Pascal-like triangles: base 2 and beyond

Joint work with Julien Leroy and Michel Rigo (ULiège)

Manon Stipulanti (ULiège)  FRIA grantee

University of Winnipeg (Canada)
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### Classical Pascal triangle

<table>
<thead>
<tr>
<th>$(\begin{pmatrix} m \ k \end{pmatrix})$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>6</td>
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<td>20</td>
<td>15</td>
<td>6</td>
<td>1</td>
<td>0</td>
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<tr>
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<td>1</td>
<td>7</td>
<td>21</td>
<td>35</td>
<td>35</td>
<td>21</td>
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</tr>
</tbody>
</table>

#### Usual binomial coefficients

The usual binomial coefficients of integers are given by:

$$\binom{m}{k} = \frac{m!}{(m-k)!k!}$$

#### Pascal’s rule

Pascal’s rule states that:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$$
- **Grid**: first $2^n$ rows and columns

\[
\begin{array}{cccccc}
0 & 1 & \cdots & 2^n - 1 & 2^n \\
0 & \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{2^n - 1} \\
1 & \binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{2^n - 1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
2^n - 1 & \binom{2^n - 1}{0} & \binom{2^n - 1}{1} & \cdots & \binom{2^n - 1}{2^n - 1} \\
2^n & & & & & \\
\end{array}
\]

\[\mathbb{N}^2 \cap [0, 2^n]^2\]
• Color the grid:
  Color the first $2^n$ rows and columns of the Pascal triangle

\[
\left(\binom{m}{k} \mod 2\right)_{0 \leq m,k < 2^n}
\]

in

• white if $\binom{m}{k} \equiv 0 \mod 2$
• black if $\binom{m}{k} \equiv 1 \mod 2$
• Color the grid:
  Color the first $2^n$ rows and columns of the Pascal triangle
  
  $$\left(\binom{m}{k} \mod 2\right)_{0 \leq m, k < 2^n}$$

  in
  • white if $\binom{m}{k} \equiv 0 \mod 2$
  • black if $\binom{m}{k} \equiv 1 \mod 2$

• Normalize by a homothety of ratio $1/2^n$
  (bring into $[0, 1] \times [0, 1]$)
  $\rightsquigarrow$ sequence belonging to $[0, 1] \times [0, 1]$
What happens for $n \in \{0, 1\}$

$n = 0$
What happens for $n \in \{0, 1\}$

$n = 0$

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
What happens for $n \in \{0, 1\}$

$n = 0$

\[
\begin{array}{c|c}
\hline
0 & 1 \\
\hline
(0) & 0 \\
\hline
1 & 1 \\
\end{array}
\]
What happens for $n \in \{0, 1\}$

$n = 0$

$\binom{0}{0}$

$\binom{1}{0}$

$\binom{1}{1}$
What happens for $n \in \{0, 1\}$

$n = 0$

$n = 1$
What happens for $n \in \{0, 1\}$

$n = 0$

$n = 1$
What happens for $n \in \{0, 1\}$

$n = 0$

$n = 1$
What happens for \( n \in \{0, 1\} \)

\( n = 0 \)

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\( n = 1 \)

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

Generalized Pascal triangles
The first six elements of the sequence

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The tenth element of the sequence

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The Sierpiński gasket
The Sierpiński gasket

Generalized Pascal triangles

Manon Stipulanti (ULiège)
The Sierpiński gasket
Folklore fact
The latter sequence converges to the Sierpiński gasket when $n$ tends to infinity (for the Hausdorff distance).
Folklore fact

The latter sequence converges to the Sierpiński gasket when $n$ tends to infinity (for the Hausdorff distance).

Definitions:

- $\epsilon$-fattening of a subset $S \subset \mathbb{R}^2$

$$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the Hausdorff distance $d_h$

$$d_h(S, S') = \min\{\epsilon \in \mathbb{R}_{\geq 0} \mid S \subset [S']_\epsilon \text{ and } S' \subset [S]_\epsilon\}$$
New idea

Replace usual binomial coefficients of integers by binomial coefficients of finite words
Definition: A finite word is a finite sequence of letters belonging to a finite set called the alphabet.

Example: 101, 101001 ∈ \{0, 1\}^*
**Definition:** A finite word is a finite sequence of letters belonging to a finite set called the alphabet.

