Some generalizations of the Pascal triangle: base 2 and beyond

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Classical Pascal triangle

$\binom{m}{k}$		k							
		0	1	2	3	4	5	6	7	$\cdots$
	0	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	2	1	2	1	0	0	0	0	0	
m	3	1	3	3	1	0	0	0	0	
	4	1	4	6	4	1	0	0	0	
	5	1	5	**10**	**10**	5	1	0	0	
	6	1	6	15	**20**	15	6	1	0	
	7	1	7	21	35	35	21	7	1	
	$\vdots$									$\ddots$

Usual binomial coefficients of integers:

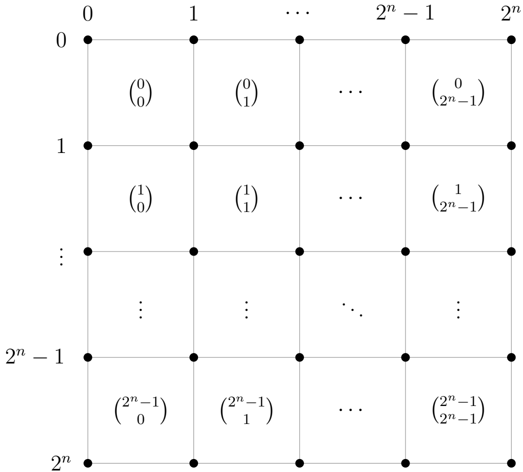
$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!}$$

Pascal's rule:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$$

A specific construction

- Grid: first 2^n rows and columns



$\mathbb{N}^2 \cap [0, 2^n]^2$

- Color the grid:
  Color the first 2^n rows and columns of the Pascal triangle

$$\left(\binom{m}{k} \bmod 2\right)_{0 \leq m,k < 2^n}$$

in

  - white if $\binom{m}{k} \equiv 0 \bmod 2$
  - black if $\binom{m}{k} \equiv 1 \bmod 2$

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  - white if $\binom{m}{k} \equiv 0 \bmod 2$
  - black if $\binom{m}{k} \equiv 1 \bmod 2$
- Normalize by a homothety of ratio $1/2^n$
  (bring into $[0,1] \times [0,1]$)
  $\rightsquigarrow$ sequence belonging to $[0,1] \times [0,1]$

What happens for $n \in \{0, 1\}$

$n = 0$

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0 1

$$\binom{0}{0}$$

1

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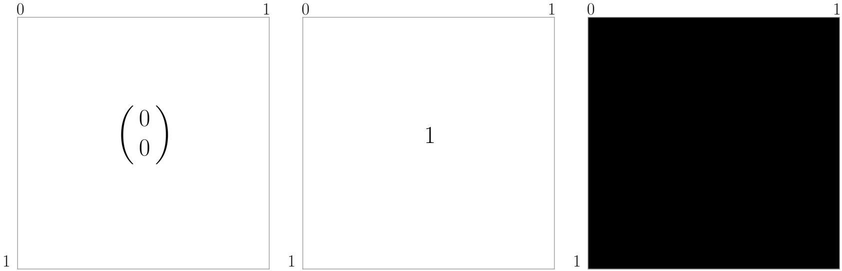
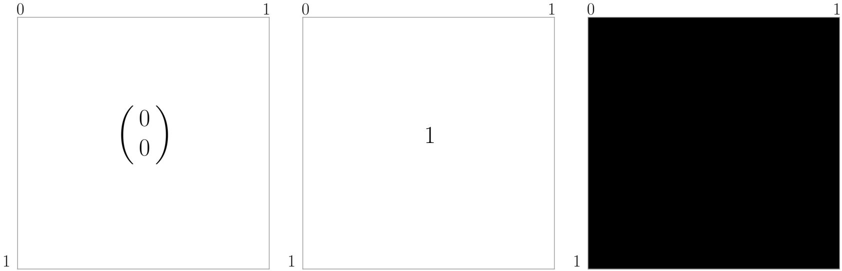
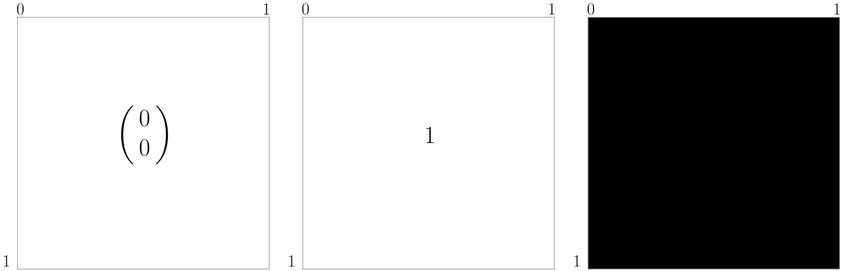


$n = 0$



$n = 1$

$n = 0$



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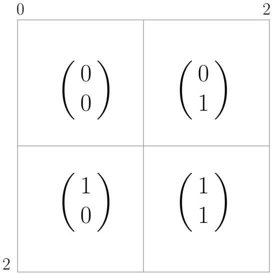
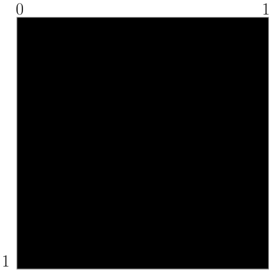


