Chapter 1
From combinatorial games to shape-symmetric morphisms

Michel Rigo

Abstract Siegel suggests in his book on combinatorial games that quite simple games provide us with challenging problems: "No general formula is known for computing arbitrary Grundy values of Wythoff's game. In general, they appear chaotic, though they exhibit a striking fractal-like pattern.". This observation is the first motivation behind this chapter. We present some of the existing connections between combinatorial game theory and combinatorics on words. In particular, multidimensional infinite words can be seen as tilings of $\mathbb{N}^d$. They naturally arise from subtraction games on d heaps of tokens. We review notions such as k-automatic, k-regular or shape-symmetric multidimensional words. The underlying general idea is to associate a finite automaton with a morphism.

1.1 Introduction

Combinatorial game theory (CGT) uses many tools from other fields: number theory, continued fractions, numeration systems, cellular automata, *etc*. We will limit ourselves to subtraction games played on heaps of tokens. When analyzing Sprague–Grundy values of some well-known and popular subtraction games — like Nim or Wythoff's game — notions such as k-automatic, k-regular and morphic (also called substitutive) multidimensional sequences enter naturally the picture. The aim of this chapter is to introduce these concepts and present the interplay existing between CGT and combinatorics on words (COW). The focus is put on infinite (multidimensional) words generated by iterated morphisms. The organization of this chapter faithfully reflects the course presented during the school[1] "*Tiling*

Michel Rigo

University of Liège, Dept. of Mathematics, Allée de la découverte 12 (B37), B-4000 Liège, Belgium, e-mail: `M.Rigo@ulg.ac.be`

[1] A video is available at `http://library.cirm-math.fr/`

Dynamical System" organized during the Jean Morlet Chair attributed to Professor Shigeki Akiyama.

After setting notation in Section 1.2, minimal requirements and basic results about CGT are given in Section 1.3. Well-known facts about *k*-automatic and *k*-regular (unidimensional) sequences are presented in Section 1.4. For some general references, see [61, 8, 72, 73]. Let $\mathbf{P}_W : \mathbb{N}^2 \to \{0,1\}$ be the bidimensional infinite word defined by $\mathbf{P}_W(m,n) = 1$ if and only if (m,n) is a $\mathscr{P}$-position, i.e., a loosing position or a zero of the Sprague–Grundy function, of Wythoff's game. Section 1.5 deals with the syntactic characterization of these $\mathscr{P}$-positions in terms of representations in the Fibonacci numeration system. This result by Fraenkel is proved using the infinite Fibonacci word and a transducer computing the successor function in the Fibonacci numeration system. In particular, we make explicit the general fact that morphic words can be generated by finite automata. We look at the sequence of states reached by all the accepted words when they have been genealogically ordered. The transformation from morphisms to automata is recurrent in this chapter: uniform and non-uniform morphisms for words, uniform and non-uniform morphisms in a multidimensional setting.

The multidimensional point of view begins with Section 1.6. We first look at *k*-automatic and *k*-regular multidimensional words. It turns out that characterizing the $\mathscr{P}$-positions of Wythoff's game leads to a *shape-symmetric* morphism as introduced and studied by A. Maes in a logical setting. The word $\mathbf{P}_W$ depicted in Table 1.5 is the fixed point of a morphism φ_W and it has the shape-symmetric property with respect to φ_W. These notions are well fitted to a volume dedicated to tilings and are presented in Section 1.7. We introduce a generalization of numeration systems whose language of representations is regular: *abstract numeration systems*. Indeed, a multidimensional infinite word is $\mathscr{S}$-automatic, for some abstract numeration system $\mathscr{S}$, if and only if it is the image by a coding of a shape-symmetric word. Hence we prove that $\mathbf{P}_W$ codes the $\mathscr{P}$-positions of the Wyhtoff's game. We conclude this chapter with a short discussion about games with a finite set of rules and some bibliographic notes.

1.2 Notation and conventions

We assume that 0 belongs to the set $\mathbb{N}$ of non-negative integers. An *alphabet* is just a finite set. A (finite) *word* over an alphabet A is a finite sequence of elements in A, i.e., a map from a finite set $\{0, \dots, \ell - 1\}$ to A. The length of the word w is denoted by $|w|$. The set of finite words over A is denoted by A^*. Equipped with the concatenation product, it is a free monoid. A (right) *infinite word* is a map from $\mathbb{N}$ to A. It is indeed more convenient (when working with automatic sequences) to start indexing with 0. We will indifferently use the terminology *infinite word* or *sequence*. Compared with other chapters of this book, we will never encounter bi-infinite words. Let $d \geq 2$ be an integer. By extension, we say that a map from $\mathbb{N}^d$ to

an alphabet A is a (multidimensional) *infinite word* or *sequence*. Infinite words will be denoted using a bold face letter such as $\mathbf{x}$ or $\mathbf{P}_W$.

Let $k \geq 2$ be an integer. We let $\text{rep}_k(n)$ denote the usual *base-k expansion* of $n \geq 0$. It is a word over the alphabet $\{0, \ldots, k-1\}$. We set $\text{rep}_k(0)$ to be the empty word denoted by ε. Except when stated otherwise (when dealing with a transducer computing the successor function in the proof of Theorem 11 or, when considering an automaton realizing addition in base 2 in the proof of Lemma 4), we use the most significant digit first convention (MSDF): if $n > 0$, the most significant digit of $\text{rep}_k(n)$ is written on the left — as a prefix — and is non-zero. If $w = w_\ell \cdots w_0$ is a word over an alphabet $A \subset \mathbb{Z}$, then the *k-numerical value* of w (w.r.t. A) is given by the map

$$\text{val}_k : A^* \to \mathbb{Z}, \ w \mapsto \sum_{i=0}^{\ell} w_i \, k^i.$$

In this chapter, we will also encounter other numeration systems, i.e., bijections from $\mathbb{N}$ to some (infinite) formal language L. We will adapt the corresponding notation to $\text{rep}_L : \mathbb{N} \to L$ and $\text{val}_L : L \to \mathbb{N}$.

