to start developing generalisation at primary level, while not introducing algebraic symbolism until secondary school (e.g., 'Learning abstraction at primary level, since the formula has to work for an unknown number, and commencement of algebraic calculation in secondary'—Primary school teacher). Others either saw no value in using this type of activity in primary school (e.g., 'There is no point in doing this in primary school. In secondary school, it can be used to move from a concrete problem to its generalisation in mathematical symbols'—Secondary school teacher), or limited the activity to enumeration in primary school and generalisation in secondary (e.g., 'In primary school, this can be used to create sequences with numbers. In secondary, it points towards generalisation with letters'—Secondary school teacher), or felt that the same objectives should be aimed for at both levels of schooling (e.g., 'At both primary and secondary level, this can be used to make links between real situations and a general formula'—Secondary school teacher).

Table 2 shows the results of this analysis.

Most primary school teachers who were aware of the algebraic nature of these activities expected to see students' progress in this area between primary and secondary school (10 out of 14), suggesting that they were aware, despite the strong arithmetical focus of the activities undertaken at their school level, that the objective was algebraic in nature, an idea advocated by the early algebra movement (Carraher & Schliemann, 2007). This tendency was much less marked among their secondary school colleagues (17 out of 29), many of whom regarded generalisation as a theme for secondary education only. Although this analysis is based on a small proportion of teachers, especially for primary school, it shows that, despite an unfavourable curricular context, primary and secondary school teachers can still conceive of students making progress in developing true algebraic thinking in the transition from primary to secondary school.

Table 2 Perceptions regarding the progression of learning on the part of teachers well informed about the value of pattern activities in algebraic thinking, according to the level of schooling

	Primary school teachers ($N = 14$)	Secondary school teachers ($N = 29$)
Progression in terms of algebraic thinking	10	8
No value in working on this kind of activity in primary school	3	12
Enumeration in primary school and generalisation in secondary school	0	5
Same objectives at both levels of schooling	1	4

5.2 Teachers' view of arithmetical generalisation (RQ2)

The analysis of the comments made by teachers on the third student response, representing an arithmetical generalisation, tried to determine the extent to which the teachers offered help that the research literature suggested would be potentially effective in moving the student's thinking towards algebraic generalisation. However, given the general nature of the questionnaire's wording, some teachers focused their comments on analysing the student's approach, while others suggested offering more targeted assistance to improve the student's response.

On the basis of the research literature, two types of comments were considered potentially effective in helping students to make progress in their approach: firstly, comments drawing the student's attention to the different types of supporting material (Radford, 2008; Rivera, 2013; Warren et al., 2016) in order either to take the student's approach further or provide a nuanced diagnosis of the quality of the response; and secondly, those prompting the student towards multiplicative thinking (Rivera, 2013).

Here are some examples of comments regarded as potentially effective.

1. Comments leading the student to take an interest in the various supporting materials for the problem (visual and digital)

   Your drawing is clear, but there is a mistake in your explanatory formula. (Nuanced diagnosis suggested by a primary school teacher)
   Check your formula for if there are 8 tables: count the number of chairs and check that your formula gives the same result. (Explicit help suggested by a secondary school teacher)

2. Comments encouraging multiplicative thinking

   Your method can't be used to work out the total number of chairs needed. If you add 2 chairs per table, the number of tables must be multiplied by 2 and not added to 2. (Primary school teacher)

   Start each time with a new starting number, and add chairs. Try starting with just 1 table: how many times do you have to add 2 chairs then? (Secondary school teacher)

Other approaches were considered potentially ineffective since the comment directed the student towards an approach other than his or her initial one (Rivera, 2013). Four categories were identified here:

3. Comments aimed at focusing the student's attention on the standard formula

   Wrong. I would cross out the response and replace it with "Number of chairs = 2 times the number of tables + 2" (Secondary school teacher)
   I would correct by using the first formula, giving an example so that he understands. (Primary school teacher)

4. The teacher misinterprets the student's response. Very often, in this case, the comment implied that the problem for the student was that he had not taken into account the fact that there were 2 chairs per table, despite the student's answer showing that the problem was not of this nature.