$n = 0$

$\binom{0}{0}$

1



$n = 1$

$\binom{0}{0}$	$\binom{0}{1}$
$\binom{1}{0}$	$\binom{1}{1}$

1	0
1	1

$n = 0$

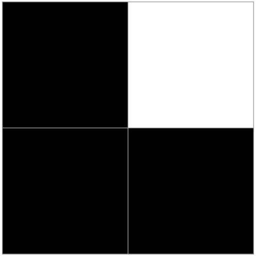
	$\binom{0}{0}$

	1

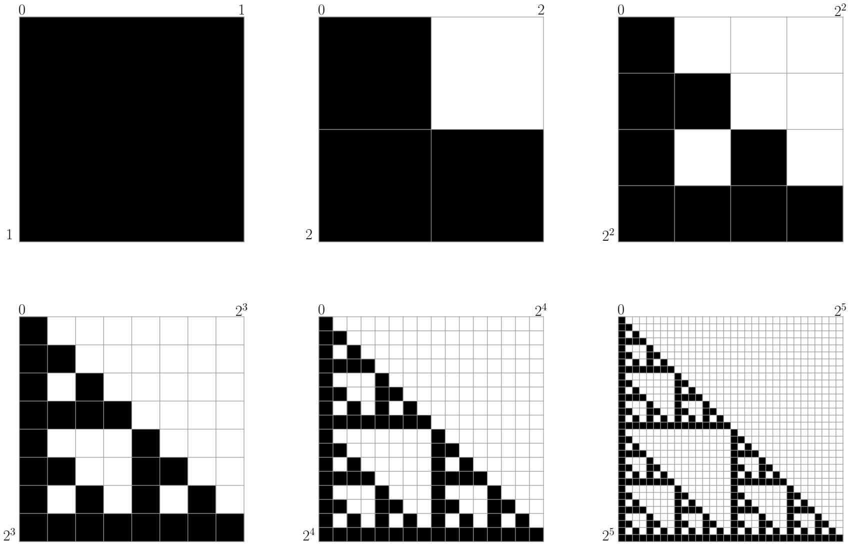
$n = 1$

$\binom{0}{0}$	$\binom{0}{1}$
$\binom{1}{0}$	$\binom{1}{1}$

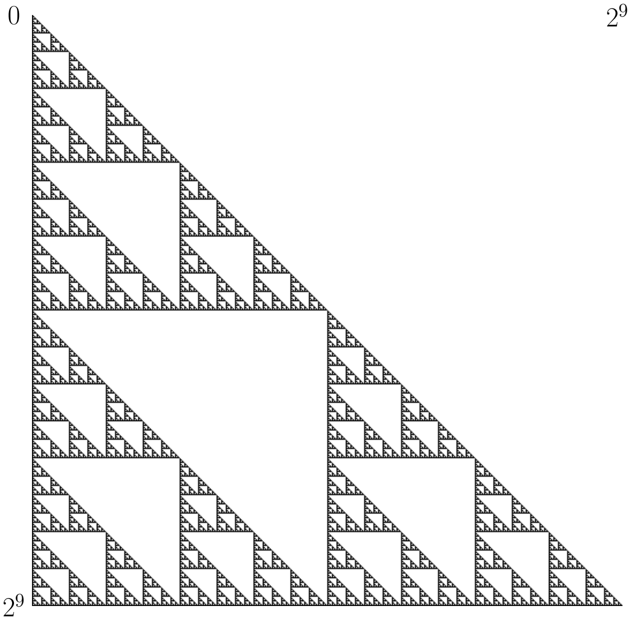
1	0
1	1



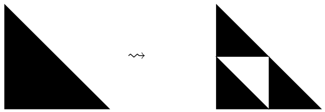
The first six elements of the sequence



The tenth element of the sequence



The Sierpiński gasket



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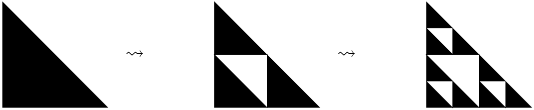

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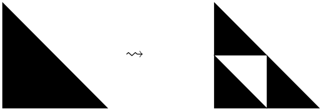

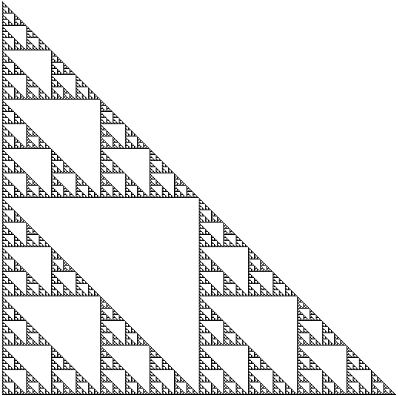

Folklore fact

The latter sequence converges to the Sierpiński gasket when n tends to infinity (for the Hausdorff distance).

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Definitions:

- *ϵ-fattening* of a subset $S \subset \mathbb{R}^2$

$$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the *Hausdorff distance* d_h

$$d_h(S, S') = \min\{\epsilon \in \mathbb{R}_{\geq 0} \mid S \subset [S']_\epsilon \quad \text{and} \quad S' \subset [S]_\epsilon\}$$

New idea

Replace usual binomial coefficients of integers by binomial coefficients of finite words

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

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Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\qquad v = 101$

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Example: $u = \mathbf{101}001$ $\quad v = 101 \quad$ 1 occurrence

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Example: $u = 101001$ $v = 101$ 2 occurrences

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Binomial coefficient of words

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Example: $u = 101001$ $\quad v = 101$ $\quad$ 3 occurrences

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Binomial coefficient of words

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Example: $u = 101001$ $v = 101$ 4 occurrences

Binomial coefficient of finite words

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Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $v = 101$ 5 occurrences

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\quad$ $v = 101$ $\quad$ 6 occurrences

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\qquad v = 101$

$$\Rightarrow \binom{101001}{101} = 6$$

Remark:

Natural generalization of binomial coefficients of integers

With a one-letter alphabet $\{a\}$

$$\binom{a^m}{a^k} = \binom{\overbrace{a \cdots a}^{m \text{ times}}}{\underbrace{a \cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall\, m, k \in \mathbb{N}$$

Base 2

Definitions:

- $\text{rep}_2(n)$ greedy base-2 expansion of $n \in \mathbb{N}_{>0}$ starting with 1
- $\text{rep}_2(0) = \varepsilon$ where ε is the empty word

n		$\text{rep}_2(n)$
0		ε
1	1×2^0	1
2	$1 \times 2^1 + 0 \times 2^0$	10
3	$1 \times 2^1 + 1 \times 2^0$	11
4	$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	100
5	$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	101
6	$1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$	110
$\vdots$	$\vdots$	$\vdots$
		$\{\varepsilon\} \cup 1\{0,1\}^*$

Generalized Pascal triangle P_2 in base 2

$\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)}$		$\mathrm{rep}_2(k)$							
		ε	1	10	11	100	101	110	111	$\cdots$
	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	11	1	**2**	0	**1**	0	0	0	0	
$\mathrm{rep}_2(m)$	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	1	3	0	**3**	0	0	0	1	
	$\vdots$									$\ddots$

Binomial coefficient of finite words: $\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)}$

Rule (not local):

