Pascal-like triangles: base 2 and beyond

Joint work with Julien Leroy and Michel Rigo (ULiège)

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Classical Pascal triangle

	$\binom{m}{k}$				k					
		0	1	2	3	4	5	6	7	$\cdots$
	0	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	2	1	2	1	0	0	0	0	0	
m	3	1	3	3	1	0	0	0	0	
	4	1	4	6	4	1	0	0	0	
	5	1	5	**10**	**10**	5	1	0	0	
	6	1	6	15	**20**	15	6	1	0	
	7	1	7	21	35	35	21	7	1	
	$\vdots$									$\ddots$

Usual binomial coefficients of integers:

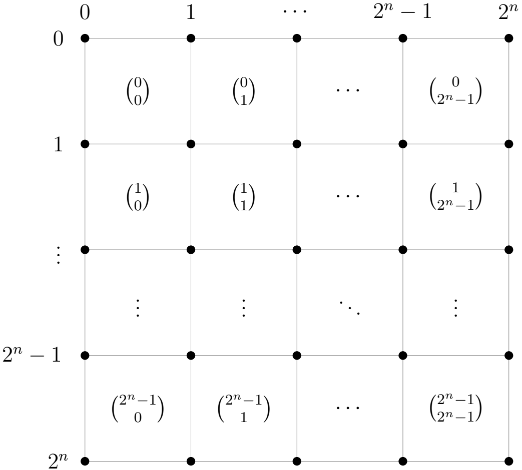
$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!}$$

Pascal's rule:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$$

A specific construction

- Grid: first 2^n rows and columns



$\mathbb{N}^2 \cap [0, 2^n]^2$

- Color the grid:
  Color the first 2^n rows and columns of the Pascal triangle

$$\left(\binom{m}{k} \bmod 2\right)_{0\leq m,k<2^n}$$

  in
  - white if $\binom{m}{k} \equiv 0 \bmod 2$
  - black if $\binom{m}{k} \equiv 1 \bmod 2$

- Color the grid:
  Color the first 2^n rows and columns of the Pascal triangle

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  in
  - white if $\binom{m}{k} \equiv 0 \bmod 2$
  - black if $\binom{m}{k} \equiv 1 \bmod 2$
- Normalize by a homothety of ratio $1/2^n$
  (bring into $[0,1] \times [0,1]$)
  $\rightsquigarrow$ sequence belonging to $[0,1] \times [0,1]$

What happens for $n \in \{0, 1\}$

$n = 0$

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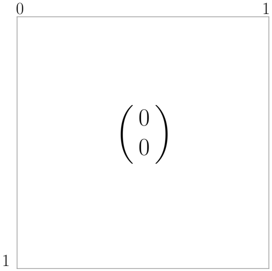

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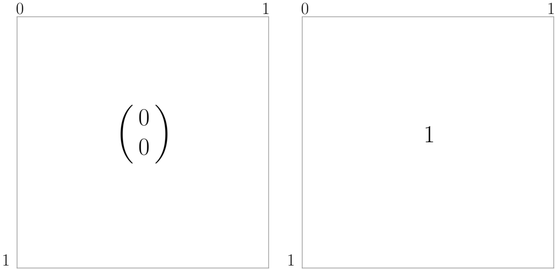

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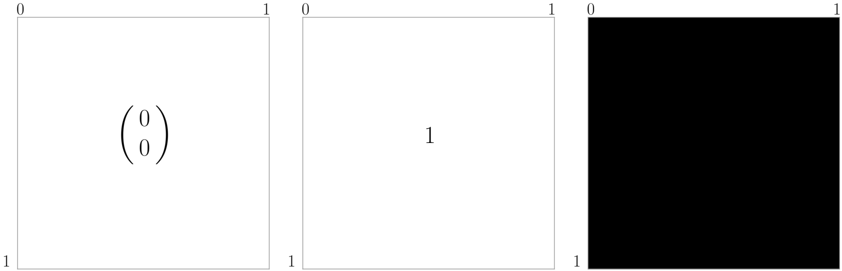

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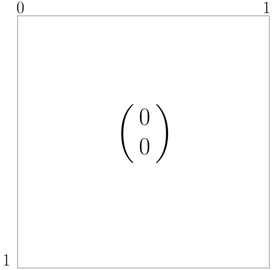
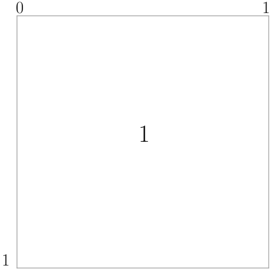
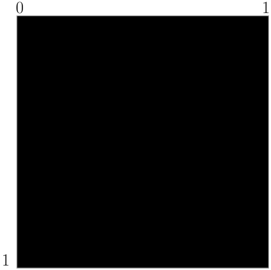