We assume that the reader is familiar with basic concepts from graph theory. If $G = (V, E)$ is a directed graph and $u, v \in V$ are two of its vertices, we write $(u, v) \in E$ or $u \to v$ to denote an edge of G. Nevertheless, we recall the definition of a finite automaton. We will encounter automata in Section 1.4. For more, see, for instance, [79, 83, 86].

Definition 1. A *deterministic finite automaton* (DFA for short) is a 5-tuple $\mathscr{M} = (Q, q_0, A, \delta, F)$ where Q is a finite set of *states*, $q_0 \in Q$ is the *initial state*, A is the alphabet of the automaton, $\delta : Q \times A \to Q$ is the *transition function* and $F \subseteq Q$ is the set of *final states*.

For a DFA $\mathscr{M}$, the transition function is extended to $\delta : Q \times A^* \to Q$ by $\delta(q, \varepsilon) = q$ and $\delta(q, aw) = \delta(\delta(q, a), w)$ for all $q \in Q$, $a \in A$, $w \in A^*$. If the automaton is clear from the context, we simply write $q \cdot w$ instead of $\delta(q, w)$. The state reached when reading w from the initial state will also be written $\mathscr{M} \cdot w$. The language *accepted* or *recognized* by $\mathscr{M}$ is

$$L(\mathscr{M}) = \{w \in A^* \mid \mathscr{M} \cdot w \in F\}.$$

A language is *regular* if it is accepted by some DFA.

Definition 2. A *deterministic finite automaton with output* (DFAO for short) is given by a 5-tuple $\mathscr{M} = (Q, q_0, A, \delta, \mu)$ where the first four components are defined as for a DFA and $\mu : Q \to B$ is the output map (where B is some alphabet). A DFAO where the output map takes at most two values is a standard DFA.

Definition 3. A *non-deterministic finite automaton* (NFA for short) over an alphabet A is given by a 5-tuple $\mathscr{N} = (Q, I, A, \Delta, F)$ where Q is a finite set of states, $I \subseteq Q$ is the set of initial states, $\Delta \subset Q \times A^* \times Q$ is the (finite) transition relation and $F \subseteq Q$ is the set of final states. A word w is *accepted* by $\mathscr{N}$ if there exist an integer i,

some (possibly empty) words $v_1, \ldots, v_i$ and a sequence of states $q_1, \ldots, q_{i+1}$ such that $w = v_1 \cdots v_i$ and

- $(q_1, v_1, q_2), (q_2, v_2, q_3), \ldots, (q_i, v_i, q_{i+1}) \in \Delta$
- $q_1 \in I, q_{i+1} \in F$.

Otherwise stated, there is at least one accepting path with label w from an initial state to some final state. The *language accepted* by $\mathcal{N}$ is the set of accepted words. One can assume that (q, ε, q) belongs to Δ for all $q \in Q$.

1.3 Bits of combinatorial game theory

They are many kinds of games that you can think of. We deliberately restrict ourselves to one of the most simple classes: *subtraction games* (taking and breaking games) [48]. In this setting, we have two players acting in turns. They have a finite number of tokens organized in piles to start with. The tokens are given in some initial position. Typically, the players have to remove some tokens complying to prescribed rules. A player may not pass: at least one token must be removed. The set of rules is known in advance and we have the same rules for both players. There is no chance involved (no randomness) and the information is completely known for both players — no hidden information for one of the two players [1, 66]. Finally we will assume a *normal play* convention: the first player unable to move loses the game. Nim game and Wythoff's game will provide us with quite enough material.

I do not assume that the reader has any particular knowledge about CGT. Some general references are [13, 85, 2]. I was also inspired by Ferguson's lecture notes from UCLA. This section serves as a self-contained introduction to the topic. It is not at all aimed to be exhaustive.

Example 1 (Finite subtraction game). Starting from a single finite pile of tokens, the players may remove either 1, 2 or 4 tokens. The player taking the last token wins the game. An example of game is given by the following sequence

$$9 \xrightarrow{-2} 7 \xrightarrow{-1} 6 \xrightarrow{-1} 5 \xrightarrow{-2} 3 \xrightarrow{-2} 1 \xrightarrow{-1} 0.$$

For instance, we can immediately see that when a player is in a position with 3 tokens left, he/she will lose the game because the only available options are to remove either 1 or 2 tokens. We will encounter finite subtraction game with several piles of tokens in Section 1.8.

Example 2 (Game of Nim). Let $p \geq 1$ be the number of piles of tokens to play with. If we assume here that the piles are ordered, a *position* is coded by a p-tuple of non-negative integers. An initial position is given: a game starts with p piles containing respectively $x_1, \ldots, x_p \geq 0$ tokens. In their turn to play, the player chooses a non-empty pile and removes a positive number of tokens from it. Thus, from a position of the form $(x_1, \ldots, x_p)$ the available *options* are given by p-tuples of the

form $(y_1, \ldots, y_p) \in \mathbb{N}^p$ where there exists j such that $y_j < x_j$ and $y_i = x_i$ for all $i \neq j$. In particular, the number of options available from $(x_1, \ldots, x_p)$ is just $\sum_{i=1}^{p} x_i$. The game is lost for the player left in the position $(0, \ldots, 0)$. Loosing positions will be determined in Theorem 1.

In this section, to avoid any confusion, generic positions are denoted by Greek letters: α, β, γ. Formally, if α and β are two positions of a subtraction game, we say that β is an *option* of α, if there is a move permitting the players to go from α to β. The set of options of α is denoted by $\mathrm{Opt}(\alpha)$. We can extend Opt to a set X of positions in a natural way: $\mathrm{Opt}(X) = \cup_{\alpha \in X} \mathrm{Opt}(\alpha)$. The reflexive and transitive closure of Opt is denoted by Opt^*: β belongs to $\mathrm{Opt}^*(\alpha)$ if and only if there exists a finite (possibly empty) sequence of moves from α to β. In graph-theoretic terms, one can speak of *successor* instead of option.