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

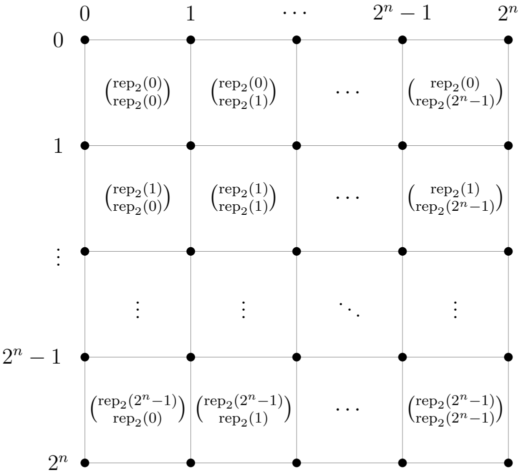
$\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)}$		$\mathrm{rep}_2(k)$								
		ε	**1**	10	**11**	100	101	110	**111**	$\cdots$
	ε	**1**	0	0	0	0	0	0	0	
	1	**1**	**1**	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	11	**1**	**2**	0	**1**	0	0	0	0	
$\mathrm{rep}_2(m)$	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	**1**	**3**	0	**3**	0	0	0	**1**	
	$\vdots$									$\ddots$

The classical Pascal triangle

Questions:

- After coloring and normalization can we expect the convergence to an analogue of the Sierpiński gasket?
- Could we describe this limit object?

- Grid: first 2^n rows and columns of P_2



$\mathbb{N}^2 \cap [0, 2^n]^2$

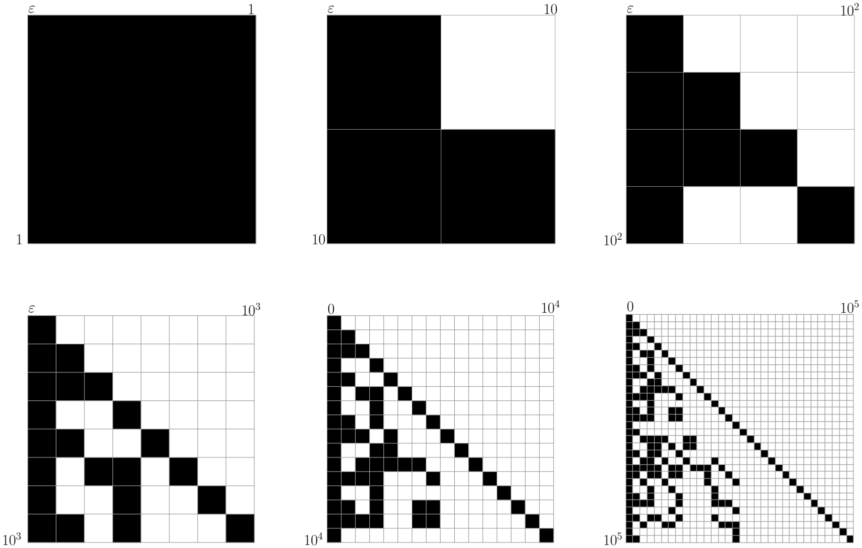
- Color the grid:
  Color the first 2^n rows and columns of the generalized Pascal triangle P_2

$$\left(\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \bmod 2\right)_{0\le m,k<2^n}$$

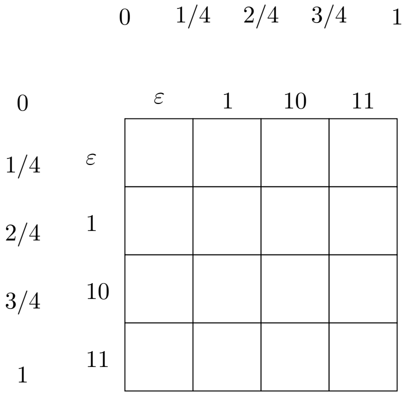
  in
  - white if $\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \equiv 0 \bmod 2$
  - black if $\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \equiv 1 \bmod 2$
- Normalize by a homothety of ratio $1/2^n$
  (bring into $[0,1]\times[0,1]$)
  $\rightsquigarrow$ sequence $(U_n)_{n\ge 0}$ belonging to $[0,1]\times[0,1]$

$$U_n = \frac{1}{2^n} \bigcup_{\substack{u,v\in\{\varepsilon\}\cup 1\{0,1\}^* \text{ s.t.} \\ |u|,|v|\le n,\ \binom{u}{v}\equiv 1 \bmod 2}} \{(\mathrm{val}_2(v), \mathrm{val}_2(u)) + Q\}$$

$$Q = [0,1]\times[0,1]$$


ε
1
1
ε
10
10
ε
10^2
10^2
ε
10^3
10^3
0
10^4
10^4
0
10^5
10^5

The element U_2



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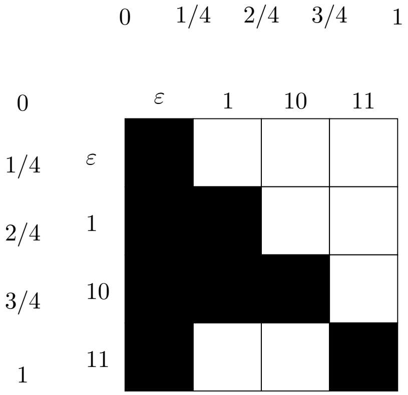
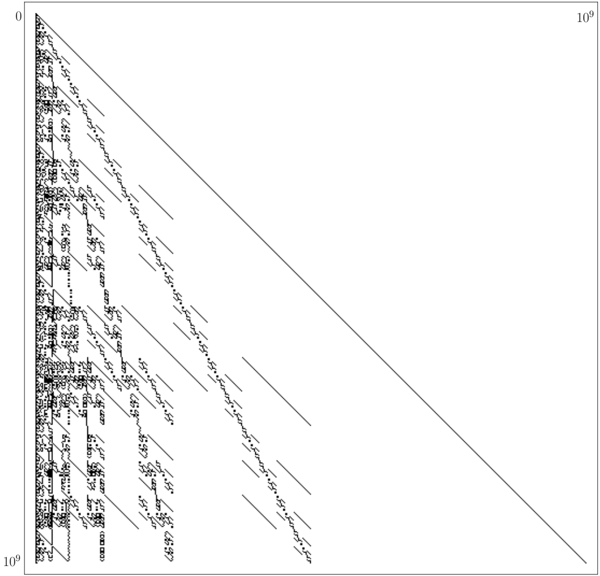


$$\varepsilon \rightsquigarrow 0, \quad 1 \rightsquigarrow 1/4, \quad 10 \rightsquigarrow 2/4 = 1/2, \quad 11 \rightsquigarrow 3/4$$

$$w \in \{\varepsilon\} \cup 1\{0,1\}^* \text{ with } |w| \leq 2 \rightsquigarrow \frac{\mathrm{val}_2(w)}{2^2}$$