$n = 1$

What happens for $n \in \{0, 1\}$

$n = 0$

0 1

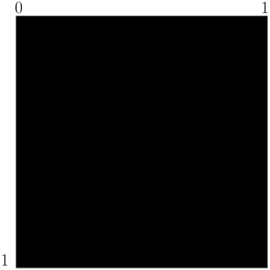
$\binom{0}{0}$

1

0 1

1

1



$n = 1$

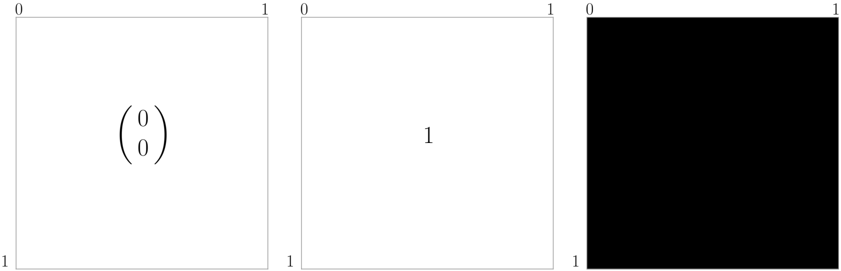
0 2

$\binom{0}{0}$	$\binom{0}{1}$
$\binom{1}{0}$	$\binom{1}{1}$

2

What happens for $n \in \{0, 1\}$

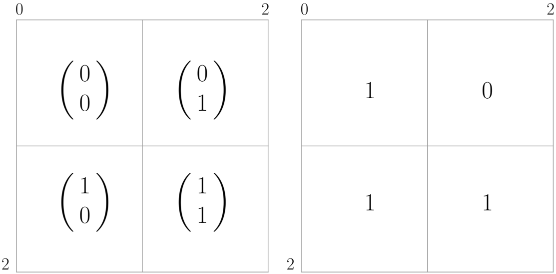
$n = 0$



$\binom{0}{0}$

1

$n = 1$



$\binom{0}{0}$	$\binom{0}{1}$
$\binom{1}{0}$	$\binom{1}{1}$

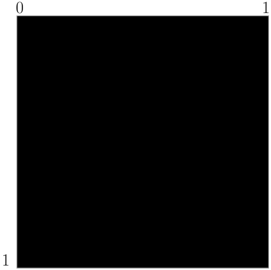
1	0
1	1

What happens for $n \in \{0, 1\}$

$n = 0$

$\binom{0}{0}$

1



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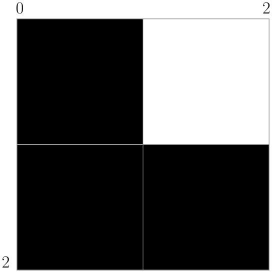
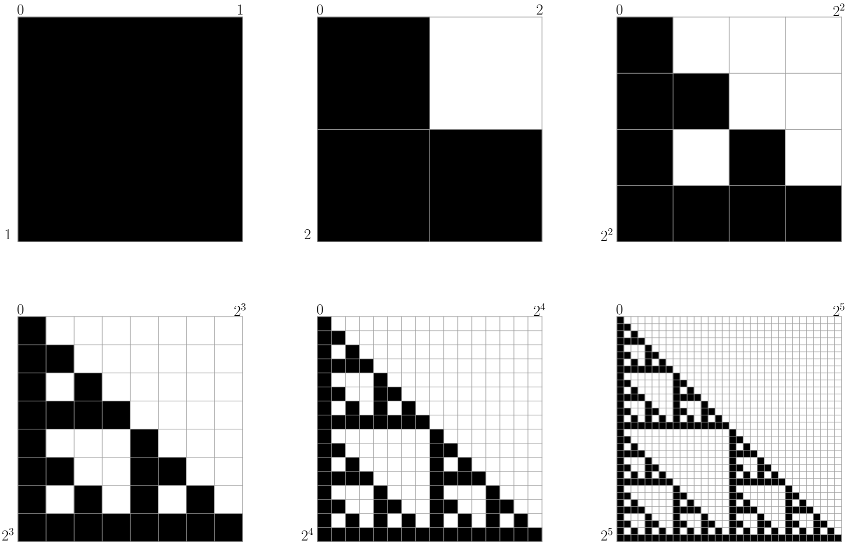
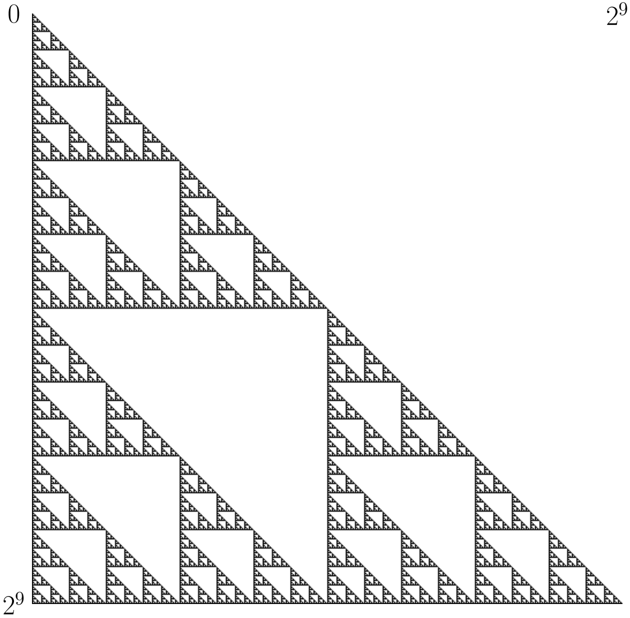