Example 3 (Wythoff's game). Consider the following modification of the game of Nim on two piles. We add extra rules to the set of Nim rules. A player can, as for Nim, either remove a positive number of tokens from one pile, or *remove simultaneously the same number of tokens from both piles*. So from a position $(x_1, x_2) \neq (0,0)$, a player can move to either $(x_1 - i, x_2)$ with $0 < i \leq x_1$, or $(x_1, x_2 - i)$ with $0 < i \leq x_2$, or $(x_1 - i, x_2 - i)$ with $0 < i \leq \min\{x_1, x_2\}$. An example of play, starting with the position $(4, 3)$, is given below:

$$(4,3) \xrightarrow{-(2,0)} (2,3) \xrightarrow{-(1,1)} (1,2) \xrightarrow{-(0,1)} (1,1) \xrightarrow{-(1,1)} (0,0).$$

The chosen move is indicated on the arrows. The reader may already notice this: the player in position $(1, 2)$ whatever the chosen move is, will give a position for the other player from which the game can always be won. Loosing positions will be determined by Theorem 2.

Note that it is a quite natural setting to encounter subtraction games whose rule-set extends the game of Nim.

Definition 4 (Game-graph). A subtraction game is given (notation will not refer to the chosen game). The *game-graph* Gr_γ, for an initial position γ, is the finite directed graph whose set $V(\mathrm{Gr}_\gamma)$ of vertices is the set of positions that can be reached from γ by a finite sequence of allowed moves, i.e., $V(\mathrm{Gr}_\gamma) = \mathrm{Opt}^*(\gamma)$. There is an edge from α to β if and only if β belongs to $\mathrm{Opt}(\alpha)$. The set of edges is denoted by $E(\mathrm{Gr}_\gamma)$.

Remark 1. In a subtraction game, consider two positions α and β. If β can be reached from α, i.e., $\beta \in \mathrm{Opt}^*(\alpha)$, then the game-graph Gr_β associated with β is a subgraph of the game-graph Gr_α associated with α. More precisely, we have

$$E(\mathrm{Gr}_\beta) = E(\mathrm{Gr}_\alpha) \cap \big(V(\mathrm{Gr}_\beta) \times V(\mathrm{Gr}_\beta)\big)$$

and there is no edge in Gr_α from $V(\mathrm{Gr}_\beta)$ to $V(\mathrm{Gr}_\alpha) \setminus V(\mathrm{Gr}_\beta)$.

For subtraction games, the corresponding game-graph is obviously acyclic: the total number of tokens left is decreasing. Thus, there exists at least one vertex with out-degree equal to zero[2].

Definition 5. Let $G = (V, E)$ be a directed graph. To stick to the game setting, we denote by $\text{Opt}(u)$ the set of vertices v such that $(u, v) \in E$. In particular, the out-degree of u is $\#\text{Opt}(u)$. A *kernel* of G is a subset K of V with the following two properties:

- K is *stable*: $\forall u \in K, \text{Opt}(u) \cap K = \emptyset$;
- K is *absorbing*: $\forall v \in V \setminus K, \text{Opt}(v) \cap K \neq \emptyset$.

Let $G = (V, E)$ be a finite directed acyclic graph (it is common in the literature to find the acronym DAG). For instance, DAG are the graphs with a topological sort of the vertices [74]. Let us describe an algorithm to compute a kernel of G [12]. Let $G_0 = G$. We will build a finite sequence of subgraphs $(G_i)_{i \geq 0}$.

- Let $\mathscr{P}_0$ be the set of vertices of G_0 with out-degree equal to zero (sometimes called *sink vertices*),

$$\mathscr{P}_0 = \{v \in V(G_0) \mid \text{Opt}(v) = \emptyset\}.$$

- Let $\mathscr{N}_0$ be the set of vertices x of G_0 such that $\text{Opt}(x) \cap \mathscr{P}_0 \neq \emptyset$.

Due to the absorbing property, the vertices of $\mathscr{P}_0$ must belong to any kernel of G. Therefore, the vertices in $\mathscr{N}_0$ do not belong to any kernel of the graph due to stability. Let G_1 be the subgraph obtained by removing the vertices in $\mathscr{P}_0 \cup \mathscr{N}_0$. This graph is again acyclic. We repeat the procedure and define $\mathscr{P}_1$ and $\mathscr{N}_1$ accordingly. If a vertex v belong to $\mathscr{P}_1$, it has no option in $V(G_1)$. In the original graph G_0, it has no option in $\mathscr{P}_0$ because otherwise we would have $v \in \mathscr{N}_0$. Thus again the vertices of $\mathscr{P}_1$ must belong to any kernel of G and the vertices in $\mathscr{N}_1$ are in no kernel of G. We define a sequence of subgraphs where G_{i+1} is obtained by removing $\mathscr{P}_i \cup \mathscr{N}_i$ from G_i. The procedure halts when we reach an empty subgraph. We directly get the next proposition.

Proposition 1. *Let $G = (V, E)$ be a finite directed acyclic graph. Applying the above algorithm, the set $\cup_i \mathscr{P}_i$ is the unique kernel of G.*

Remark 2. If the graph is not acyclic, the situation is not so nice. Consider a simple (oriented) cycle of length n. It is easy to see that if n is odd (resp. even), then the graph has no kernel (resp. two kernels). Moreover, the reader may notice that the previous algorithm does not work anymore.

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[2] Consider a simple path $v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_r$ of maximal length r in a finite acyclic graph. Then v_r has out-degree zero. Proceed by contradiction and assume that there is an edge starting from v_r. Either it goes to one of the v_i's with $i < r$ and it creates a cycle. Or, it goes to some other vertex and we may build a longer simple path. Both situations lead to a contradiction.

Consider a subtraction game. For a game-graph Gr_β, since it is acyclic, we have a partition of the set of vertices of Gr_β into two subsets: those belonging to the kernel and those out of it. The first ones will be called $\mathscr{P}$-positions, the other ones $\mathscr{N}$-positions. Moreover, thanks to Remark 1, we see that the status of a position is *invariant with respect to the initial position*: if α, β are two positions such that $\beta \in \mathrm{Opt}^*(\alpha)$, then the kernel of Gr_β is a subset of the kernel of Gr_α. The following definition is thus meaningful.

Definition 6 ($\mathscr{N}$- and $\mathscr{P}$-positions). Consider a subtraction game. A position is a *losing position*, or $\mathscr{P}$-position, if all its options are $\mathscr{N}$-positions. A position is a *winning position*, or $\mathscr{N}$-position, if there exists an option which is a $\mathscr{P}$-position. The set of $\mathscr{P}$-positions (resp. $\mathscr{N}$-positions) is denoted $\mathscr{P}$ (resp. $\mathscr{N}$).