The element U_9



Lines of different slopes: $1, 2, 4, 8, 16, \ldots$

The $(\star)$ condition

$(\star)$

$$(u,v) \text{ satisfies } (\star) \text{ iff } \begin{cases} u, v \neq \varepsilon \\ \binom{u}{v} \equiv 1 \bmod 2 \\ \binom{u}{v0} = 0 = \binom{u}{v1} \end{cases}$$

Example: $(u,v) = (101, 11)$ satisfies $(\star)$

$$\binom{101}{11} = 1 \qquad \binom{101}{110} = 0 \qquad \binom{101}{111} = 0$$

Lemma: Completion

(u,v) satisfies $(\star) \Rightarrow (u0,v0),(u1,v1)$ satisfy $(\star)$

Proof: Since (u,v) satisfies $(\star)$

$$\binom{u}{v} \equiv 1 \bmod 2, \quad \binom{u}{v0} = 0 = \binom{u}{v1}$$

Proof for $(u0,v0)$:

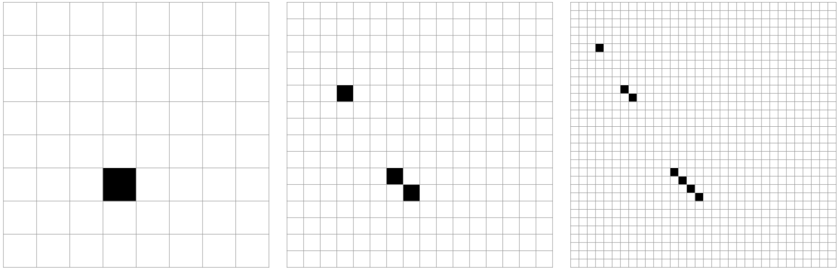
$$\binom{u0}{v0} = \underbrace{\binom{u}{v0}}_{=0 \text{ since } (\star)} + \underbrace{\binom{u}{v}}_{\equiv 1 \bmod 2} \equiv 1 \bmod 2$$

If $\binom{u0}{v00} > 0$ or $\binom{u0}{v01} > 0$, then $v0$ is a subsequence of u.
This contradicts $(\star)$.

Same proof for $(u1,v1)$.

□

Example: $(u,v) = (101, 11)$ satisfies $(\star) \Rightarrow \binom{u}{v} \equiv 1 \bmod 2$



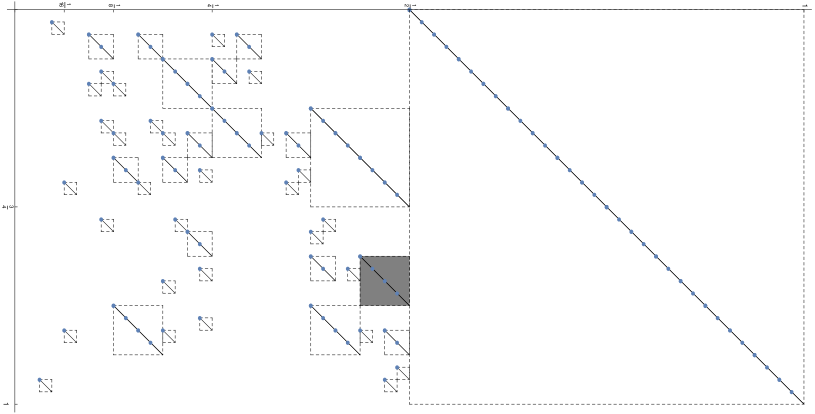
U_3 U_4 U_5

$\rightsquigarrow$ Creation of segments of slope 1
Endpoint $(3/8, 5/8) = (\mathrm{val}_2(11)/2^3, \mathrm{val}_2(101)/2^3)$
Length $\sqrt{2} \cdot 2^{-3}$

$S_{u,v} \subset [0,1] \times [1/2, 1]$ endpoint $(\mathrm{val}_2(v)/2^{|u|}, \mathrm{val}_2(u)/2^{|u|})$
length $\sqrt{2} \cdot 2^{-|u|}$

Definition: Set of segments of slope 1

$$\mathcal{A}_0 = \overline{\bigcup_{\substack{(u,v) \\ \text{satisfying}(\star)}} S_{u,v}} \subset [0,1] \times [1/2,1]$$

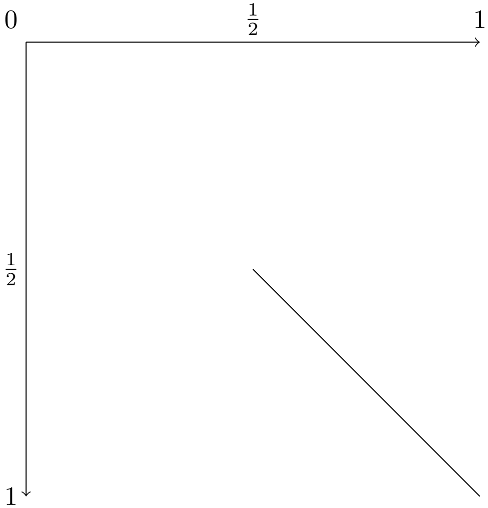


Modifying the slope

Example: $(1,1)$ satisfies $(\star)$

Segment $S_{1,1}$ endpoint $(1/2, 1/2)$ length $\sqrt{2} \cdot 2^{-1}$

$c : (x,y) \mapsto (x/2, y/2) \qquad h : (x,y) \mapsto (x, 2y)$



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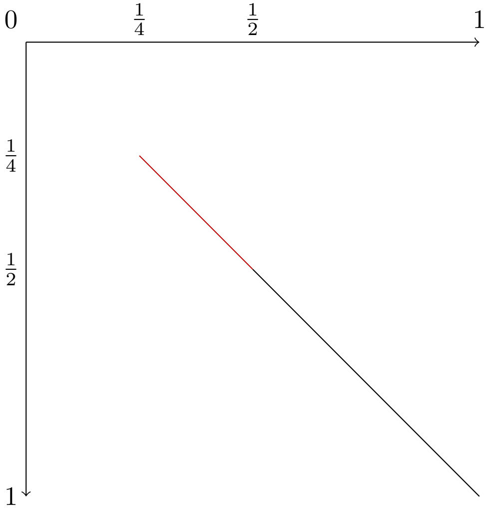