**Example:** $101, 101001 \in \{0, 1\}^*$

**Binomial coefficient of words**

Let $u, v$ be two finite words. The binomial coefficient $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = 101001$ $v = 101$
**Definition:** A finite word is a finite sequence of letters belonging to a finite set called the alphabet.

**Example:** 101, 101001 ∈ \{0, 1\}*

**Binomial coefficient of words**

Let \(u, v\) be two finite words. The binomial coefficient \(\binom{u}{v}\) of \(u\) and \(v\) is the number of times \(v\) occurs as a subsequence of \(u\) (meaning as a “scattered” subword).

**Example:** \(u = 101001\) \(v = 101\) 1 occurrence
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

**Example:** 101, 101001 ∈ \{0, 1\}*

**Binomial coefficient of words**

Let \( u, v \) be two finite words. The *binomial coefficient* \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

**Example:** \( u = 101001 \quad v = 101 \quad 2 \text{ occurrences} \)
**Definition**: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

**Example**: 101, 101001 ∈ \{0, 1\}*

**Binomial coefficient of words**

Let \( u, v \) be two finite words. The *binomial coefficient* \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

**Example**: \( u = 101001 \) \( v = 101 \) 3 occurrences
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet.*

**Example:** 101, 101001 ∈ \{0, 1\}*

**Binomial coefficient of words**

Let \( u, v \) be two finite words. The *binomial coefficient* \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

**Example:** \( u = 101001 \) \( v = 101 \) 4 occurrences
Definition: A finite word is a finite sequence of letters belonging to a finite set called the alphabet.

Example: 101, 101001 ∈ \{0, 1\}^*

Binomial coefficient of words

Let \( u, v \) be two finite words. The binomial coefficient \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

Example: \( u = 101001 \quad v = 101 \quad 5 \) occurrences
**Definition**: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

**Example**: $101, 101001 \in \{0, 1\}^*$

---

**Binomial coefficient of words**

Let $u, v$ be two finite words. The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example**: $u = 101001$ $v = 101$ 6 occurrences
**Definition:** A finite word is a finite sequence of letters belonging to a finite set called the alphabet.

**Example:** 101, 101001 ∈ \{0, 1\}*

---

**Binomial coefficient of words**

Let \( u, v \) be two finite words.

The binomial coefficient \( (u \atop v) \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

**Example:** \( u = 101001 \quad v = 101 \)

\[ \Rightarrow \binom{101001}{101} = 6 \]
Remark:
Natural generalization of binomial coefficients of integers

With a one-letter alphabet \( \{a\} \)

\[
\binom{a^m}{a^k} = \underbrace{a \cdot \cdots \cdot a}_{m \text{ times}} = \binom{m}{k} \quad \forall m, k \in \mathbb{N}
\]
The base 2 case

Definitions:

- \( \text{rep}_2(n) \) greedy base-2 expansion of \( n \in \mathbb{N}_{>0} \) starting with 1
- \( \text{rep}_2(0) = \varepsilon \) where \( \varepsilon \) is the empty word

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \text{rep}_2(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>1</td>
<td>( 1 \times 2^0 )</td>
</tr>
<tr>
<td>2</td>
<td>( 1 \times 2^1 + 0 \times 2^0 )</td>
</tr>
<tr>
<td>3</td>
<td>( 1 \times 2^1 + 1 \times 2^0 )</td>
</tr>
<tr>
<td>4</td>
<td>( 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 )</td>
</tr>
<tr>
<td>5</td>
<td>( 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 )</td>
</tr>
<tr>
<td>6</td>
<td>( 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

\( \{\varepsilon\} \cup 1\{0,1\}^* \)
### Generalized Pascal triangle in base 2

Table:

<table>
<thead>
<tr>
<th>( \text{rep}_2(m) )</th>
<th>( \varepsilon )</th>
<th>1</th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
<th>( \ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varepsilon )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>1</td>
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<tr>
<td>110</td>
<td>1</td>
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<td>1</td>
<td>0</td>
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<td>1</td>
<td>0</td>
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</tr>
<tr>
<td>111</td>
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<td>3</td>
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<td>0</td>
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<td>1</td>
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<td>( \vdots )</td>
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</tbody>
</table>

#### Binomial coefficient of finite words:

\[
\binom{u}{v}
\]

#### Rule (not local):

\[
\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b} \binom{u}{v}
\]
### Generalized Pascal triangle in base 2

The classical Pascal triangle can be generalized to any base, including base 2. Consider the table below, where each row represents a number in base 2, and each column represents the corresponding values in the generalized Pascal triangle.