The classical meaning of $\mathscr{P}$ or $\mathscr{N}$ in CGT refers to the "previous" player or the "next" move. In a $\mathscr{P}$-position, the previous player (so not the one being in that actual position) is able to win the game. In a $\mathscr{N}$-position, the actual player may choose the correct next move to win the game. This choice is the notion of a *winning strategy*: selecting for every $\mathscr{P}$-position one available $\mathscr{N}$-position among the options.

Remark 3. In a subtraction game on p piles of tokens, if there is a move from $(x_1, \ldots, x_p)$ to $(y_1, \ldots, y_p)$, then $\sum_i x_i > \sum_i y_i$. Let $C \geq 0$ be an integer. If we know the status $\mathscr{N}$ or $\mathscr{P}$ of all positions $(x_1, \ldots, x_p)$ such that $\sum_i x_i \leq C$, then we can determine the status of any position $(x_1, \ldots, x_p)$ such that $\sum_i x_i = C + 1$.

Example 4. Consider the finite subtraction game of Example 1. Prove that n is a $\mathscr{P}$-position if and only if $n \equiv 0 \pmod 3$.

Even though the status $\mathscr{N}$ or $\mathscr{P}$ of positions can be determined by some naive methods, let us stress the fact that in terms of algorithmic complexity, computing the kernel of the game-graph or, equivalently using the above remark, has an exponential cost compared to the size of the input (size of the game position) [44]. Indeed, when playing a game like Wythoff's game, a position (x_1, x_2) is coded by a word (let us think about the base-2 expansions of x_1 and x_2 that are the inputs given to the algorithm) whose length is proportional to $\log x_1 + \log x_2 = \log(x_1 . x_2)$. But the game-graph has a number of vertices equal to $(x_1 + 1).(x_2 + 1)$ which is exponentially larger than the size of the input. One can therefore ask for an algorithm determining the status of a position whose complexity is a polynomial in the length of the coding. We will see in Section 1.5 that deciding the status of a position of the Wythoff's game can be done in polynomial time.

Definition 7. Let m, n be two non-negative integers. Let $x = x_\ell \cdots x_0$ and $y = y_\ell \cdots y_0$ be two words over $\{0, 1\}$ such that $\mathrm{val}_2(x) = m$ and $\mathrm{val}_2(y) = n$. In other words, x and y are the base-2 expansions of m and n up to some leading zeroes ensuring that we have two words of the same length. The *Nim-sum* of m and n is the integer

$$m \oplus n = \sum_{j=0}^{\ell} (x_j + y_j \mod 2) \, 2^j.$$

As an example, $7 \oplus 2 \oplus 9 = 12$. It is an easy exercise to show that $(\mathbb{N}, \oplus)$ is an abelian group.

Theorem 1 (Bouton's theorem [15]). *For the game of Nim played on $p \geq 1$ piles of tokens, a position $(x_1, \ldots, x_p)$ is in $\mathscr{P}$ if and only if*

$$\bigoplus_{i=1}^{p} x_i = 0.$$

Proof. From Definition 6, we have to prove that the set K of p-tuples with zero Nim-sum is stable and absorbing. If we consider a p-tuple whose Nim-sum is zero, playing one move of Nim will change exactly one of the piles of tokens. Thus we modify exactly one term, let us say x_j, of the Nim-sum. Therefore, at least one bit of $\mathrm{rep}_2(x_j)$ is modified and thus the Nim-sum is no more equal to zero. This shows that the set K is stable.

Now we have to prove that K is absorbing. Consider a p-tuple whose Nim-sum s is non-zero. There exists $\ell \geq 0$ such that $2^{\ell} \leq s < 2^{\ell+1}$. Let x_j be a component such that the base-2 expansion of x_j has a non-zero digit in the position corresponding to 2^{ℓ} (such an element exists), i.e., $\mathrm{rep}_2(x_j) = v1w$ where $|w| = \ell$. We can thus replace x_j with a smaller integer in such a way that the total Nim-sum is zero. $\square$

Two years after Bouton's result, Wythoff proposed a modification of the game of Nim and characterized the corresponding set of $\mathscr{P}$-position [87].

Theorem 2 (Wythoff's theorem). *A pair (x, y) is a $\mathscr{P}$-position of Wythoff's game if and only if it is of the form*

$$\left(\lfloor n\varphi \rfloor, \lfloor n\varphi^2 \rfloor \right) \ or \ \left(\lfloor n\varphi^2 \rfloor, \lfloor n\varphi \rfloor \right), \quad for \ some \ n$$

where φ is the Golden ratio $(1 + \sqrt{5})/2$.

For a proof, see the original paper of Wythoff. It is enough to prove that for the game-graph associated with Wythoff's game, the above set is stable and absorbing. This can be considered as a "difficult" exercise (one has to deal with integer and fractional parts of multiple of the Golden ratio).

Definition 8 (MeX). If S is a strict subset of $\mathbb{N}$, then $\mathrm{MeX}(S) = \min(\mathbb{N} \setminus S)$. So it is the least non-negative integer not in S. It stands for "minimal excluded" value. In particular, $\mathrm{MeX}(\emptyset) = 0$.

For a subtraction game, we introduce a map associating a non-negative integer with each position. This map is defined recursively. Its use will be clear when we introduce the sum of games.

Definition 9. The *Sprague–Grundy function* $\mathscr{G}$ (of a game with an acyclic game-graph) is defined by

$$\mathscr{G}(x) = \mathrm{MeX}\left(\{ \mathscr{G}(y) \mid y \in \mathrm{Opt}(x) \} \right).$$

Example 5 (Continuing Example 1). If the players are allowed to remove $1, 2, 4$ tokens from a single pile, the first values of the Sprague–Grundy function are given in Table 1.1. It is an easy exercise (left to the reader) to prove that this function is periodic of period 3: $\mathscr{G}(n) = n \mod 3$. We will reconsider games with a finite set of moves in Section 1.8. See Proposition 10.

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\mathscr{G}(n)$	0	1	2	0	1	2	0	1	2	0	1	$\cdots$

Table 1.1 First few values of $\mathscr{G}$ for a finite subtraction game.