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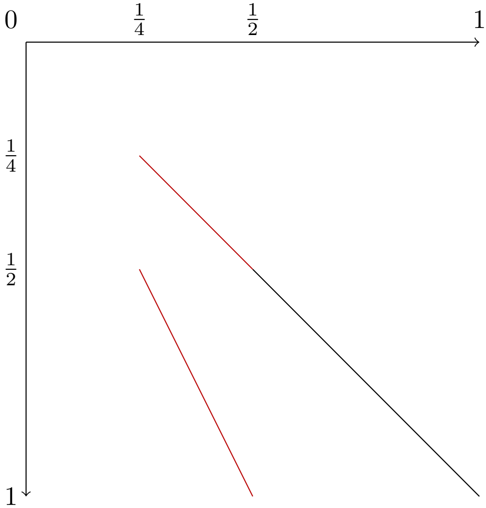

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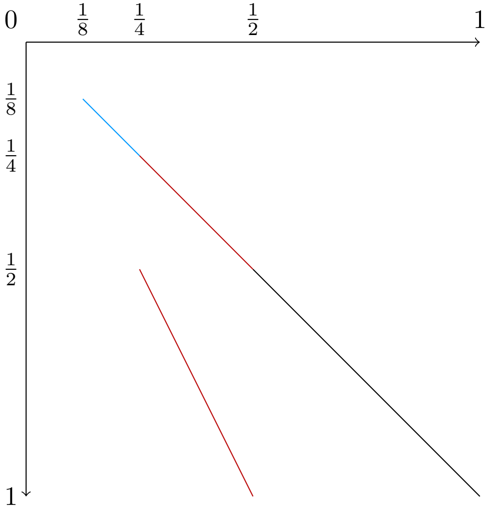
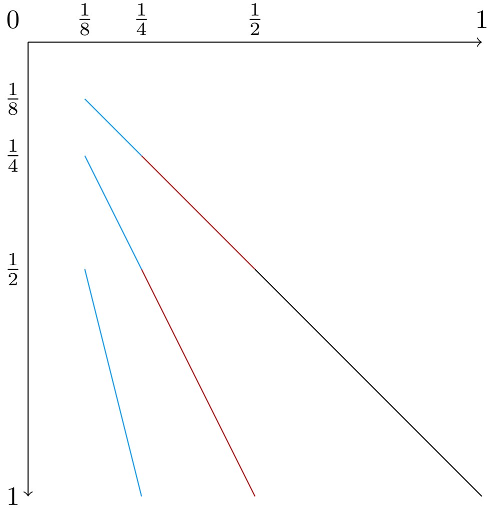


Modifying the slope

<u>Example</u>: $(1,1)$ satisfies $(\star)$

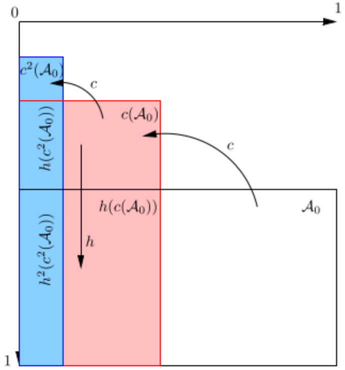
Segment $S_{1,1}$ endpoint $(1/2,1/2)$ length $\sqrt{2}\cdot 2^{-1}$

$c:(x,y)\mapsto(x/2,y/2)\qquad h:(x,y)\mapsto(x,2y)$



Definition: Set of segments of different slopes
$c : (x, y) \mapsto (x/2, y/2)$
$h : (x, y) \mapsto (x, 2y)$

$$\mathcal{A}_n = \bigcup_{0 \le j \le i \le n} h^j(c^i(\mathcal{A}_0))$$



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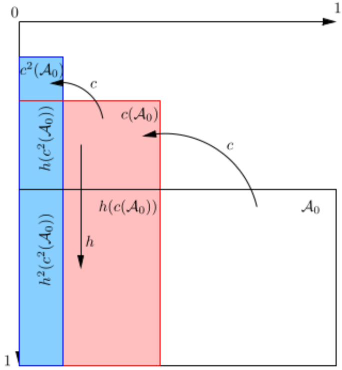

Lemma: $(\mathcal{A}_n)_{n \ge 0}$ is a Cauchy sequence

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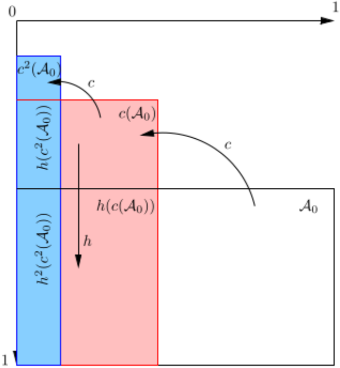
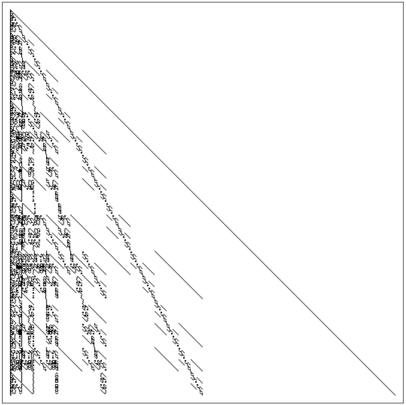


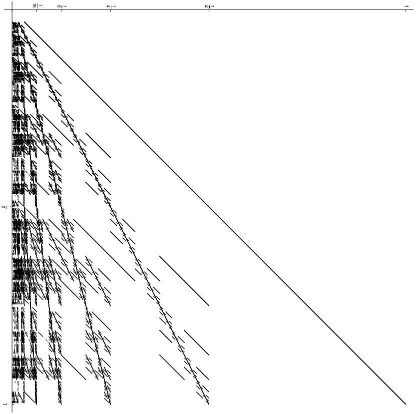
Lemma: $(\mathcal{A}_n)_{n \geq 0}$ is a Cauchy sequence

Definition: Limit object $\mathcal{L}$

Theorem (Leroy, Rigo, S., 2016)

The sequence $(U_n)_{n\geq 0}$ of compact sets converges to the compact set $\mathcal{L}$ when n tends to infinity (for the Hausdorff distance).