<table>
<thead>
<tr>
<th>rep₂(m)</th>
<th>rep₂(k)</th>
<th>ε</th>
<th>1</th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
<th>⋯</th>
</tr>
</thead>
<tbody>
<tr>
<td>ε</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>⋯</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>⋯</td>
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<tr>
<td>10</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>⋯</td>
</tr>
<tr>
<td>11</td>
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<td>0</td>
<td>1</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
<td>⋯</td>
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<tr>
<td>100</td>
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<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>⋯</td>
</tr>
<tr>
<td>101</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>0</td>
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<td>0</td>
<td>0</td>
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<td>⋯</td>
</tr>
<tr>
<td>110</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
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<td>0</td>
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<td>⋯</td>
</tr>
<tr>
<td>111</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>⋯</td>
</tr>
</tbody>
</table>

The classical Pascal triangle is recovered when base 2 representation is considered.

---

**Manon Stipulanti (ULiège)**
Questions:

- After coloring and normalization can we expect the convergence to an analogue of the Sierpiński gasket?
- Could we describe this limit object?
- Grid: first $2^n$ rows and columns

\[
\begin{array}{ccccccc}
0 & 1 & \cdots & 2^n - 1 & 2^n \\
0 & \begin{pmatrix} \text{rep}_2(0) \\ \text{rep}_2(0) \end{pmatrix} & \begin{pmatrix} \text{rep}_2(0) \\ \text{rep}_2(1) \end{pmatrix} & \cdots & \begin{pmatrix} \text{rep}_2(0) \\ \text{rep}_2(2^n - 1) \end{pmatrix} \\
1 & \begin{pmatrix} \text{rep}_2(1) \\ \text{rep}_2(0) \end{pmatrix} & \begin{pmatrix} \text{rep}_2(1) \\ \text{rep}_2(1) \end{pmatrix} & \cdots & \begin{pmatrix} \text{rep}_2(1) \\ \text{rep}_2(2^n - 1) \end{pmatrix} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
2^n - 1 & \begin{pmatrix} \text{rep}_2(2^n - 1) \\ \text{rep}_2(0) \end{pmatrix} & \begin{pmatrix} \text{rep}_2(2^n - 1) \\ \text{rep}_2(1) \end{pmatrix} & \cdots & \begin{pmatrix} \text{rep}_2(2^n - 1) \\ \text{rep}_2(2^n - 1) \end{pmatrix} \\
2^n & \begin{pmatrix} \text{rep}_2(2^n) \\ \text{rep}_2(0) \end{pmatrix} & \begin{pmatrix} \text{rep}_2(2^n) \\ \text{rep}_2(1) \end{pmatrix} & \cdots & \begin{pmatrix} \text{rep}_2(2^n) \\ \text{rep}_2(2^n - 1) \end{pmatrix} \\
\end{array}
\]
• Color the grid:
  Color the first $2^n$ rows and columns of the generalized Pascal triangle
  \[
  \left(\left(\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \mod 2\right)_{0 \leq m, k < 2^n}\right)
  \]
  in
  • white if \(\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 0 \mod 2\)
  • black if \(\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 1 \mod 2\)

• Normalize by a homothety of ratio $1/2^n$
  (bring into $[0, 1] \times [0, 1]$)
  \(\leadsto\) sequence \((U_n)_{n \geq 0}\) belonging to $[0, 1] \times [0, 1]$
The elements $U_0, \ldots, U_5$
The element $U_9$

Lines of different slopes...

Generalized Pascal triangles
A key result

**Theorem (Leroy, Rigo, S., 2016)**

The sequence \((U_n)_{n \geq 0}\) of compact sets converges to a limit compact set \(L\) when \(n\) tends to infinity (for the Hausdorff distance).