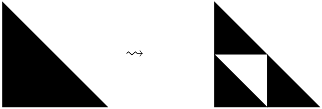
The first six elements of the sequence



The tenth element of the sequence



The Sierpiński gasket



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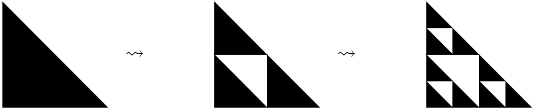

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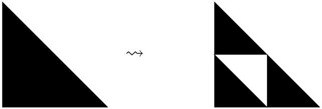

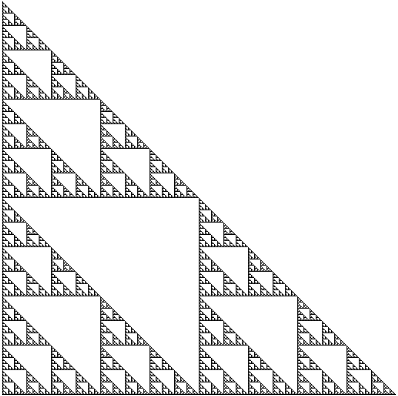

Folklore fact

The latter sequence converges to the Sierpiński gasket when n tends to infinity (for the Hausdorff distance).

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Definitions:

- *ϵ-fattening* of a subset $S \subset \mathbb{R}^2$

$$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the *Hausdorff distance* d_h

$$d_h(S, S') = \min\{\epsilon \in \mathbb{R}_{\geq 0} \mid S \subset [S']_\epsilon \quad \text{and} \quad S' \subset [S]_\epsilon\}$$

New idea

Replace usual binomial coefficients of integers by binomial coefficients of finite words

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

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Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\qquad$ $v = 101$

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Example: $u = \mathbf{101}001$ $\quad v = 101 \quad$ 1 occurrence

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Binomial coefficient of words

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Example: $u = \mathbf{10}100\mathbf{1}$ $v = 101$ 2 occurrences

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Example: $u = \mathbf{1}\mathbf{0}1\mathbf{0}0\mathbf{1}$ $v = 101$ 3 occurrences

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Binomial coefficient of words

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Example: $u = \mathbf{1}\mathbf{0}\mathbf{1}00\mathbf{01}$ $\qquad v = 101 \qquad$ 4 occurrences

Binomial coefficient of finite words

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Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a “scattered” subword).

Example: $u = 101001$ $\qquad v = 101$ $\qquad$ 5 occurrences

Binomial coefficient of finite words

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Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $v = 101$ 6 occurrences

Binomial coefficient of finite words

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called the *alphabet*.

Example: $101, 101001 \in \{0,1\}^*$

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a “scattered” subword).

Example: $u = 101001 \qquad v = 101$

$$\Rightarrow \binom{101001}{101} = 6$$

Remark:

Natural generalization of binomial coefficients of integers

With a one-letter alphabet $\{a\}$

$$\binom{a^m}{a^k} = \binom{\overbrace{a\cdots a}^{m \text{ times}}}{\underbrace{a\cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall\, m,k \in \mathbb{N}$$

The base 2 case

Definitions:

- $\mathrm{rep}_2(n)$ greedy base-2 expansion of $n \in \mathbb{N}_{>0}$ starting with 1
- $\mathrm{rep}_2(0) = \varepsilon$ where ε is the empty word

n		$\mathrm{rep}_2(n)$
0		ε
1	1×2^0	1
2	$1 \times 2^1 + 0 \times 2^0$	10
3	$1 \times 2^1 + 1 \times 2^0$	11
4	$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	100
5	$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	101
6	$1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$	110
$\vdots$	$\vdots$	$\vdots$
		$\{\varepsilon\} \cup 1\{0,1\}^*$

Generalized Pascal triangle in base 2

					$\mathrm{rep}_2(k)$					
	$\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)}$	ε	1	10	11	100	101	110	111	$\cdots$
	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
$\mathrm{rep}_2(m)$	11	1	**2**	0	**1**	0	0	0	0	
	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	1	3	0	**3**	0	0	0	1	
	$\vdots$									$\ddots$

Binomial coefficient of finite words: $\binom{u}{v}$

Rule (not local):

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

Generalized Pascal triangle in base 2

$\mathrm{rep}_2(k)$

		ε	**1**	10	**11**	100	101	110	**111**	$\cdots$
	ε	**1**	0	0	0	0	0	0	0	
	1	**1**	**1**	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
$\mathrm{rep}_2(m)$	**11**	**1**	**2**	0	**1**	0	0	0	0	
	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	**1**	**3**	0	**3**	0	0	0	**1**	
	$\vdots$									$\ddots$

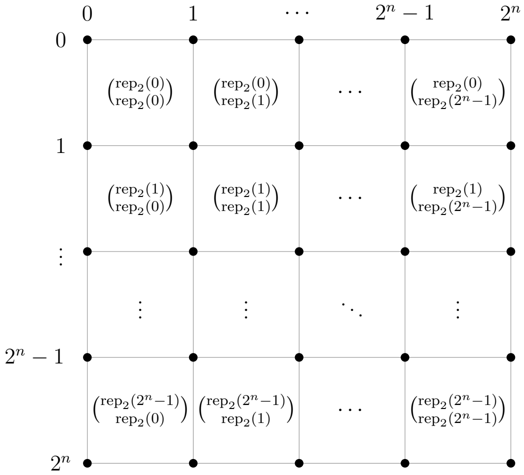
The classical Pascal triangle

Questions:

- After coloring and normalization can we expect the convergence to an analogue of the Sierpiński gasket?
- Could we describe this limit object?