Proposition 2. *Consider a subtraction game. A position α is a $\mathscr{P}$-position if and only if $\mathscr{G}(\alpha) = 0$.*

Proof. One has to prove that $\{\alpha \mid \mathscr{G}(\alpha) = 0\}$ is the kernel of the game-graph. This is a direct consequence of the use of the MeX operator in the definition of $\mathscr{G}$. $\quad\square$

The reader may wonder why Sprague–Grundy functions, or simply Grundy functions, could be useful. It turns out that we can consider the (disjunctive) *sum* of games. Assume that we have several games $G_1, \ldots, G_n$ in front of the two players, e.g. a Wythoff's game G_1 in position $(3, 4)$, a game of Nim G_2 in position $(1, 2, 3)$ and another Wythoff's game G_3 in position $(2, 2)$. The idea is that, at each turn, the actual player chooses the game where he/she plays a move. We use the same normal convention: when there is no more available move (in any of the n games), the game is lost.

Definition 10 (Sum of games). Let $\mathsf{Gr}_1, \ldots, \mathsf{Gr}_n$ be the game-graphs of n games $G_1, \ldots, G_n$ respectively. The game-graph of the *sum* of $G_1, \ldots, G_n$ has the Cartesian product

$$V(\mathsf{Gr}_1) \times \cdots \times V(\mathsf{Gr}_n)$$

as set of vertices. If $(\alpha_1, \ldots, \alpha_n)$ is a vertex, i.e., α_i is a position of G_i, for all i, then there is an edge to $(\beta_1, \ldots, \beta_n)$ whenever there exists j such that $\beta_j \in \mathrm{Opt}(\alpha_j)$ (being understood that β_j is an option of α_j in the game G_j) and $\beta_i = \alpha_i$ for all $i \neq j$.

This theory has been independently developed by R. P. Sprague and P. M. Grundy[3].

Theorem 3 (Sprague–Grundy theorem). *Let G_i be impartial combinatorial games with $\mathscr{G}_i$ as respective Grundy functions, $i = 1, \ldots, n$. Then the sum of games $G_1 + \cdots + G_n$ has a Grundy function $\mathscr{G}$ given by*

$$\mathscr{G}(\alpha_1, \ldots, \alpha_n) = \mathscr{G}_1(\alpha_1) \oplus \cdots \oplus \mathscr{G}_n(\alpha_n)$$

where, for all i, α_i is a position in G_i.

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[3] The following proof is inspired by the one found in Thomas S. Ferguson's lecture notes on CGT.

Proof. Let $(\alpha_1, \ldots, \alpha_n)$ be an arbitrary position in the sum of the n games. Let b be the integer $\mathscr{G}_1(\alpha_1) \oplus \cdots \oplus \mathscr{G}_n(\alpha_n)$. We will prove that

1. for all non-negative integers $a < b$, there is an option from $(\alpha_1, \ldots, \alpha_n)$ with $\mathscr{G}$-value a;
2. no option of $(\alpha_1, \ldots, \alpha_n)$ has $\mathscr{G}$-value b.

If the two items are proved, by definition of the Sprague–Grundy function, this means that $\mathscr{G}(\alpha_1, \ldots, \alpha_n) = b = \mathscr{G}_1(\alpha_1) \oplus \cdots \oplus \mathscr{G}_n(\alpha_n)$.

Let $d = a \oplus b$ and ℓ be the length of the base-2 expansion of d, i.e., $2^{\ell-1} \le d < 2^\ell$. Since $a < b$, the base-2 expansion of b must have a 1 in position ℓ and a must have 0 in that position. Indeed, if the most significant digits of a and b are the same, then these digits will cancel each other in the Nim-sum; the position corresponding to $2^{\ell-1}$ is the most significant one where they differ:

$$\mathrm{rep}_2(a) = u0v, \quad \mathrm{rep}_2(b) = u1v', \quad |v| = |v'| = \ell - 1 \;.$$

But $b = \mathscr{G}_1(\alpha_1) \oplus \cdots \oplus \mathscr{G}_n(\alpha_n)$. Hence, there exists i such that the base-2 expansion of $\mathscr{G}_i(\alpha_i)$ has a 1 in position ℓ. Hence, $d \oplus \mathscr{G}_i(\alpha_i) < \mathscr{G}_i(\alpha_i)$. In the ith game, by definition of the Sprague–Grundy function $\mathscr{G}_i$, there is thus a move from α_i to some position α_i' such that $\mathscr{G}_i(\alpha_i') = d \oplus \mathscr{G}_i(\alpha_i)$. In the sum of the games, there is a move from $(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ to $(\alpha_1, \ldots, \alpha_i', \ldots, \alpha_n)$ and

$$\mathscr{G}_1(\alpha_1) \oplus \cdots \oplus \mathscr{G}_i(\alpha_i') \oplus \cdots \oplus \mathscr{G}_n(\alpha_n) = d \oplus \mathscr{G}_1(\alpha_1) \oplus \cdots \oplus \mathscr{G}_i(\alpha_i) \oplus \cdots \oplus \mathscr{G}_n(\alpha_n)$$
$$= d \oplus b = a.$$

Let us consider the second item and proceed by contradiction. Assume that the position $(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ has an option $(\alpha_1, \ldots, \alpha_i', \ldots, \alpha_n)$ with the same $\mathscr{G}$-value. Hence,

$$\mathscr{G}_1(\alpha_1) \oplus \cdots \oplus \mathscr{G}_i(\alpha_i) \oplus \cdots \oplus \mathscr{G}_n(\alpha_n) = \mathscr{G}_1(\alpha_1) \oplus \cdots \oplus \mathscr{G}_i(\alpha_i') \oplus \cdots \oplus \mathscr{G}_n(\alpha_n)$$

and we conclude that $\mathscr{G}_i(\alpha_i) = \mathscr{G}_i(\alpha_i')$ which contradicts the fact that $\mathscr{G}_i$ is a Sprague–Grundy function and that α_i' is an option of α_i (in the ith game). $\square$

Remark 4. A game of Nim on p piles of tokens can be seen as the sum of p games of Nim on a single pile. Consequently, if we have p piles with respectively $x_1, \ldots, x_n$ tokens, then the corresponding Grundy value for this position is $\oplus_{i=1}^p x_i$. Also, with a single pile, the Grundy value of the position $n \ge 0$ is simply n. Thus, Bouton's theorem can retrospectively be seen as a corollary of Proposition 2 and Sprague–Grundy theorem.

