Bayesian inference to determine elastoplastic parameter distributions

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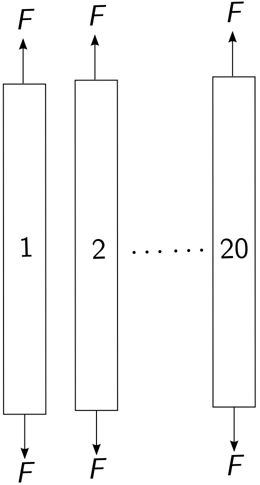
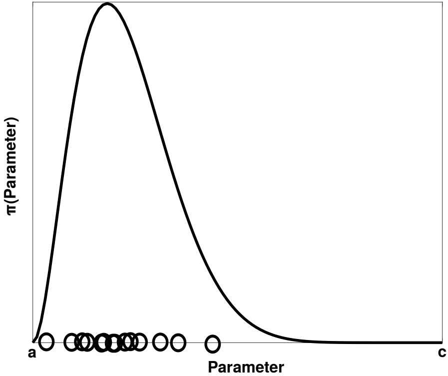
Introduction



Introduction

Objective

Find the distribution from which the parameters of the specimens are coming with **limited** number of specimens



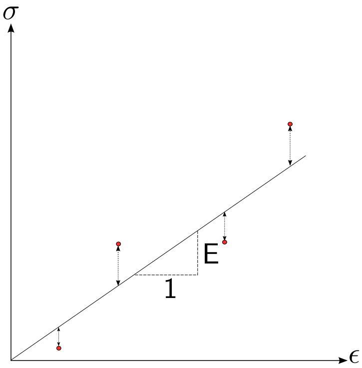
A simple approach

Least squares method

$\sigma = E\epsilon$

$$J = \frac{1}{2}\sum_{i=1}^{n_m}(\sigma_i - E\epsilon_i)^2$$

$$\overline{E} = \underset{E}{\operatorname{argmin}}\ J(E)$$

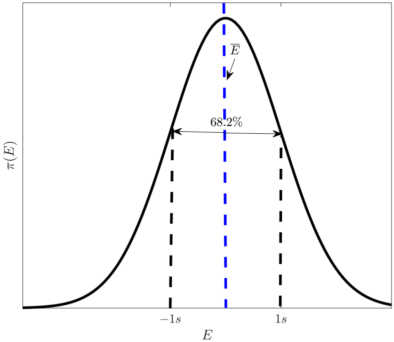


A simple approach

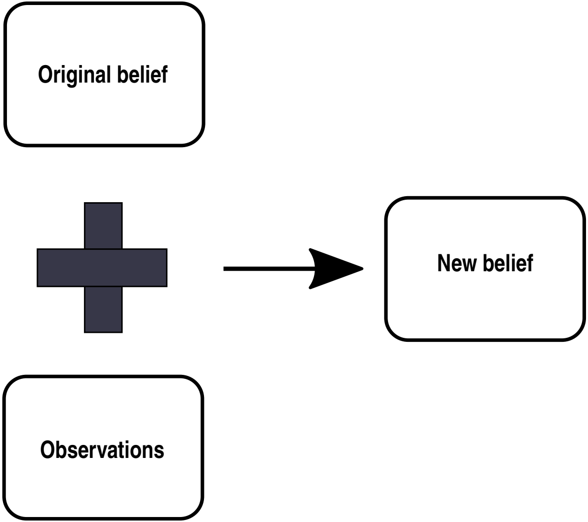
Now that we have $\{\overline{E}_1, \cdots, \overline{E}_{20}\}$

$E_{\text{mean}} = \sum_{i=1}^{20} \frac{\overline{E}_i}{20}$

s(standard deviation)$= \sqrt{\frac{\sum_{i=1}^{20}(\overline{E}_i - E_{\text{mean}})^2}{20-1}}$