Same construction

- Grid: first 2^n rows and columns



$\binom{\mathrm{rep}_2(0)}{\mathrm{rep}_2(0)}$	$\binom{\mathrm{rep}_2(0)}{\mathrm{rep}_2(1)}$	$\cdots$	$\binom{\mathrm{rep}_2(0)}{\mathrm{rep}_2(2^n-1)}$
$\binom{\mathrm{rep}_2(1)}{\mathrm{rep}_2(0)}$	$\binom{\mathrm{rep}_2(1)}{\mathrm{rep}_2(1)}$	$\cdots$	$\binom{\mathrm{rep}_2(1)}{\mathrm{rep}_2(2^n-1)}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\binom{\mathrm{rep}_2(2^n-1)}{\mathrm{rep}_2(0)}$	$\binom{\mathrm{rep}_2(2^n-1)}{\mathrm{rep}_2(1)}$	$\cdots$	$\binom{\mathrm{rep}_2(2^n-1)}{\mathrm{rep}_2(2^n-1)}$

$\mathbb{N}^2 \cap [0, 2^n]^2$

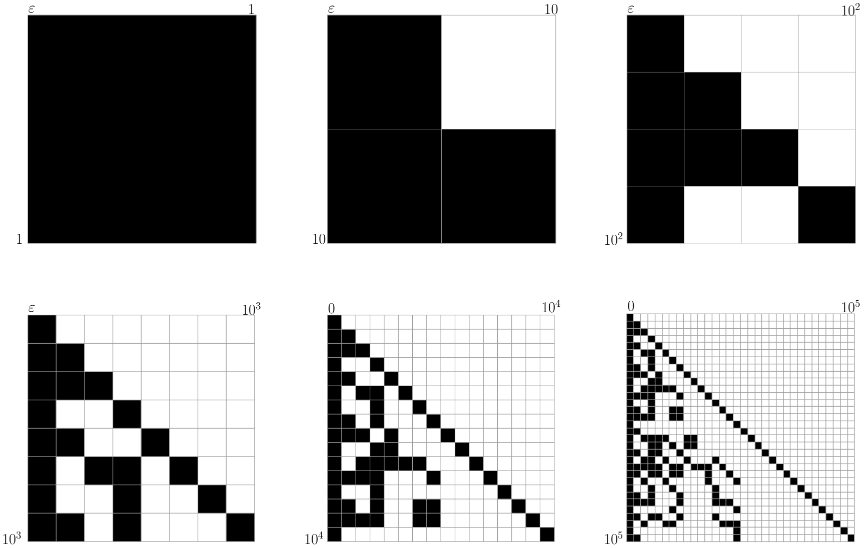
- Color the grid:
  Color the first 2^n rows and columns of the generalized Pascal triangle

  $$\left(\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \bmod 2\right)_{0\leq m,k<2^n}$$

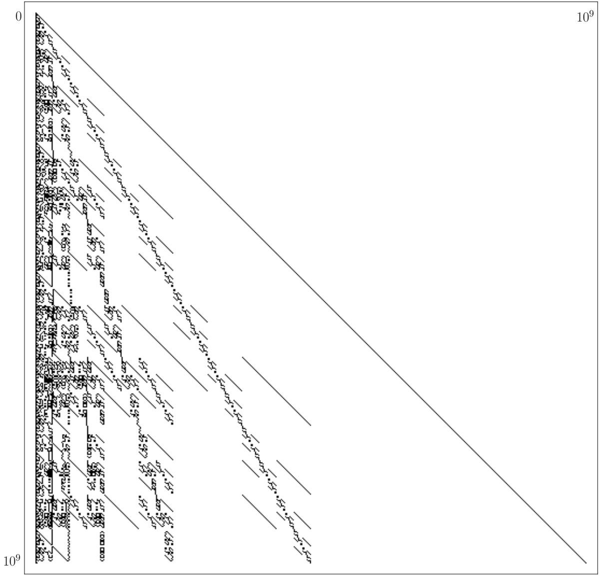
  in

  - white if $\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \equiv 0 \bmod 2$
  - black if $\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \equiv 1 \bmod 2$
- Normalize by a homothety of ratio $1/2^n$
  (bring into $[0,1] \times [0,1]$)
  $\rightsquigarrow$ sequence $(U_n)_{n\geq 0}$ belonging to $[0,1] \times [0,1]$

The elements $U_0, \ldots, U_5$



The element U_9

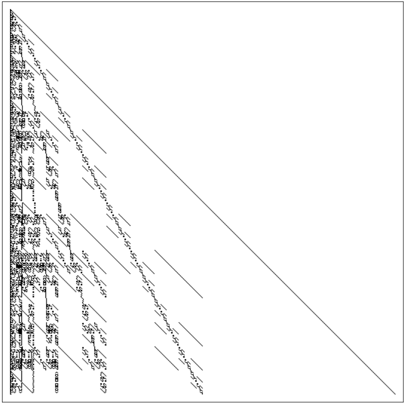


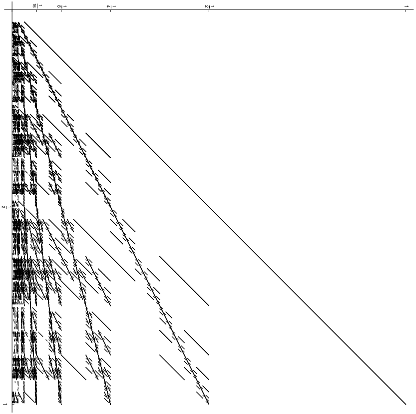
Lines of different slopes...