In Table 1.2, we give the first few values of the Sprague–Grundy sequence $(\mathscr{G}_{NIM}(m,n))_{m,n \ge 0}$ for Nim on two piles. Our interest in this example comes from Exercises 21 and 22 found in [8, Section 16.6, p. 451].

Of course, values in this table can easily be obtained by computing a Nim-sum but we would like to know more about the structure of this table. Can we find some

9	9	8	11	10	13	12	15	14	1	0
8	8	9	10	11	12	13	14	15	0	1
7	7	6	5	4	3	2	1	0	15	14
6	6	7	4	5	2	3	0	1	14	15
5	5	4	7	6	1	0	3	2	13	12
4	4	5	6	7	0	1	2	3	12	13
3	3	2	1	0	7	6	5	4	11	10
2	2	3	0	1	6	7	4	5	10	11
1	1	0	3	2	5	4	7	6	9	8
0	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

Table 1.2 First few values of the Grundy function for Nim on two piles.

general pattern or recurrence occurring in it? We can also ask the same question for Wythoff's game. The following table can easily be computed using Remark 3. It will turn out that the analysis of Table 1.2 and Table 1.3 are quite different.

9	9	10	11	12	8	7	13	14	15	16
8	8	6	7	10	1	2	5	3	4	15
7	7	8	6	9	0	1	4	5	3	14
6	6	7	8	1	9	10	3	4	5	13
5	5	3	4	0	6	8	10	1	2	7
4	4	5	3	2	7	6	9	0	1	8
3	3	4	5	6	2	0	1	9	10	12
2	2	0	1	5	3	4	8	6	7	11
1	1	2	0	4	5	3	7	8	6	10
0	0	1	2	3	4	5	6	7	8	9
	0	1	2	3	4	5	6	7	8	9

Table 1.3 First few values of the Grundy function for Wythoff's game.

It is indeed an old open problem to obtain non-trivial information about the values given in Table 1.3. Not much is known, even for the positions with $\mathscr{G}$-value equal to 1. See Section 1.6.2.

1.4 Automatic and regular sequences

The question concluding the previous section motivates us for introducing first, k-automatic sequences and then, k-regular sequences ($k \geq 2$ is an integer). The main difference between the two concepts is that the first one is limited to sequences taking values in an alphabet. We will see, in Section 1.6 where concepts are gen-

eralized to a multidimensional setting, that the bidimensional infinite sequence $(\mathscr{G}_{NIM}(m,n))_{m,n\geq 0}$ is 2-regular. This will provide us with a general construction scheme and recurrence relations for $(\mathscr{G}_{NIM}(m,n))_{m,n\geq 0}$.

There are many good references dealing with automatic and regular sequences. To cite just a few, we mention Allouche and Shallit's book [8], the original paper by Cobham [31] and the survey [17].

Many characterizations of these sequences do exist. A *k*-automatic sequence can be defined as the image under a coding, i.e., a mapping[4] on a smaller alphabet, of a fixed point of a *k*-uniform morphism (see Definition 14). It is the sequence of outputs of a DFA fed with base-*k* expansions (see Theorem 5). For every letter, the set of indices corresponding to this letter is defined by a first order formula in $\langle \mathbb{N},+,V_k \rangle$. This is the so-called Büchi–Bruyère theorem [17]. When *k* is a prime power, it can be defined in terms of algebraic formal power series (this is the so-called Christol–Kamae–Mendès France–Rauzy theorem [28]). It also appears as a column of the space-time diagram of a cellular automaton [78], see Theorem 6 below. Finally, it can be defined in terms of finiteness of the its *k*-kernel (see Theorem 7).

1.4.1 Generalities on sequences and morphisms

Let us take some time to define properly some important notions. We first define a distance turning $A^{\mathbb{N}}$ into a complete ultrametric space for which the notion of convergence is usual. For general references, see, for instance, [8, 27, 61, 72].

Definition 11. Let $\mathbf{w}, \mathbf{x}$ be two distinct infinite words in $A^{\mathbb{N}}$. We let $\Lambda(\mathbf{w},\mathbf{x})$ denote the longest common prefix of $\mathbf{w}$ and $\mathbf{x}$ and we define a map $d : A^{\mathbb{N}} \times A^{\mathbb{N}} \to [0,+\infty)$ by

$$d(\mathbf{w},\mathbf{x}) = 2^{-|\Lambda(\mathbf{w},\mathbf{x})|}.$$

Note that $|\Lambda(\mathbf{w},\mathbf{x})| = \inf\{i \in \mathbb{N} \mid w_i \neq x_i\}$. Moreover, we set $d(\mathbf{w},\mathbf{w}) = 0$. Hence, the longer prefix two words share, the closer they are. It is obvious that $d(\mathbf{w},\mathbf{x}) > 0$ whenever $\mathbf{w} \neq \mathbf{x}$.

Definition 12. Let $(\mathbf{w}_n)_{n\geq 0}$ be a sequence of infinite words over the alphabet A. This sequence *converges* to the word $\mathbf{z} \in A^{\mathbb{N}}$ if $d(\mathbf{w}_n,\mathbf{z}) \to 0$ whenever $n \to +\infty$. Otherwise stated, for all $\ell \in \mathbb{N}$, there exists N such that, for all $n \geq N$, $\mathbf{w}_n$ and $\mathbf{z}$ share a common prefix of length at least ℓ. We say that $\mathbf{z}$ is the *limit* of the converging sequence $(\mathbf{w}_n)_{n\geq 0}$.

Let $(w_n)_{n\geq 0}$ be a sequence of finite words over A. If # is an extra symbol that does not belong to A, then any word $u \in A^*$ is in one-to-one correspondence with the infinite word $u\#^{\omega} \in (A \cup \{\#\})^{\mathbb{N}}$ where $\#^{\omega}$ means the concatenation of infinitely many copies of #. We say that the sequence $(w_n)_{n\geq 0}$ *converges* to the infinite word $\mathbf{w}$ if the sequence of infinite words $(w_n\#^{\omega})_{n\geq 0}$ converges to $\mathbf{w}$.