“Simple” characterization of \(L\): topological closure of a union of segments described through a “simple” combinatorial property.
### Counting subword occurrences

**Generalized Pascal triangle in base 2**

<table>
<thead>
<tr>
<th></th>
<th>ε</th>
<th>1</th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
<th>n</th>
<th>$S_2(n)$</th>
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<td>0</td>
<td>1</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

**Definition:**

\[
S_2(n) = \# \left\{ m \in \mathbb{N} \mid \left( \frac{\text{rep}_2(n)}{\text{rep}_2(m)} \right) > 0 \right\} \quad \forall n \geq 0
\]

\[
= \# \text{ (scattered) subwords in } \{\varepsilon\} \cup 1\{0, 1\}^* \text{ of } \text{rep}_2(n)
\]
The sequence \( (S_2(n))_{n \geq 0} \) in the interval \([0, 256]\)

Palindromic structure \( \rightsquigarrow \) regularity

Generalized Pascal triangles

Manon Stipulanti (ULiège)
• 2-kernel of $s = (s(n))_{n \geq 0}$

\[ \mathcal{K}_2(s) = \{(s(n))_{n \geq 0}, (s(2n))_{n \geq 0}, (s(2n + 1))_{n \geq 0}, (s(4n))_{n \geq 0}, (s(4n + 1))_{n \geq 0}, (s(4n + 2))_{n \geq 0}, \ldots \} \]

\[ = \{(s(2^i n + j))_{n \geq 0} \mid i \geq 0 \text{ and } 0 \leq j < 2^i \} \]
• 2-kernel of $s = (s(n))_{n \geq 0}$

$$\mathcal{K}_2(s) = \{(s(n))_{n \geq 0}, (s(2n))_{n \geq 0}, (s(2n + 1))_{n \geq 0}, (s(4n))_{n \geq 0},$$

$$\quad (s(4n + 1))_{n \geq 0}, (s(4n + 2))_{n \geq 0}, \ldots \}$$

$$= \{(s(2^i n + j))_{n \geq 0} \mid i \geq 0 \text{ and } 0 \leq j < 2^i \}$$

• $s = (s(n))_{n \geq 0}$ is 2-regular if there exist

$$(t_1(n))_{n \geq 0}, \ldots, (t_\ell(n))_{n \geq 0}$$

s.t. each $(t(n))_{n \geq 0} \in \mathcal{K}_2(s)$ is a $\mathbb{Z}$-linear combination of the $t_j$’s
The case of \((S_2(n))_{n \geq 0}\)

**Theorem (Leroy, Rigo, S., 2017)**

The sequence \((S_2(n))_{n \geq 0}\) satisfies, for all \(n \geq 0\),

\[
S_2(2n + 1) = 3S_2(n) - S_2(2n)
\]
\[
S_2(4n) = -S_2(n) + 2S_2(2n)
\]
\[
S_2(4n + 2) = 4S_2(n) - S_2(2n).
\]

**Corollary (Leroy, Rigo, S., 2017)**

\((S_2(n))_{n \geq 0}\) is 2-regular.

**Example:** \(\mathbb{Z}\)-linear combination of \((S_2(n))_{n \geq 0}\) and \((S_2(2n))_{n \geq 0}\)

\[
S_2(4n + 1) = S_2(2(2n) + 1) = 3S_2(2n) - S_2(4n)
\]
\[
= 3S_2(2n) - (-S_2(n) + 2S_2(2n)) = S_2(n) + S_2(2n)
\]
Example: $s(n)$ number of 1’s in $\text{rep}_2(n)$

\[ s(2n) = s(n) \quad s(2n+1) = s(n) + 1 \]

$s$ is 2-regular

Summatory function $A$:

\[ A(0) = 0 \]
\[ A(n) = \sum_{j=0}^{n-1} s(j) \quad \forall n \geq 1 \]

**Theorem (Delange, 1975)**

\[
\frac{A(n)}{n} = \frac{1}{2} \log_2(n) + \mathcal{G}(\log_2(n)) \tag{1}
\]

where $\mathcal{G}$ continuous, nowhere differentiable, periodic of period 1.
Theorem (Allouche, Shallit, 2003)
Under some hypotheses, the summatory function of any $k$-regular sequence has a behavior analogous to (1).