"Simple" characterization of $\mathcal{L}$: $(\star)$ condition

First step: coloring the cells of the grids regarding the parity

Extension using Lucas' theorem

Everything still holds for binomial coefficients $\equiv r \bmod p$ with

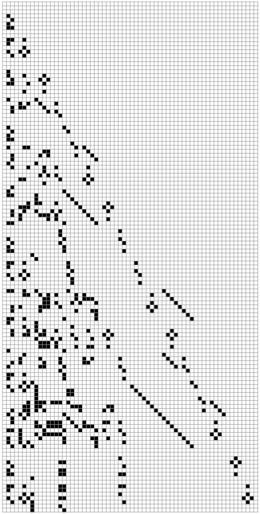
- base-2 expansions of integers
- p a prime
- $r \in \{1, \dots, p-1\}$

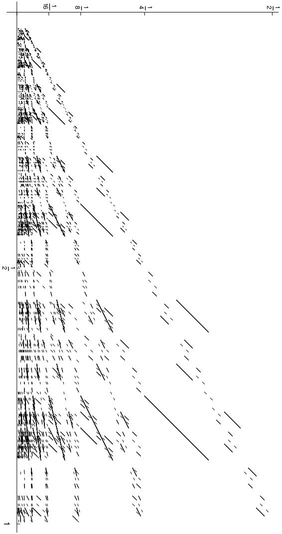
Theorem (Lucas, 1878)

Let p be a prime number.
If $m = m_k p^k + \cdots + m_1 p + m_0$ and $n = n_k p^k + \cdots + n_1 p + n_0$ then

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \bmod p.$$

Left: binomial coefficients $\equiv 2 \bmod 3$
Right: estimate of the corresponding limit object



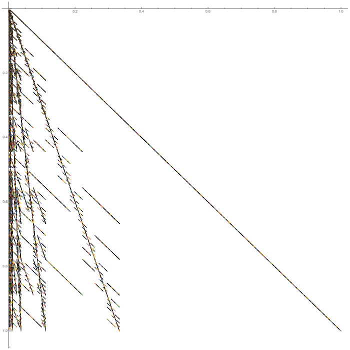


Extension to any integer base

Everything still holds for binomial coefficients $\equiv r \bmod p$ with

- base-b expansions of integers with $b \geq 2$
- p a prime
- $r \in \{1, \dots, p-1\}$

Example: base 3, $\equiv 1 \bmod 2$



Definitions:

- Fibonacci numbers $(F(n))_{n\geq 0}$:
  $F(0)=1,\ F(1)=2,\ F(n+2)=F(n+1)+F(n)\quad \forall\, n\geq 0$
  $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$
- $\mathrm{rep}_F(n)$ greedy Fibonacci representation of $n\in\mathbb{N}_{>0}$ starting with 1
- $\mathrm{rep}_F(0)=\varepsilon$ where ε is the empty word

n		$\mathrm{rep}_F(n)$
0		ε
1	$1\times F(0)$	1
2	$1\times F(1)+0\times F(0)$	10
3	$1\times F(2)+0\times F(1)+0\times F(0)$	100
4	$1\times F(2)+0\times F(1)+1\times F(0)$	101
5	$1\times F(3)+0\times F(2)+0\times F(1)+0\times F(0)$	1000
6	$1\times F(3)+0\times F(2)+0\times F(1)+1\times F(0)$	1001
$\vdots$	$\vdots$	$\vdots$
		$\{\varepsilon\}\cup 1\{0,01\}^*$

Generalized Pascal triangle P_F in Fibonacci base

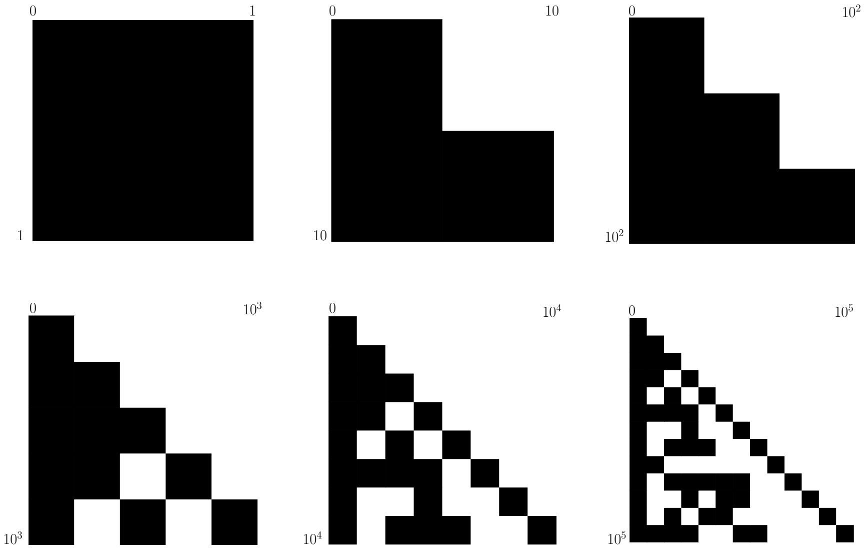
$\binom{\text{rep}_F(m)}{\text{rep}_F(k)}$		$\text{rep}_F(k)$							
		ε	1	10	100	101	1000	1001	1010	…
	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
	100	1	1	2	1	0	0	0	0	
$\text{rep}_F(m)$	101	1	2	1	0	1	0	0	0	
	1000	1	1	3	3	0	1	0	0	
	1001	1	2	2	1	2	0	1	0	
	1010	1	2	3	1	1	0	0	1	
	⋮									⋱

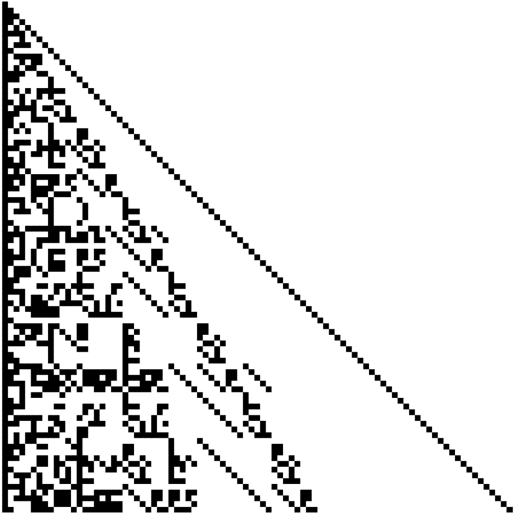
Binomial coefficient of finite words: $\binom{\text{rep}_F(m)}{\text{rep}_F(k)}$

Rule (not local):