Bayesian inference



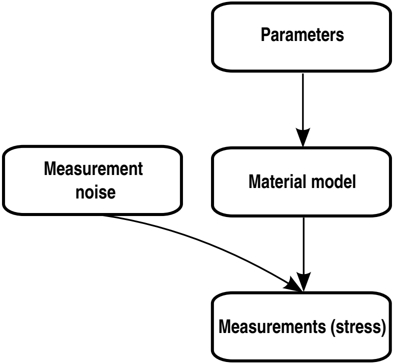
Bayesian inference

$$\overbrace{\pi(x|y)}^{posterior} = \frac{\overbrace{\pi(x)}^{prior} \times \overbrace{\pi(y|x)}^{likelihood}}{\underbrace{\pi(y)}_{evidence}} \implies \pi(x|y) \propto \pi(x) \times \pi(y|x)$$

$y :=$ observation
$x :=$ parameter
$\pi(x) :=$ original belief
$\pi(y|x) :=$ given by the mathematical model that relates y to x
$\pi(y) :=$ is a constant number

Bayesian inference

$$\pi(x|y) \propto \pi(x) \times \pi(y|x)$$



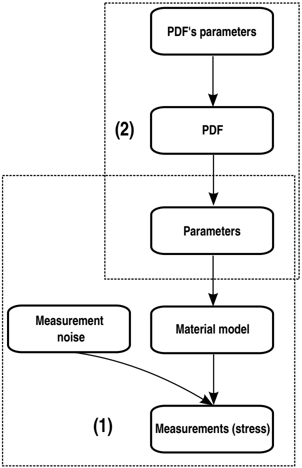
Full Bayesian hierarchical model

$$y \sim \pi(y|x_m) \longrightarrow \ (1)$$

$$x_m \sim \pi(x_m|x_d) \longrightarrow \ (2)$$

$$x_d \sim \pi(x_d) \longrightarrow \ \text{prior}$$

$$\pi(x_m, x_d|y) \propto \pi(y|x_m)\pi(x_m|x_d)\pi(x_d)$$



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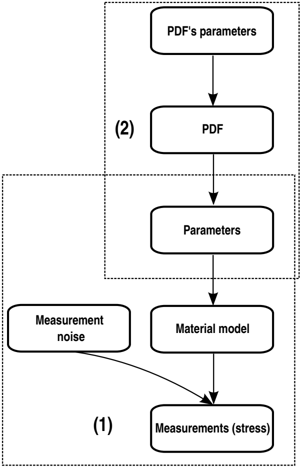


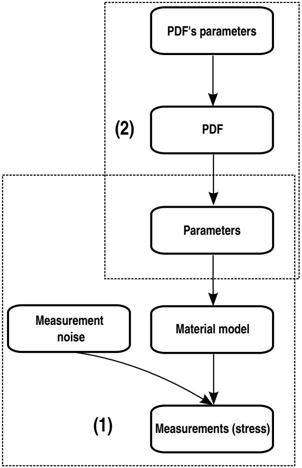
Hard to incorporate when the parameter distribution or model is not standard

Bayesian updating-Least squares

(1) $\longrightarrow$ Least squares

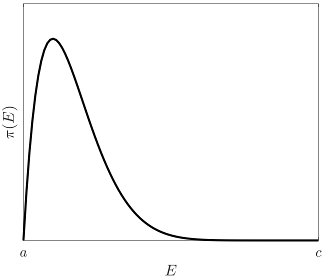
(2) $\longrightarrow$ Bayesian updating

$\pi(x_d|\overline{x}_m) \propto \pi(x_d|\overline{x}_m)\pi(x_d)$



Linear elasticity with Beta distribution as parameter distribution

$\mathbf{E} = \{E_1, \cdots, E_{20}\}$
and
$E_i \sim \text{Beta}(\alpha, \beta, a, c) = \frac{(E_i - a)^{\alpha-1}(c - E_i)^{\beta-1}}{(c-a)^{\alpha+\beta+1}B(\alpha,\beta)}$



Linear elasticity with Beta distribution as parameter distribution

Bayesian inference

$$\pi(\alpha, \beta, a, c|E_i) \propto \pi(E_i|\alpha, \beta, a, c)\pi(\alpha)\pi(\beta)\pi(a)\pi(c)$$

Linear elasticity with Beta distribution as parameter distribution

Bayesian inference

$$\pi(\alpha, \beta, a, c|E_i) \propto \pi(E_i|\alpha, \beta, a, c)\pi(\alpha)\pi(\beta)\pi(a)\pi(c)$$

and for 20 specimens:

$$\pi(\alpha, \beta, a, c|\mathbf{E}) \propto \prod_{i=1}^{20} \pi(E_i|\alpha, \beta, a, c)\pi(\alpha)\pi(\beta)\pi(a)\pi(c)$$