$\Rightarrow$ Replacing $s$ by $S_2$: same behavior as (1) but does not satisfy the hypotheses of the theorem
Definition: $A_2(0) = 0$

$$A_2(n) = \sum_{j=0}^{n-1} S_2(j) \quad \forall n \geq 1$$

First few values:

0, 1, 3, 6, 9, 13, 18, 23, 27, 32, 39, 47, 54, 61, 69, 76, 81, 87, 96, 107, ...
Summatory function of \((S_2(n))_{n \geq 0}\)

**Definition:** \(A_2(0) = 0\)

\[
A_2(n) = \sum_{j=0}^{n-1} S_2(j) \quad \forall n \geq 1
\]

First few values:

0, 1, 3, 6, 9, 13, 18, 23, 27, 32, 39, 47, 54, 61, 69, 76, 81, 87, 96, 107, …

**Lemma (Leroy, Rigo, S., 2017)**

For all \(n \geq 0\), \(A_2(2^n) = 3^n\).
Lemma (Leroy, Rigo, S., 2017)

Let $\ell \geq 1$.

- If $0 \leq r \leq 2^{\ell-1}$, then
  \[ A_2(2^\ell + r) = 2 \cdot 3^{\ell-1} + A_2(2^{\ell-1} + r) + A_2(r). \]

- If $2^{\ell-1} < r < 2^\ell$, then
  \[ A_2(2^\ell + r) = 4 \cdot 3^\ell - 2 \cdot 3^{\ell-1} - A_2(2^{\ell-1} + r') - A_2(r') \text{ where } r' = 2^\ell - r. \]
Lemma (Leroy, Rigo, S., 2017)

Let $\ell \geq 1$.

- If $0 \leq r \leq 2^{\ell-1}$, then
  \[
  A_2(2^\ell + r) = 2 \cdot 3^{\ell-1} + A_2(2^{\ell-1} + r) + A_2(r).
  \]

- If $2^{\ell-1} < r < 2^\ell$, then
  \[
  A_2(2^\ell + r) = 4 \cdot 3^\ell - 2 \cdot 3^{\ell-1} - A_2(2^{\ell-1} + r') - A_2(r') \quad \text{where } r' = 2^\ell - r.
  \]

$\Rightarrow$ 3-decomposition: particular decomposition of $A_2(n)$ based on powers of 3
Lemma (Leroy, Rigo, S., 2017)

Let $\ell \geq 1$.

- If $0 \leq r \leq 2^{\ell-1}$, then
  \[ A_2(2^\ell + r) = 2 \cdot 3^{\ell-1} + A_2(2^{\ell-1} + r) + A_2(r). \]

- If $2^{\ell-1} < r < 2^\ell$, then
  \[ A_2(2^\ell + r) = 4 \cdot 3^{\ell} - 2 \cdot 3^{\ell-1} - A_2(2^{\ell-1} + r') - A_2(r') \quad \text{where} \quad r' = 2^\ell - r. \]

$\Rightarrow 3$-decomposition: particular decomposition of $A_2(n)$ based on powers of 3

$\Rightarrow$ two numeration systems: base 2 and base 3
Theorem (Leroy, Rigo, S., 2017)

There exists a continuous and periodic function $\mathcal{H}_2$ of period 1 such that, for all $n \geq 1$,

$$A_2(n) = 3^{\log_2(n)} \mathcal{H}_2(\log_2(n)).$$
The Fibonacci case

Definitions:
- Fibonacci numbers \( (F(n))_{n \geq 0} \): \( F(0) = 1 \), \( F(1) = 2 \) and \( F(n + 2) = F(n + 1) + F(n) \) \( \forall n \geq 0 \)
- \( \text{rep}_F(n) \) greedy Fibonacci representation of \( n \in \mathbb{N}_{>0} \) starting with 1
- \( \text{rep}_F(0) = \varepsilon \) where \( \varepsilon \) is the empty word

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \text{rep}_F(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>1</td>
<td>( 1 \times F(0) )</td>
</tr>
<tr>
<td>2</td>
<td>( 1 \times F(1) + 0 \times F(0) )</td>
</tr>
<tr>
<td>3</td>
<td>( 1 \times F(2) + 0 \times F(1) + 0 \times F(0) )</td>
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<td>4</td>
<td>( 1 \times F(2) + 0 \times F(1) + 1 \times F(0) )</td>
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<td>5</td>
<td>( 1 \times F(3) + 0 \times F(2) + 0 \times F(1) + 0 \times F(0) )</td>
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<td>6</td>
<td>( 1 \times F(3) + 0 \times F(2) + 0 \times F(1) + 1 \times F(0) )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
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</tbody>
</table>
# Generalized Pascal triangle in base Fibonacci