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

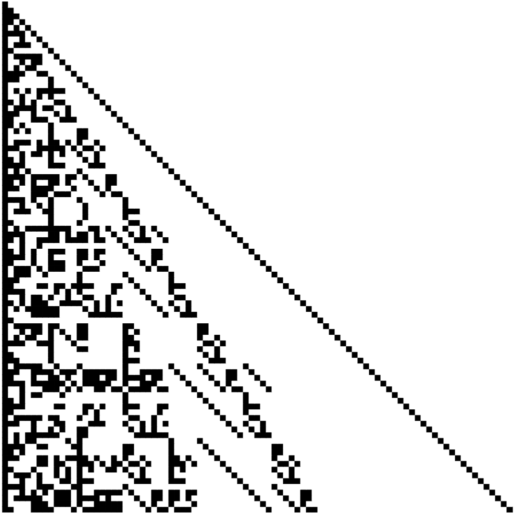
The first six elements of the sequence $(U'_n)_{n\geq 0}$





Lines of different slopes:

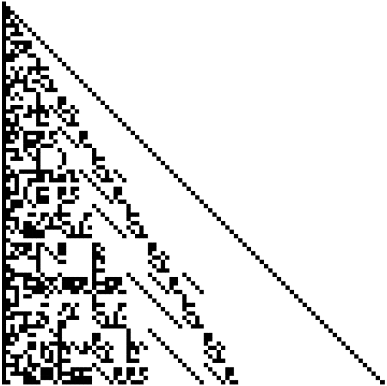
The tenth element

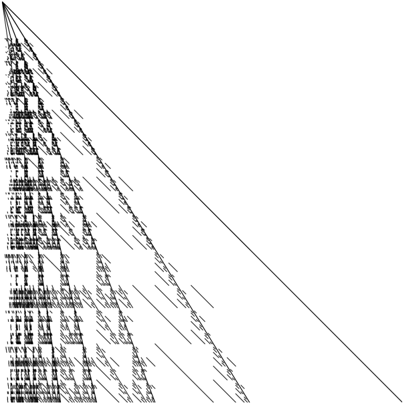


Lines of different slopes: φ^n, $n \geq 0$, with $\varphi = \frac{1+\sqrt{5}}{2}$ Golden Ratio

Theorem (S., 2018)

The sequence $(U'_n)_{n\geq 0}$ of compact sets converges to a limit compact set $\mathcal{L}'$ when n tends to infinity (for the Hausdorff distance).





"Simple" characterization of $\mathcal{L}'$: $(\star')$ condition

$\beta \in \mathbb{R}_{>1} \qquad A_\beta = \{0, 1, \ldots, \lceil\beta\rceil - 1\}$

$$x \in [0,1) \rightsquigarrow x = \sum_{j=1}^{+\infty} c_j \beta^{-j}, \quad c_j \in A_\beta$$

Greedy way: $d_\beta(x) = c_1 c_2 c_3 \cdots$

$$d_\beta(1) = \begin{cases} (\beta - 1)^\omega & \text{if } \beta \in \mathbb{N} \\ (\lceil\beta\rceil - 1) d_\beta(1 - (\lceil\beta\rceil - 1)/\beta) & \text{if } \beta \notin \mathbb{N} \end{cases}$$

Definition

$\beta \in \mathbb{R}_{>1}$ is a *Parry number* if $d_\beta(1)$ is ultimately periodic.

Example: $b \in \mathbb{N}_{>1}$: $d_b(1) = (b-1)^\omega$
φ: $d_\varphi(1) = 110^\omega$

Parry number $\rightsquigarrow$ algebraic integer whose conjugates have modulus less than β (Perron number)

Linear numeration system

Definition

A *linear numeration system* is a sequence $(U(n))_{n\geq 0}$ such that

- U increasing
- $U(0) = 1$
- $\sup_{n\geq 0} \frac{U(n+1)}{U(n)}$ bounded by a constant
- U linear recurrence relation
  $\exists k \geq 1, \exists a_0, \ldots, a_{k-1} \in \mathbb{Z}$ such that

$$U(n+k) = a_{k-1}\, U(n+k-1) + \cdots + a_0\, U(n) \quad \forall n \geq 0$$

Greedy representation in $(U(n))_{n\geq 0}$:

$$n = \sum_{j=0}^{\ell} c_j U(j) \qquad \mathrm{rep}_U(n) = c_\ell \cdots c_0 \in L_U = \mathrm{rep}_U(\mathbb{N})$$

Example: integer base $(b^n)_{n\geq 0}$ with $b \in \mathbb{N}_{>1}$,
Fibonacci numeration system $(F(n))_{n\geq 0}$

Parry numeration system

Parry number $\beta \in \mathbb{R}_{>1} \rightsquigarrow$ linear numeration system $(U_\beta(n))_{n\geq 0}$

- $d_\beta(1) = t_1 \cdots t_m 0^\omega$

$$\begin{array}{rcll} U_\beta(0) & = & 1 & \\ U_\beta(i) & = & t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 & \forall\, 1 \leq i \leq m-1 \\ U_\beta(n) & = & t_1 U_\beta(n-1) + \cdots + t_m U_\beta(n-m) & \forall\, n \geq m \end{array}$$

- $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+k})^\omega$

$$\begin{array}{rcll} U_\beta(0) & = & 1 & \\ U_\beta(i) & = & t_1 U_\beta(i-1) + \cdots + t_i U_\beta(0) + 1 & \forall\, 1 \leq i \leq m+k-1 \\ U_\beta(n) & = & t_1 U_\beta(n-1) + \cdots + t_{m+k} U_\beta(n-m-k) & \forall\, n \geq m+k \\ & & + U_\beta(n-k) - t_1 U_\beta(n-k-1) - \cdots & \\ & & - t_m U_\beta(n-m-k) & \end{array}$$

Examples:

$b \in \mathbb{N}_{>1} \rightsquigarrow (b^n)_{n\geq 0}$ base b

$\varphi \rightsquigarrow (F(n))_{n\geq 0}$ Fibonacci numeration system

Bertrand numeration system

Definition

A linear numeration system $(U(n))_{n\geq 0}$ is a *Bertrand numeration system* if

$$w \in L_U \Leftrightarrow w0 \in L_U \quad \forall\, w \neq \varepsilon.$$

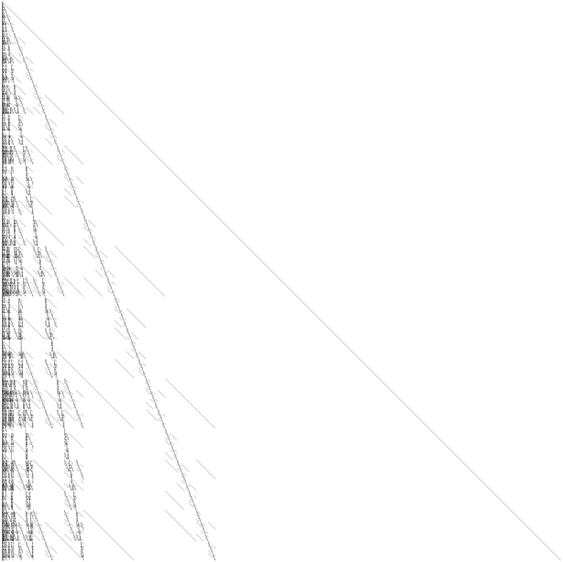
Proposition (Bertrand-Mathis, 1989)

If $\beta \in \mathbb{R}_{>1}$ is a Parry number, then $(U_\beta(n))_{n\geq 0}$ is a Bertrand numeration system.