Linear elasticity with Beta distribution as parameter distribution

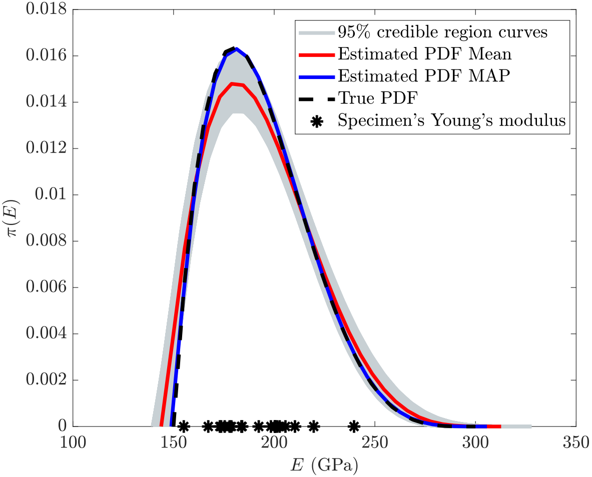
$$\pi(\alpha, \beta, a, c|\mathbf{E}) = \prod_{i=1}^{20} \mathrm{Beta}(E_i; \alpha, \beta, a, c) N(\overline{\alpha}, s_\alpha^2) N(\overline{\beta}, s_\beta^2) N(\overline{a}, s_a^2) N(\overline{c}, s_c^2)$$

where

$$N(x, s^2) = \frac{1}{\sqrt{2\pi s}} \exp\left(\frac{(x-\overline{x})^2}{2s^2}\right)$$

If the priors are not informative much more specimens will be needed to make the problem numerically tractable

Linear elasticity with Beta distribution as parameter distribution



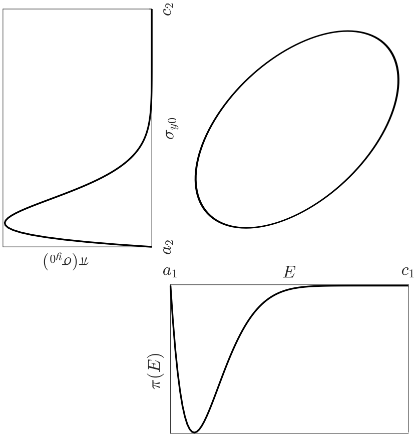
Linear elasticity-perfect plasticity

$\mathbf{E} = \{E_1, \cdots, E_{20}\}$

$E_i \sim \text{Beta}(\alpha_1, \beta_1, a_1, c_1)$

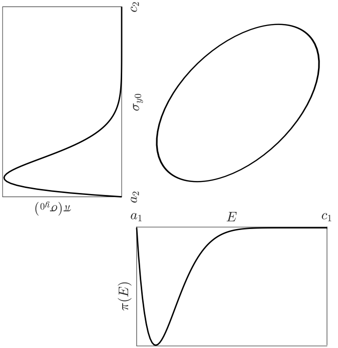
$\boldsymbol{\sigma}_{y0} = \{\sigma_{y0_1}, \cdots, \sigma_{y0_{20}}\}$

$\sigma_{y0_i} \sim \text{Beta}(\alpha_2, \beta_2, a_2, c_2)$

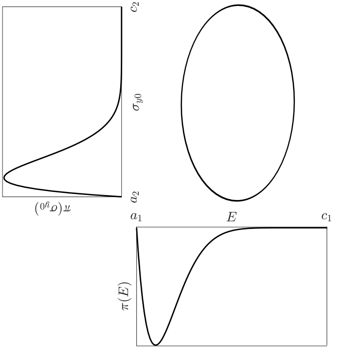


Copula

Copulas are tools enable us to model dependence of several random variables in terms of their marginal distribution.