You are forgetting that there are 2 chairs per table. (Secondary school teacher)

5. The comment does not point out the limited nature of the approach and therefore suggests that the response is equivalent to the two previous ones.

Perfect, very good explanation. (Primary school teacher)

6. The teacher seems to think that the response is completely wrong: in this case, the comment fails to point out the partially correct character of the response.

Wrong. (Secondary school teacher)

Table 3 shows the proportion of teachers who suggested each type of comment. As before, only teachers who responded to the question are shown here.

Helping students progress in arithmetical generalisation was no easy matter for the majority of teachers, whether they were working at primary level (34/47) or at secondary level (23/44). Many misinterpreted the students' response, most often because they thought that the student had not taken into account the fact that there were 2 chairs per table (11/47 in primary and 7/44 in secondary) or that their response was incorrect, without identifying the partially correct nature of the approach (17/47 in primary and 12/44 in secondary).

In the final analysis, only 13 primary school teachers among 47 and 23 secondary school teachers among 44 offered approaches in their comments that could be effective in this context in light of the research literature.

These results suggest that greater mastery of mathematical content is not a sufficient condition for using pedagogical content knowledge (Baumert et al., 2010; Depaepe et al., 2015). Although secondary school teachers have a stronger mathematical background than their non-specialist primary school colleagues, which undoubtedly makes it easier for them to identify the variety of correct approaches taken by students, many teachers at both primary and secondary level produced a limited response to an algebraic generalisation: they pointed out the incorrect nature of the formula without considering giving any more specific help, either by nuancing their comments or by helping the student more directly to take his or her thinking

Table 3 Analysis of teachers' comments in response to arithmetical generalisation

	Comments by primary school teachers (grades 5 and 6) ($N = 47$)	Comments by secondary school teachers (grades 7 and 8) ($N = 44$)
Those suggesting help that was potentially helpful in encouraging the student to take his analysis of the sequence further	13	21
1) By leading the student to take an interest in the various supporting materials for the problem (visual and digital)	12	20
2) By encouraging multiplicative thinking	1	1
Those redirecting the student's current thought processes inappropriately	34	23
3) By focusing the student's attention on the classical formula	1	4
4) By misinterpreting the student's response	11	7
5) By failing to point out the limited nature of the approach	5	0

further. These reactions lead us to believe that there are shortcomings on the part of both primary and secondary teachers in terms of pedagogical content knowledge.

6 Discussion and conclusion

An inadequate command of the knowledge required to teach algebraic thinking is presented by several authors as a factor in explaining the weaknesses of students in algebra, despite the progress made in recent years in the development and inclusion in textbooks of activities which are potentially helpful in developing algebraic thinking (Cai et al., 2011; Demosthenous & Stylianides, 2014; Warren et al., 2016).

However, there is currently very little information available on teachers' knowledge, especially with regard to pattern activities (Warren et al., 2016). The purpose of the exploratory study presented in this article was to make a contribution to this body of research by surveying teachers working in late primary and early secondary education on the relevance of pattern activities and the type of input they would give in response to an arithmetical generalisation, which is a key milestone on the road to algebraic generalisation (Radford, 2014; Rivera, 2013).

This study clearly has its limitations. First of all, the teachers who took part in it were not chosen at random: they volunteered to take part in a study on the transition between primary and secondary school in mathematics. We can therefore assume that they had some interest in this subject, which makes the gaps noted in their knowledge even more striking. Another important limitation is that the analysis is based on written statements relating to a single pattern problem. Some questions may have been misinterpreted by some of the teachers. In particular, the second question invited them to write a comment about each student's response. Some teachers confined themselves to simply correcting the response, but it is possible that they would have been more forthcoming and nuanced in an interview, or indeed in a real classroom situation. It would therefore definitely be useful to supplement this exploratory research with interviews and direct classroom observations, to gain a better understanding of the potential value that teachers ascribe to this activity, to see more clearly how they actually deal with incorrect responses given by students and, finally, to shed more light on how they react in real life when they use this type of activity in the classroom.