Extension to Parry–Bertrand numeration systems

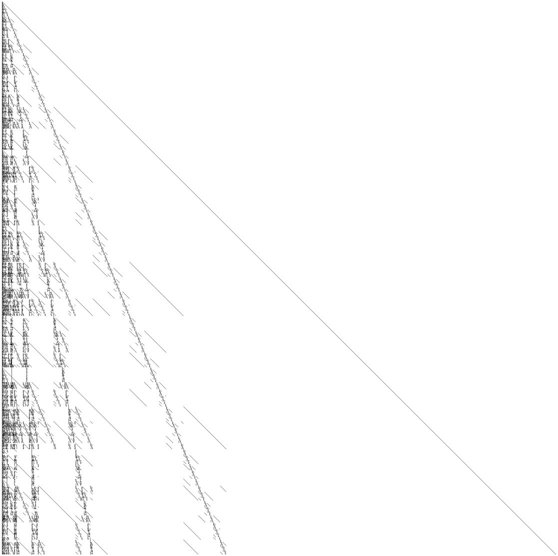
- $\beta \in \mathbb{R}_{>1}$ Parry number
- $(U_\beta(n))_{n\geq 0}$ Parry–Bertrand numeration system
- Generalized Pascal triangle P_β in $(U_\beta(n))_{n\geq 0}$ indexed by words of L_{U_β}
- Sequence of compact sets extracted from P_β (first $U_\beta(n)$ rows and columns of P_β)
- Convergence to a limit object (same technique)
  - Lines of different slopes: β^n, $n \geq 0$
  - $(\star')$ condition
- Works modulo any prime number

φ^2



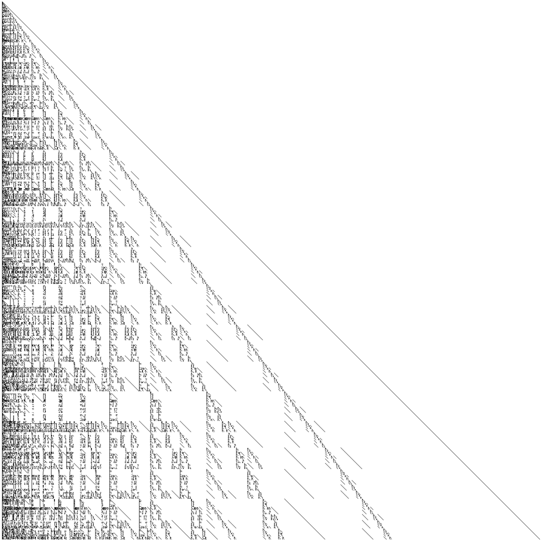
Example 2

$\beta_1 \approx 2.47098$ dominant root of $P(X) = X^4 - 2X^3 - X^2 - 1$



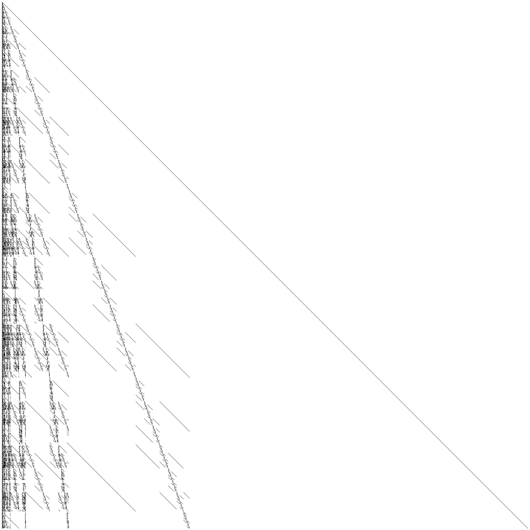
Example 3

$\beta_2 \approx 1.38028$ dominant root of $P(X) = X^4 - X^3 - 1$



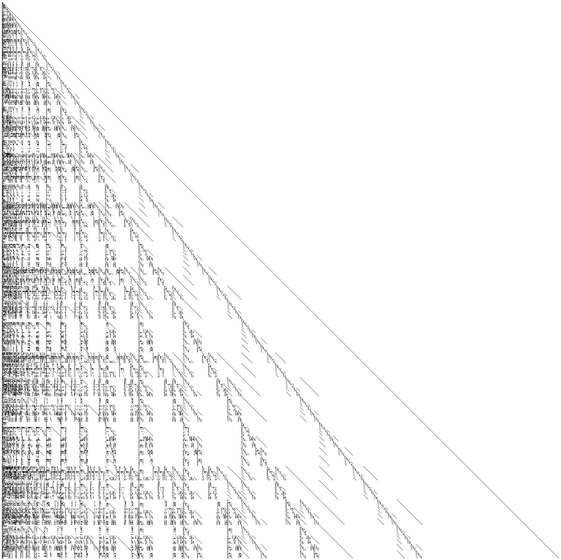
Example 4

$\beta_3 \approx 2.80399$ dominant root of $P(X) = X^4 - 2X^3 - 2X^2 - 2$



Example 5

$\beta_4 \approx 1.32472$ dominant root of polynomial $P(X) = X^5 - X^4 - 1$



Conclusion

In this talk:

	Generalized Pascal triangle	Convergence mod p
base 2	✓	✓
integer base	✓	✓
Fibonacci	✓	✓
Parry–Bertrand	✓	✓

Other related work

- Regularity of the sequence counting subword occurrences: result for any integer base b and the Fibonacci numeration system
- Behavior of the summatory function: result for any integer base b (exact behavior) and the Fibonacci numeration system (asymptotics)

Conus textile or Cloth of gold cone

Color pattern of its shell ⇝ Sierpiński gasket

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