A key result

Theorem (Leroy, Rigo, S., 2016)

The sequence $(U_n)_{n\geq 0}$ of compact sets converges to a limit compact set $\mathcal{L}$ when n tends to infinity (for the Hausdorff distance).





"Simple" characterization of $\mathcal{L}$: topological closure of a union of segments described through a "simple" combinatorial property

Counting subword occurrences

Generalized Pascal triangle in base 2

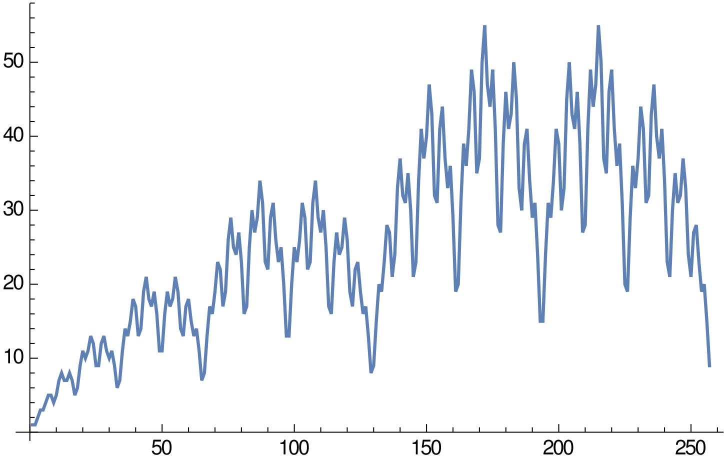
	ε	1	10	11	100	101	110	111	n	$S_2(n)$
ε	1	0	0	0	0	0	0	0	0	**1**
1	1	1	0	0	0	0	0	0	1	**2**
10	1	1	1	0	0	0	0	0	2	**3**
11	1	2	0	1	0	0	0	0	3	**3**
100	1	1	2	0	1	0	0	0	4	**4**
101	1	2	1	1	0	1	0	0	5	**5**
110	1	2	2	1	0	0	1	0	6	**5**
111	1	3	0	3	0	0	0	1	7	**4**

Definition:

$$S_2(n) = \# \left\{ m \in \mathbb{N} \mid \binom{\operatorname{rep}_2(n)}{\operatorname{rep}_2(m)} > 0 \right\} \quad \forall n \geq 0$$

$$= \# \text{ (scattered) subwords in } \{\varepsilon\} \cup 1\{0,1\}^* \text{ of } \operatorname{rep}_2(n)$$

The sequence $(S_2(n))_{n\geq 0}$ in the interval $[0, 256]$



Palindromic structure $\rightsquigarrow$ regularity

Regularity in the sense of Allouche and Shallit (1992)

- 2-*kernel* of $s = (s(n))_{n\geq 0}$

$$\begin{aligned}\mathcal{K}_2(s) =& \{(s(n))_{n\geq 0}, (s(2n))_{n\geq 0}, (s(2n+1))_{n\geq 0}, (s(4n))_{n\geq 0},\\ &(s(4n+1))_{n\geq 0}, (s(4n+2))_{n\geq 0}, \ldots\}\\ =& \{(s(2^i n + j))_{n\geq 0} \mid i \geq 0 \text{ and } 0 \leq j < 2^i\}\end{aligned}$$

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- 2-*kernel* of $s = (s(n))_{n\geq 0}$

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- $s = (s(n))_{n\geq 0}$ is 2-*regular* if there exist

$$(t_1(n))_{n\geq 0}, \ldots, (t_\ell(n))_{n\geq 0}$$

s.t. each $(t(n))_{n\geq 0} \in \mathcal{K}_2(s)$ is a $\mathbb{Z}$-linear combination of the t_j's

The case of $(S_2(n))_{n\geq 0}$

Theorem (Leroy, Rigo, S., 2017)

The sequence $(S_2(n))_{n\geq 0}$ satisfies, for all $n \geq 0$,

$$\begin{aligned} S_2(2n+1) &= 3\,S_2(n) - S_2(2n) \\ S_2(4n) &= -S_2(n) + 2\,S_2(2n) \\ S_2(4n+2) &= 4\,S_2(n) - S_2(2n). \end{aligned}$$

Corollary (Leroy, Rigo, S., 2017)

$(S_2(n))_{n\geq 0}$ is 2-regular.