Despite these limitations, the overall picture which emerges from this exploratory study is striking. It seems to confirm that although pattern activities are included in textbooks (Demosthenous & Stylianides, 2014), the use of these activities—which are potentially helpful for algebraic thinking—in primary and secondary classes may be problematic if teachers' poor knowledge of how to teach these activities hinders their effective use (Cooper & Warren, 2011; Moss & McNab, 2011; Radford, 2008, 2014). More generally, the results give rise to a series of further reflections, which are presented here in two main sections before a brief conclusion.

6.1 The idea that algebraic thinking should develop continuously between primary and secondary school is not yet widely shared by teachers

There is serious doubt about the extent to which primary and early secondary school teachers accept the view that algebra can be detached from symbolism and therefore developed from primary school onwards. In line with the conclusions of Stephens (2008), Bishop and Stump (2000, cited by Kieran, 2007) and Menzel and Clarke (1998, cited by Kieran, 2007), our results reveal that many primary school teachers do not fully realise the algebraic nature of these problems, and instead focus on a sometimes vaguer aspect such as the development of logical thinking or problem-solving skills, or even a completely different objective (concerning intervals or proportionality, for example). These results lead to the worrying conclusion that this group of teachers is not yet sufficiently conscious of the vital role they should be playing in their students' first encounter with algebraic thinking. As for secondary school teachers, although more of these recognise the algebraic relevance of these problems involving figural linear patterns, many do not believe that they can be worked on progressively in the transition from primary to early secondary education. On the contrary, they think that algebraic learning should only begin at age 13, thereby greatly underestimating the ability of primary students to cope with these activities (Cooper & Warren, 2011; Radford, 2008, 2014) and deviating from the approach of the early algebra movement, which seeks continuity of learning between primary and secondary school (Carraher & Schliemann, 2007; Warren et al., 2016).

6.2 Significant shortcomings in the knowledge required to teaching these pattern activities risks compromising the development of students' algebraic thinking

The overall picture which emerges from our findings also leads us to believe that many primary school teachers have significant gaps in their specific content knowledge for pattern activities. When analysing student responses, many fail to recognise the correctness of non-standard algebraic generalisation. Conversely, others fail to point out the limited nature of arithmetical generalisation and consider it to be completely correct. These weaknesses in the content knowledge of these primary school teachers confirm the results of other studies conducted with the same group (Depaepe et al., 2015; Krauss et al., 2008; Senk et al., 2012). This seems particularly problematic since, as Baumert et al. (2010) and Depaepe et al. (2015) have shown, content knowledge is a necessary condition for developing pedagogical content knowledge. It is therefore a matter of urgent importance for teachers at this level to improve both their content knowledge and their pedagogical content knowledge.

With respect to secondary school teachers, the picture that emerges from our study points to the fact that while their command of content knowledge is generally much better, this is often not accompanied by pedagogical content knowledge. This finding reinforces the results of quantitative research in the field (Depaepe et al., 2015; Kleickmann et al., 2013; Krauss et al., 2008; Senk et al., 2012). Thus, in our study, when teachers considered the third student response, less than half of them suggested help that was potentially effective at guiding the student to develop his approach further. Although these teachers had the knowledge to assess the correctness or otherwise of student responses, they found it harder to engage with students' thought processes, even in order to provide them with a nuanced analysis of their response. Yet being able to engage with students' current approach is seen as a crucial element in helping them to progress in the development of their algebraic thinking (Rivera, 2013).

6.3 Conclusion

Despite the limitations mentioned above, this exploratory study demonstrates the value of continuing investigation in this area. It also highlights the urgent need to communicate more effectively to teachers at both levels of schooling the results of research into the teaching and learning of algebraic thinking.

If more students are to benefit from the very promising research results of the early algebra approach, the points that emerge from our exploratory study provide support for the findings of Warren et al. (2016) that it is urgent to continue investigating teachers' knowledge, to enable progress to be made with the ambitious project of algebra for all. In addition, it is urgent to look at the origin of the sometimes mistaken conceptions that influence teachers in their practices, but also to improve our understanding of when and how they actually make use of their pedagogical content knowledge in their actual classroom practices as they manage student learning 'in the heat of action' (Depaepe et al., 2013).

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