with dependence



without dependence

Copula

for two parameters

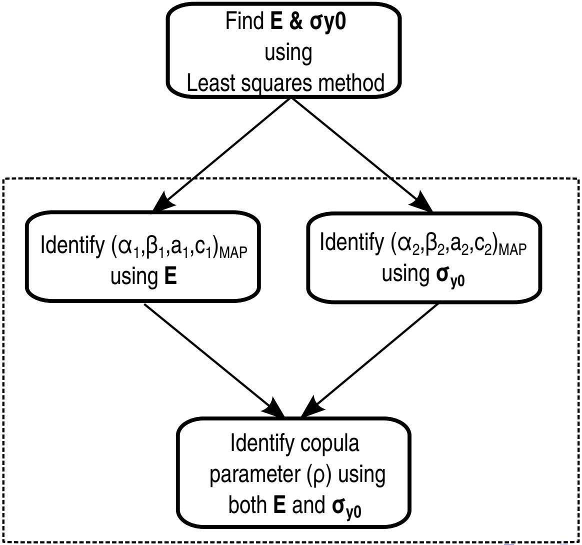
$h(x_1, x_2) = c_h(F_1(x_1), F_2(x_2), \rho) f_1(x_1) f_2(x_2)$

$h :=$ the joint PDF of the random variables

$f_1(x_1)$ and $f_2(x_2) :=$ the marginal PDFs of the random variables

$F_1(x_1)$ and $F_2(x_2) :=$ the marginal CDFs of the random variables

Linear elasticity-perfect plasticity-Modular Bayesian



Linear elasticity-perfect plasticity-Modular Bayesian

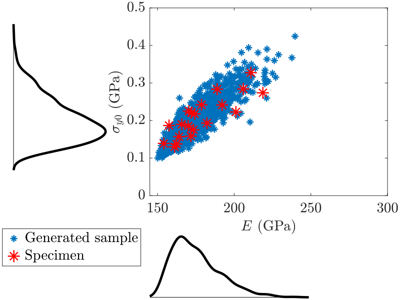
Find **E & σy0**
using
Least squares method

$$\pi(\alpha_1,\beta_1,a_1,c_1,\alpha_2,\beta_2,a_2,c_2,\rho|\mathbf{E},\boldsymbol{\sigma_{y0}}) \propto$$

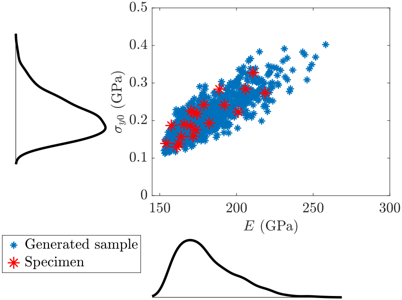
$$\prod_{i=1}^{20} \mathrm{Beta}(\alpha_1,\beta_1,a_1,c_1)\mathrm{Beta}(\alpha_2,\beta_2,a_2,c_2)c_h(F_E(E),F_{\sigma_{y0}}(\sigma_{y0})) \times \mathrm{priors}$$

Numerically intractable to sample from

Linear elasticity-perfect plasticity-Modular Bayesian

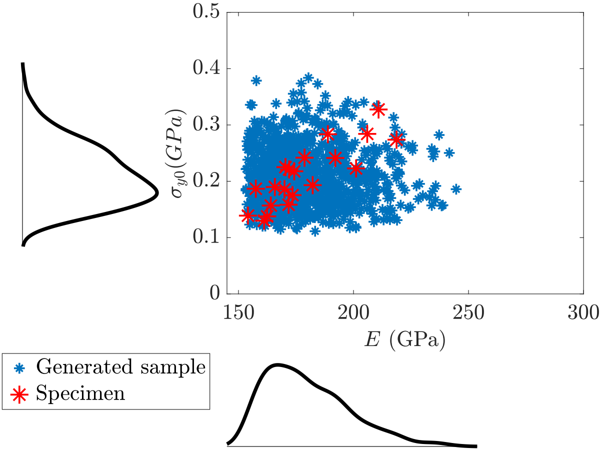


True joint distribution



Estimated joint distribution

Linear elasticity-perfect plasticity-Modular Bayesian



Summary

- We used Bayesian inference for identification of the material parameters distribution with limited number of specimens
- As the number of measurements for each specimen is large enough the effect of measurement uncertainty is smaller than the parameter variability
- This means that we can break the full Bayesian problem to a least squares and Bayesian inference problem
- To be able to overcome sampling problems we had to use modular Bayesian as the number of specimens was limited