Example: $\mathbb{Z}$-linear combination of $(S_2(n))_{n\geq 0}$ and $(S_2(2n))_{n\geq 0}$

$$\begin{aligned} S_2(4n+1) = S_2(2(2n)+1) &= 3S_2(2n) - S_2(4n) \\ &= 3S_2(2n) - (-S_2(n) + 2S_2(2n)) = S_2(n) + S_2(2n) \end{aligned}$$

Summatory function

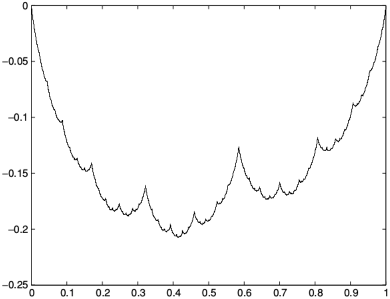
Example: $s(n)$ number of 1's in $\mathrm{rep}_2(n)$

$s(2n) = s(n)$ $s(2n+1) = s(n)+1$
s is 2-regular

Summatory function A:

$$A(0) = 0$$

$$A(n) = \sum_{j=0}^{n-1} s(j) \quad \forall n \geq 1$$



Theorem (Delange, 1975)

$$\frac{A(n)}{n} = \frac{1}{2}\log_2(n) + \mathcal{G}(\log_2(n)) \qquad (1)$$

where $\mathcal{G}$ continuous, nowhere differentiable, periodic of period 1.

Theorem (Allouche, Shallit, 2003)

Under some hypotheses, the summatory function of any k-regular sequence has a behavior analogous to (1).

$\rightsquigarrow$ Replacing s by S_2: same behavior as (1) but does not satisfy the hypotheses of the theorem

Summatory function of $(S_2(n))_{n\geq 0}$

Definition: $A_2(0) = 0$

$$A_2(n) = \sum_{j=0}^{n-1} S_2(j) \quad \forall n \geq 1$$

First few values:

$0, \mathbf{1}, \mathbf{3}, 6, \mathbf{9}, 13, 18, 23, \mathbf{27}, 32, 39, 47, 54, 61, 69, 76, \mathbf{81}, 87, 96, 107, \ldots$

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Lemma (Leroy, Rigo, S., 2017)

For all $n \geq 0$, $A_2(2^n) = 3^n$.

Using several numeration systems

Lemma (Leroy, Rigo, S., 2017)

Let $\ell \geq 1$.

- If $0 \leq r \leq 2^{\ell-1}$, then

$$A_2(2^\ell + r) = 2 \cdot 3^{\ell-1} + A_2(2^{\ell-1} + r) + A_2(r).$$

- If $2^{\ell-1} < r < 2^\ell$, then

$$A_2(2^\ell+r) = 4\cdot3^\ell-2\cdot3^{\ell-1}-A_2(2^{\ell-1}+r')-A_2(r') \quad \text{where } r' = 2^\ell-r.$$

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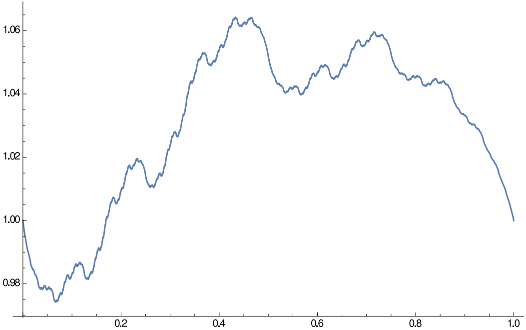
$\rightsquigarrow$ 3-*decomposition*: particular decomposition of $A_2(n)$ based on powers of 3

$\rightsquigarrow$ two numeration systems: base 2 and base 3

Theorem (Leroy, Rigo, S., 2017)

There exists a continuous and periodic function $\mathcal{H}_2$ of period 1 such that, for all $n \geq 1$,

$$A_2(n) = 3^{\log_2(n)}\ \mathcal{H}_2(\log_2(n)).$$



The Fibonacci case

Definitions:

- Fibonacci numbers $(F(n))_{n\geq 0}$: $F(0)=1$, $F(1)=2$ and $F(n+2)=F(n+1)+F(n)\ \forall\, n\geq 0$
- $\text{rep}_F(n)$ greedy Fibonacci representation of $n\in\mathbb{N}_{>0}$ starting with 1
- $\text{rep}_F(0)=\varepsilon$ where ε is the empty word

n		$\text{rep}_F(n)$
0		ε
1	$1\times F(0)$	1
2	$1\times F(1)+0\times F(0)$	10
3	$1\times F(2)+0\times F(1)+0\times F(0)$	100
4	$1\times F(2)+0\times F(1)+1\times F(0)$	101
5	$1\times F(3)+0\times F(2)+0\times F(1)+0\times F(0)$	1000
6	$1\times F(3)+0\times F(2)+0\times F(1)+1\times F(0)$	1001
$\vdots$	$\vdots$	$\vdots$
		$\{\varepsilon\}\cup 1\{0,01\}^*$