Binomial coefficient of finite words:

\[
\binom{u}{v}
\]

Rule (not local):

\[
\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b} \binom{u}{v}
\]

<table>
<thead>
<tr>
<th>(\text{rep}_F(m))</th>
<th>(\varepsilon)</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>101</th>
<th>1000</th>
<th>1001</th>
<th>1010</th>
<th>\ldots</th>
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<td>\ldots</td>
</tr>
<tr>
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</table>
The first six elements of the sequence $(U_n^f)_{n \geq 0}$
The sequence \((U'_n)_{n \geq 0}\) of compact sets converges to a limit compact set \(\mathcal{L}'\) when \(n\) tends to infinity (for the Hausdorff distance).

“Simple” characterization of \(\mathcal{L}'\): topological closure of a union of segments described through a “simple” combinatorial property.
Counting subword occurrences

Generalized Pascal triangle in base Fibonacci

<table>
<thead>
<tr>
<th>ε</th>
<th>0</th>
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<th>0</th>
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<th>0</th>
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<td>1</td>
<td>7</td>
<td>6</td>
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</tbody>
</table>

Definition:

$$S_F(n) = \left\{ m \in \mathbb{N} \mid \left( \frac{\text{rep}_F(n)}{\text{rep}_F(m)} \right) > 0 \right\} \quad \forall n \geq 0$$

$$= \# \text{ (scattered) subwords in } \{ \varepsilon \} \cup 1\{0,01\}^* \text{ of } \text{rep}_F(n)$$
The sequence $(S_F(n))_{n \geq 0}$ in the interval $[0, 233]$
2-kernel $K_2(s)$ of a sequence $s$

- Select all the nonnegative integers whose base-2 expansion (with leading zeroes) ends with $w \in \{0, 1\}^*$
- Evaluate $s$ at those integers
- Let $w$ vary in $\{0, 1\}^*$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{rep}_2(n)$</th>
<th>$s(n)$</th>
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<tr>
<td>0</td>
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<td>$s(4)$</td>
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<tr>
<td>5</td>
<td>101</td>
<td>$s(5)$</td>
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</table>

$F$-kernel $K_F(s)$ of a sequence $s$

- Select all the nonnegative integers whose Fibonacci representation (with leading zeroes) ends with $w \in \{0, 1\}^*$
- Evaluate $s$ at those integers
- Let $w$ vary in $\{0, 1\}^*$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{rep}_F(n)$</th>
<th>$s(n)$</th>
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<td>0</td>
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<tr>
<td>5</td>
<td>1000</td>
<td>$s(5)$</td>
</tr>
</tbody>
</table>
\( s = (s(n))_{n \geq 0} \) is \( F \)-regular if there exist
\[
(t_1(n))_{n \geq 0}, \ldots, (t_\ell(n))_{n \geq 0}
\]
s.t. each \( (t(n))_{n \geq 0} \in K_F(s) \) is a \( \mathbb{Z} \)-linear combination of the \( t_j \)’s
$s = (s(n))_{n \geq 0}$ is $F$-regular if there exist

$$(t_1(n))_{n \geq 0}, \ldots, (t_\ell(n))_{n \geq 0}$$

s.t. each $(t(n))_{n \geq 0} \in \mathcal{K}_F(s)$ is a $\mathbb{Z}$-linear combination of the $t_j$’s

**Proposition (Leroy, Rigo, S., 2017)**

$(S_F(n))_{n \geq 0}$ is $F$-regular.
$s = (s(n))_{n \geq 0}$ is $F$-regular if there exist

$$(t_1(n))_{n \geq 0}, \ldots, (t_\ell(n))_{n \geq 0}$$

s.t. each $(t(n))_{n \geq 0} \in \mathcal{K}_F(s)$ is a $\mathbb{Z}$-linear combination of the $t_j$'s

**Proposition (Leroy, Rigo, S., 2017)**

$(S_F(n))_{n \geq 0}$ is $F$-regular.