---

[4] A *coding* is a morphism from A^* to B^* where the image of every letter has length 1.

Let $f : A^* \to A^*$ be a morphism, i.e., a map such that $f(uv) = f(u)f(v)$ for all words $u, v \in A^*$. Observe that a morphism is completely defined from the images of the letters. Let a be a letter in the alphabet A. If there exists a finite word u such that $f(a) = au$, then

$$\lim_{n \to +\infty} f^n(a) = a\, u\, f(u)\, f^2(u)\, f^3(u) \cdots . \tag{1.1}$$

Note that this limit denoted by $f^\omega(a)$ is well defined in the above topology. It can possibly be equal to a finite word if, for some $k \geq 1$, we have $f^k(u) = \varepsilon$. To avoid this situation, a morphism $f : A^* \to A^*$ is said to be *prolongable* on the letter $a \in A$, not only if there exists a finite word u such that $f(a) = au$ but also, if

$$\lim_{n \to +\infty} |f^n(a)| = +\infty .$$

We will encounter a more general situation, where a second morphism is applied to the infinite word obtained by iterating a first morphism. If $g : A^* \to B^*$ is a *non-erasing* morphism, i.e., for all $a \in A$, $g(a) \neq \varepsilon$, it can be extended to a map from $A^\mathbb{N}$ to $B^\mathbb{N}$ as follows. If $\mathbf{x} = x_0 x_1 \cdots$ is an infinite word over A, then the sequence of words $(g(x_0 \cdots x_{n-1}))_{n \geq 0}$ converges to an infinite word over B. Its limit is denoted by $g(\mathbf{x}) = g(x_0)g(x_1)g(x_2)\cdots$.

Note that we can always consider non-erasing prolongable morphisms and codings.

Theorem 4. *Let $f : A^* \to A^*$ be a (possibly erasing) morphism that is prolongable on a letter $a \in A$. Let $g : A^* \to B^*$ be a (possibly erasing) morphism. If the word $g(f^\omega(a))$ is infinite, there exists a non-erasing morphism $h : C^* \to C^*$ prolongable on a letter $c \in C$ and a coding $j : C^* \to B^*$ such that $g(f^\omega(a)) = j(h^\omega(c))$.*

This result was already stated by Cobham in 1968 [29]. For a proof, see [8]. An alternative short proof is given in [52]. This result is also discussed in details in [21] and [23].

Definition 13 (Morphic word). An infinite word obtained by iterating a prolongable morphism is said to be *purely substitutive* or *purely morphic*. In the literature, one also finds the term *pure morphic*. If $\mathbf{x} \in A^\mathbb{N}$ is pure morphic and if $g : A \to B$ is a coding, then the word $\mathbf{y} = g(\mathbf{x})$ is said to be *morphic* or *substitutive*.

1.4.2 Iterating a constant-length morphism

It is now time to present k-automatic sequences. See the original paper of Cobham [30] and the comprehensive book [8].

Definition 14 (Uniform morphism). Let $k \geq 2$. A morphism $f : A^* \to B^*$ satisfying $|f(a)| = k$, for all $a \in A$, is said to be of *constant length k* or *k-uniform*. A 1-uniform morphism is a *coding*.

Definition 15. Let A, B be two alphabets. An infinite word $\mathbf{w} \in B^{\mathbb{N}}$ is *k-automatic* if there exist a *k*-uniform morphism $f : A^* \to A^*$ prolongable on $a \in A$, and a 1-uniform morphism $g : A^* \to B^*$, such that $\mathbf{w} = g(f^{\omega}(a))$.

The quotient $\lfloor j/k \rfloor$ of the Euclidean division of j by k is denoted by j DIV k. So to speak, for any symbol x_j occurring in $\mathbf{x} = f^{\omega}(a)$, we can track its history, see Figure 1.1: x_j has been produced by f from $x_{j \text{ DIV } k}$. The latter symbol appears itself in the image by f of $x_{(j \text{ DIV } k) \text{ DIV } k}$, and so on and so forth. Note that if the base-k expansion of j is $\text{rep}_k(j) = c_i \cdots c_1 c_0$, then the base-$k$ expansion of j DIV k is $c_i \cdots c_1$. This simple observation permits one to easily track the past of a given
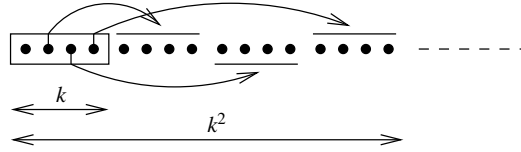
Fig. 1.1 Tracking a symbol in a *k*-automatic word.

symbol x_j by considering the prefixes of $\text{rep}_k(j)$. In other words, we have the next result.

Lemma 1. *Let* $f : A^* \to A^*$ *be a k-uniform morphism prolongable on a and* $\mathbf{x} = x_0 x_1 x_2 \cdots = f^{\omega}(a)$. *Let j such that* $k^m \leq j < k^{m+1}$, *for some* $m \geq 0$. *Then* $j = kq + r$ *with* $k^{m-1} \leq q < k^m$ *and* $0 \leq r < k$ *and the symbol* x_j *is the* $(r+1)$st *symbol occurring in* $f(x_q)$.

The next construction will be crucial in this chapter. In particular, it explains where the term "automatic sequence" comes from. The automaton that we introduce encodes exactly the same information as the morphism f. The fact that the morphism has constant length implies that the DFA is *complete*, i.e., the transition function is defined for all $(b, i) \in A \times \{0, \ldots, k-1\}$.

Definition 16 (DFA associated with a morphism). We associate with a *k*-uniform morphism $f : A^* \to A^*$ and a letter $a \in A$, a DFA $\mathscr{A}_f = (A, a, \{0, \ldots, k-1\}, \delta_f, A)$ where $\delta_f(b, i) = w_{b,i}$ if $f(b) = w_{b,0} \cdots w_{b,k-1}$. We assume that the letter a is clear from the context, if it is not the case, one can use a notation such as $\mathscr{A}_{f,a}$. If a state b has a loop labeled by 0, then the morphism is prolongable on the letter b.