Generalized Pascal triangle in base Fibonacci

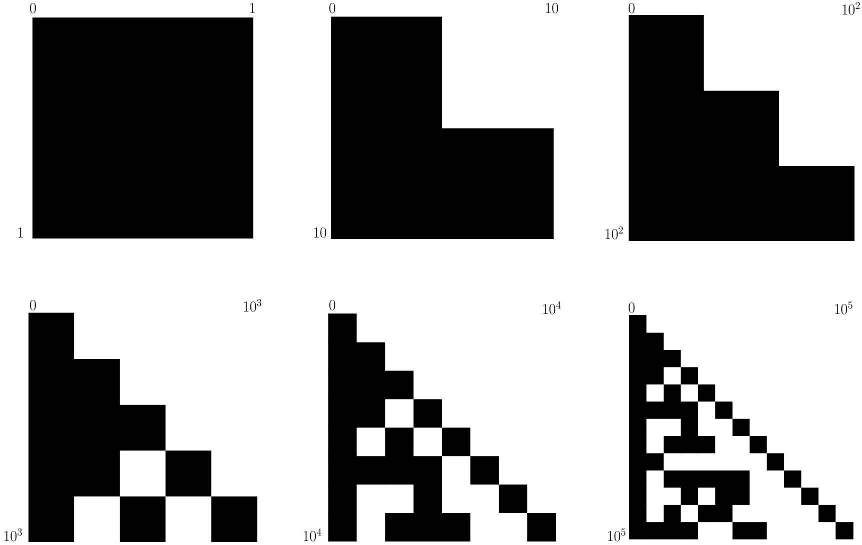
	$\binom{\mathrm{rep}_F(m)}{\mathrm{rep}_F(k)}$				$\mathrm{rep}_F(k)$					
		ε	1	10	100	101	1000	1001	1010	$\dots$
	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
$\mathrm{rep}_F(m)$	100	1	1	2	1	0	0	0	0	
	101	1	2	1	0	1	0	0	0	
	1000	1	1	3	3	0	1	0	0	
	1001	1	2	2	1	2	0	1	0	
	1010	1	2	3	1	1	0	0	1	
	$\vdots$									$\ddots$

Binomial coefficient of finite words: $\binom{u}{v}$

Rule (not local):

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

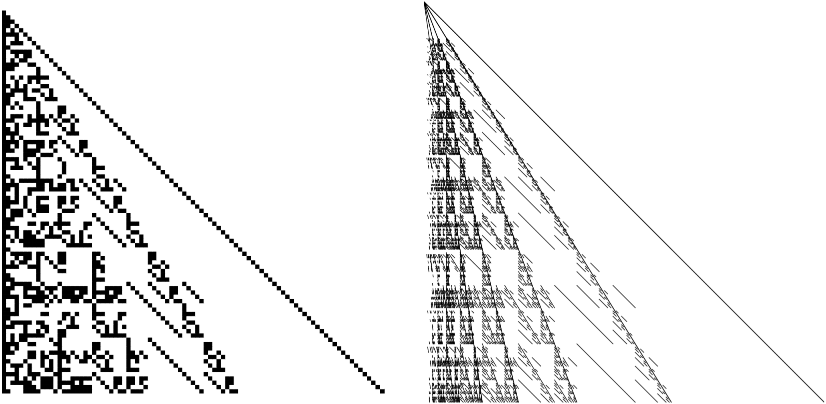
The first six elements of the sequence $(U'_n)_{n\geq 0}$



Similar convergence result

Theorem (S., 2018)

The sequence $(U'_n)_{n\geq 0}$ of compact sets converges to a limit compact set $\mathcal{L}'$ when n tends to infinity (for the Hausdorff distance).



"Simple" characterization of $\mathcal{L}'$: topological closure of a union of segments described through a "simple" combinatorial property

Counting subword occurrences

Generalized Pascal triangle in base Fibonacci

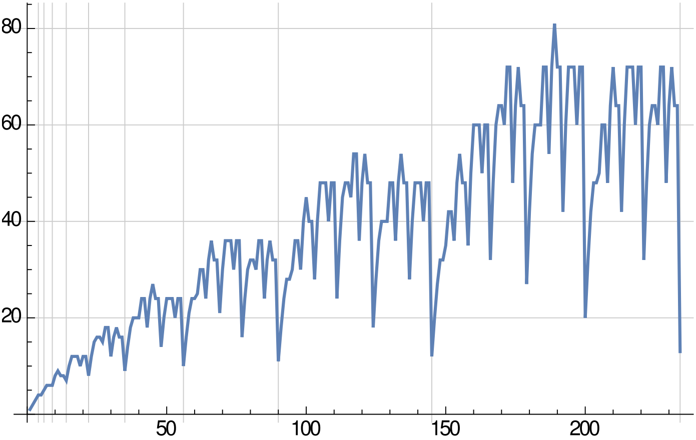
	ε	1	10	100	101	1000	1001	1010	n	$S_F(n)$
ε	1	0	0	0	0	0	0	0	0	**1**
1	1	1	0	0	0	0	0	0	1	**2**
10	1	1	1	0	0	0	0	0	2	**3**
100	1	1	2	1	0	0	0	0	3	**4**
101	1	2	1	0	1	0	0	0	4	**4**
1000	1	1	3	3	0	1	0	0	5	**5**
1001	1	2	2	1	2	0	1	0	6	**6**
1010	1	2	3	1	1	0	0	1	7	**6**

Definition:

$$S_F(n) = \left\{ m \in \mathbb{N} \mid \binom{\mathrm{rep}_F(n)}{\mathrm{rep}_F(m)} > 0 \right\} \quad \forall n \geq 0$$

$$= \#\text{ (scattered) subwords in } \{\varepsilon\} \cup 1\{0,01\}^* \text{ of } \mathrm{rep}_F(n)$$