In the literature, not so many sequences have this kind of property
Summatory function of \((S_F(n))_{n\geq 0}\)

**Definition:** \(A_F(0) = 0\)

\[
A_F(n) = \sum_{j=0}^{n-1} S_F(j) \quad \forall n \geq 1
\]

First few values:

0, 1, 3, 6, 10, 14, 19, 25, 31, 37, 45, 54, 62, 70, 77, 87, 99, 111, 123, 133, ...
Using several numeration systems

First few values (again):

0, 1, 3, 6, 10, 14, 19, 25, 31, 37, 45, 54, 62, 70, 77, 87, 99, 111, 123, 133, \ldots

**Proposition (Leroy, Rigo, S., 2017)**

For all \( n \geq 0 \), \( A_F(F(n)) = B(n) \).

**Definition:** \( B(0) = 1, B(1) = 3, B(2) = 6 \)

\[ B(n + 3) = 2B(n + 2) + B(n + 1) - B(n) \quad \forall n \geq 0 \]

First few values: 1, 3, 6, 14, 31, 70, 157, 353, 793, 1782, \ldots
Using several numeration systems

First few values (again):

0, 1, 3, 6, 10, 14, 19, 25, 31, 37, 45, 54, 62, 70, 77, 87, 99, 111, 123, 133, \ldots

Proposition (Leroy, Rigo, S., 2017)

For all \( n \geq 0 \), \( A_F(F(n)) = B(n) \).

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B(n + 3) = 2B(n + 2) + B(n + 1) - B(n) \quad \forall n \geq 0
\]

First few values: 1, 3, 6, 14, 31, 70, 157, 353, 793, 1782, \ldots

\( \leadsto B\text{-decomposition} \): particular decomposition of \( A_F(n) \) based on linear combination of \( B(n) \)
Using several numeration systems

First few values (again):

0, 1, 3, 6, 10, 14, 19, 25, 31, 37, 45, 54, 62, 70, 77, 87, 99, 111, 123, 133, …

**Proposition (Leroy, Rigo, S., 2017)**

For all $n \geq 0$, $A_F(F(n)) = B(n)$.

**Definition:** $B(0) = 1, B(1) = 3, B(2) = 6$

$$B(n + 3) = 2B(n + 2) + B(n + 1) - B(n) \quad \forall n \geq 0$$

First few values: 1, 3, 6, 14, 31, 70, 157, 353, 793, 1782, …

$\rightsquigarrow$ **$B$-decomposition**: particular decomposition of $A_F(n)$ based on linear combination of $B(n)$

$\rightsquigarrow$ two numeration systems: base $F(n)$ and base $B(n)$
Let $\beta$ be the dominant root of $X^3 - 2X^2 - X + 1$, which is the characteristic polynomial of $(B(n))_{n \geq 0}$. Let $c$ be a constant such that $\lim_{n \to \infty} B(n)/c\beta^n = 1$. There exists a continuous and periodic function $H_F$ of period 1 such that, for all $n \geq 3$,

$$A_F(n) = c \cdot \beta^\log F(n) \cdot H_F(\log F(n)) + o(\beta^\lfloor \log F(n) \rfloor).$$
In this talk:

**Base 2:**

- Generalization of the Pascal triangle in base 2 modulo 2
- 2-regularity of the sequence \((S_2(n))_{n \geq 0}\) counting subword occurrences
- Exact behavior of the summatory function \((A_2(n))_{n \geq 0}\) of the sequence \((S_2(n))_{n \geq 0}\)

**Base Fibonacci:**

- Generalization of the Pascal triangle in base Fibonacci modulo 2
- \(F\)-regularity of the sequence \((S_F(n))_{n \geq 0}\) counting subword occurrences
- Asymptotics of the summatory function \((A_F(n))_{n \geq 0}\) of the sequence \((S_F(n))_{n \geq 0}\)
Other work

Done:

- Generalization of the Pascal triangle modulo a prime number: extension to any Parry–Bertrand numeration system
- Regularity of the sequence counting subword occurrences: extension to any integer base
- Behavior of the summatory function: extension to any integer base (exact behavior)

To do:
Parry–Bertrand numeration systems,
Apply the methods for sequences not related to Pascal triangles, etc.


• J. Leroy, M. Rigo, M. Stipulanti, Counting Subword Occurrences in Base-$b$ Expansions, *accepted in December 2017*.