Future study

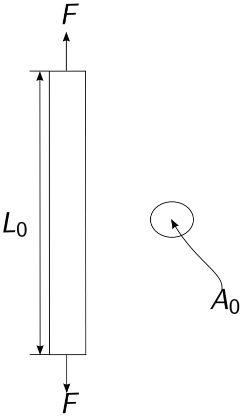
Studying the effect of parameter variability and parameters dependence in presence of geometrical randomness

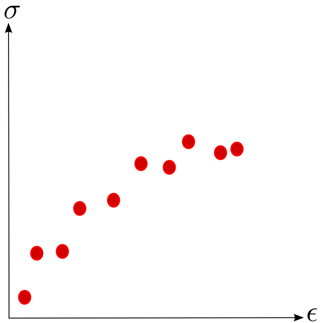
A special acknowledgement

Computational Science Priority, University of Luxembourg

The End

Introduction





stress: $\sigma = \frac{F}{A_0}$

strain: $\epsilon = \frac{\Delta L}{L_0}$

Copula

Sklar's theorem

$H(x_1, x_2) = C_H(F_1(x_1), F_2(x_2))$

$H :=$ the joint cumulative distribution function (CDF) of the random variables

$F_1(x_1)$ and $F_2(x_2) :=$ the marginal CDFs of the random variables

$C_H :=$ the copula function such that $[0, 1]^2 \longrightarrow [0, 1]$

Copula

Probability density function

$h(x_1, x_2) = c_h(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2)$

$h :=$ the joint probability density function (PDF) of the random variables

$f_1(x_1)$ and $f_2(x_2) :=$ the marginal PDFs of the random variables

$c = \frac{\partial^2 C}{\partial F_1 \partial F_2}$

Linear elasticity-perfect plasticity

$$\pi(E, \sigma_{y0}) = Beta(\alpha_1, \beta_1, a_1, c_1) Beta(\alpha_2, \beta_2, a_2, c_2) c_h(F_E(E), F_{\sigma_{y0}}(\sigma_{y0}))$$

$$F_E(E) = \int_{a_1}^{E} Beta(t; \alpha_1, \beta_1, a_1, c_1) dt$$

$$F_{\sigma_{y0}}(\sigma_{y0}) = \int_{a_2}^{\sigma_{y0}} Beta(t; \alpha_2, \beta_2, a_2, c_2) dt$$

Linear elasticity-perfect plasticity

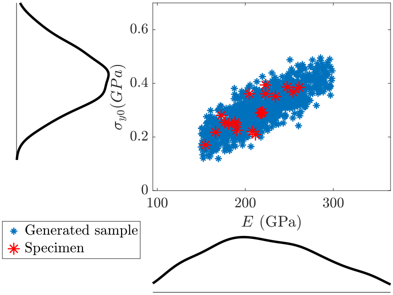
$$c_h^{\text{Gauss}}(F_E(E), F_{\sigma_{y0}}(\sigma_{y0})) =$$
$$\frac{1}{\sqrt{(|\mathbf{R}|)}}\exp\left(\begin{bmatrix} \Phi^{-1}(F_E) \\ \Phi^{-1}(F_{\sigma_{y0}}) \end{bmatrix}^T (\mathbf{R}^{-1} - \mathbf{I})(\begin{bmatrix} \Phi^{-1}(F_E) \\ \Phi^{-1}(F_{\sigma_{y0}}) \end{bmatrix} \right), \quad \mathbf{R} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Linear elasticity-perfect plasticity

Parameters to be identified

$\alpha_1, \beta_1, a_1, c_1, \alpha_2, \beta_2, a_2, c_2, \rho$

Linear elasticity-perfect plasticity-Modular Bayesian



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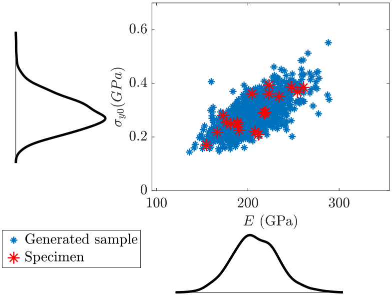

Estimated joint distribution

Linear elasticity-perfect plasticity-Modular Bayesian