The sequence $(S_F(n))_{n\geq 0}$ in the interval $[0, 233]$



F-kernel

2-kernel $\mathcal{K}_2(s)$ of a sequence s

- Select all the nonnegative integers whose base-2 expansion (with leading zeroes) ends with $w \in \{0,1\}^*$
- Evaluate s at those integers
- Let w vary in $\{0,1\}^*$

$\mathbf{w = 0}$

n	$\mathrm{rep}_2(n)$	$s(n)$
0	ε	**s(0)**
1	1	$s(1)$
2	**10**	**s(2)**
3	11	$s(3)$
4	**100**	**s(4)**
5	101	$s(5)$

F-kernel $\mathcal{K}_F(s)$ of a sequence s

- Select all the nonnegative integers whose Fibonacci representation (with leading zeroes) ends with $w \in \{0,1\}^*$
- Evaluate s at those integers
- Let w vary in $\{0,1\}^*$

n	$\mathrm{rep}_F(n)$	$s(n)$
0	ε	**s(0)**
1	1	$s(1)$
2	**10**	**s(2)**
3	**100**	**s(3)**
4	101	$s(4)$
5	**1000**	**s(5)**

F-regularity

$s = (s(n))_{n\geq 0}$ is F-*regular* if there exist

$$(t_1(n))_{n\geq 0}, \ldots, (t_\ell(n))_{n\geq 0}$$

s.t. each $(t(n))_{n\geq 0} \in \mathcal{K}_F(s)$ is a $\mathbb{Z}$-linear combination of the t_j's

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Proposition (Leroy, Rigo, S., 2017)

$(S_F(n))_{n\geq 0}$ is F-regular.

In the literature, not so many sequences have this kind of property

Summatory function of $(S_F(n))_{n\geq 0}$

Definition: $A_F(0) = 0$

$$A_F(n) = \sum_{j=0}^{n-1} S_F(j) \quad \forall n \geq 1$$

First few values:

$0, 1, 3, 6, 10, 14, 19, 25, 31, 37, 45, 54, 62, 70, 77, 87, 99, 111, 123, 133, \ldots$

Using several numeration systems

First few values (again):

$0, \mathbf{1}, \mathbf{3}, \mathbf{6}, 10, 14, 19, 25, \mathbf{31}, 37, 45, 54, 62, \mathbf{70}, 77, 87, 99, 111, 123, 133, \ldots$

Proposition (Leroy, Rigo, S., 2017)

For all $n \geq 0$, $A_F(F(n)) = B(n)$.

Definition: $B(0) = 1, B(1) = 3, B(2) = 6$

$$B(n+3) = 2B(n+2) + B(n+1) - B(n) \quad \forall\, n \geq 0$$

First few values: $1, 3, 6, 14, 31, 70, 157, 353, 793, 1782, \ldots$

Using several numeration systems

First few values (again):

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$\rightsquigarrow$ *B-decomposition*: particular decomposition of $A_F(n)$ based on linear combination of $B(n)$

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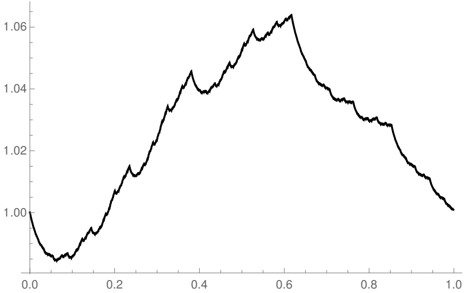
⇝ *B-decomposition*: particular decomposition of $A_F(n)$ based on linear combination of $B(n)$

⇝ two numeration systems: base $F(n)$ and base $B(n)$

Theorem (Leroy, Rigo, S., 2017)

Let β be the dominant root of $X^3 - 2X^2 - X + 1$, which is the characteristic polynomial of $(B(n))_{n\geq 0}$.
Let c be a constant such that $\lim_{n\to\infty} B(n)/c\beta^n = 1$.
There exists a continuous and periodic function $\mathcal{H}_F$ of period 1 such that, for all $n \geq 3$,

$$A_F(n) = c \cdot \beta^{\log_F(n)}\ \mathcal{H}_F(\log_F(n)) + o(\beta^{\lfloor \log_F(n) \rfloor}).$$



Conclusion

In this talk:

Base 2:

- Generalization of the Pascal triangle in base 2 modulo 2
- 2-regularity of the sequence $(S_2(n))_{n\geq 0}$ counting subword occurrences
- Exact behavior of the summatory function $(A_2(n))_{n\geq 0}$ of the sequence $(S_2(n))_{n\geq 0}$

Base Fibonnaci:

- Generalization of the Pascal triangle in base Fibonacci modulo 2
- F-regularity of the sequence $(S_F(n))_{n\geq 0}$ counting subword occurrences
- Asymptotics of the summatory function $(A_F(n))_{n\geq 0}$ of the sequence $(S_F(n))_{n\geq 0}$

Other work

Done:

- Generalization of the Pascal triangle modulo a prime number: extension to any Parry–Bertrand numeration system
- Regularity of the sequence counting subword occurrences: extension to any integer base
- Behavior of the summatory function: extension to any integer base (exact behavior)

To do:
Parry–Bertrand numeration systems,
Apply the methods for sequences not related to Pascal triangles,
etc.

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