It is a bit tricky, but the alphabet A is the set of states of this automaton.

Example 6. Consider the morphism f and the associated automaton depicted in Figure 1.2.

Proposition 3. *Let* $\mathbf{x} = f^{\omega}(a) = x_0 x_1 \cdots$ *with f a k-uniform morphism prolongable on the letter a. With the above notation, for the DFA* $\mathscr{A}_f$ *associated with f, we have, for all* $j \geq 0$,

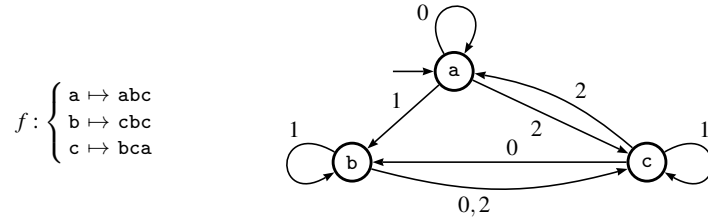$$x_j = \delta_f(a, \text{rep}_k(j)) = \mathscr{A}_f \cdot \text{rep}_k(j) .$$

$$f : \begin{cases} \texttt{a} \mapsto \texttt{abc} \\ \texttt{b} \mapsto \texttt{cbc} \\ \texttt{c} \mapsto \texttt{bca} \end{cases}$$

Fig. 1.2 A 3-uniform morphism f and the associated automaton $\mathscr{A}_f$.

Proof. This is a direct consequence of Lemma 1. Reading an extra symbol in $\mathscr{A}_f$ corresponds to append a digit on the right (least significant digit) of a base-k expansion. The reader can also look at the proof of Theorem 10 expressed in a more general setting. □

The converse also holds.

Proposition 4. *Let* $(A, a, \{0, \ldots, k-1\}, \delta, A)$ *be a DFA such that* $\delta(a, 0) = a$. *Then the word* $\mathbf{x} = x_0 x_1 x_2 \cdots$ *defined by* $x_j = \delta(a, \mathrm{rep}_k(j))$, *for all* $j \geq 0$, *is the fixed point of a* k-*uniform morphism* f *prolongable on* a *where* $f(b) = \delta(b, 0) \cdots \delta(b, k-1)$ *for all* $b \in A$.

Proof. This is a direct consequence of Lemma 1. □

Theorem 5. *Let* $\mathbf{w} = w_0 w_1 w_2 \cdots$ *be an infinite word over an alphabet B. It is of the form* $g(f^{\omega}(a))$ *where* $f : A^* \to A^*$ *is a* k-*uniform morphism prolongable on* $a \in A$ *and* $g : A^* \to B^*$ *is a coding if and only if there exists a DFAO*

$$(A, a, \{0, \ldots, k-1\}, \delta, \mu : A \to B)$$

such that $\delta(a, 0) = a$ *and, for all* $j \geq 0$, $w_j = \mu(\delta(a, \mathrm{rep}_k(j)))$.

A proof of this result can essentially be derived from Lemma 1. One has simply to put together the previous two propositions and consider an extra coding and thus an output function for the DFA. Nevertheless, we will prove later on (see Theorem 10, Theorem 16 and Proposition 9) a general statement that includes the case discussed above.

The next notion will permit us to easily discuss recognizable series.

Definition 17 (Transtion matrices). With a DFA or a DFAO over the alphabet $A = \{0, \ldots, k-1\}$, we may associate k square matrices of $\mathbb{N}^{A \times A}$, a transition matrix M_i for each label $i < k$. The matrix M_i is defined by

$$[M_i]_{a,b} = \begin{cases} 1 \text{ , if } a \cdot i = b; \\ 0 \text{ , otherwise.} \end{cases}$$

Note that each row of these matrices contains exactly one 1. Multiplying such a matrix on the left by a row vector of the form $\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$ gives a vector of the same form.

Example 7. Let us continue Example 6. The corresponding three matrices are given by

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The next observation is a prelude to the notion of a linear representation of a recognizable series.

Remark 5. If the initial state is the first one, then the state reached when reading a word $w_1 \cdots w_\ell$ in the automaton (from the initial state) can be determined by the following matrix product:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} M_{w_1} \cdots M_{w_\ell}.$$

We obtain the characteristic vector of the reached state (all entries are 0 except one). This is easily shown by induction on the length of the input word. If the input word is empty, the matrix product is empty hence equal to the identity matrix and we get the initial characteristic vector. If $\begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} M_{w_1} \cdots M_{w_i}$ is the characteristic vector of the state s reached when reading $w_1 \cdots w_i$ then the multiplication by $M_{w_{i+1}}$ is equal to $s \cdot w_{i+1}$ by definition of the matrix.

We would like to mention a connection with cellular automata.

Theorem 6 (Rowland–Yassawi [78]). *A sequence over a finite field $\mathbb{F}_q$ of characteristic p is p-automatic if and only if it occurs as a column of the space-time diagram, with eventually periodic initial conditions, of a linear cellular automaton with memory over $\mathbb{F}_q$.*

1.4.3 k-*kernel of a sequence*

The next characterization of k-automatic sequences will be extensively used in this chapter. It will lead to the notion of k-regular sequence which is based on a property of the module generated by the k-kernel of a sequence. Note that the next notion introduced by Eilenberg [39] is not related to the graph-theoretic notion introduced in Definition 5 (the standard terminology is a bit unfortunate).

Definition 18. Let $k \geq 2$. The k-*kernel* of a sequence $\mathbf{x} = (x(n))_{n \geq 0}$ is the set of subsequences:

$$\mathrm{Ker}_k(\mathbf{x}) = \left\{ (x(k^i n + s))_{n \geq 0} \mid i \geq 0, \ 0 \leq s < k^i \right\}.$$

An alternative way to define the k-kernel is first to introduce k operators of k-*decimation*, for $r \in \{0, \ldots, k-1\}$, we set

$$\partial_{k,r}((x(n))_{n \geq 0}) := (x(kn+r))_{n \geq 0}. \tag{1.2}$$

The k-kernel of $\mathbf{x}$ is thus the set of sequences of the form

$$\partial_{k,r_1} \circ \cdots \circ \partial_{k,r_m}((x(n))_{n \geq 0}), \quad r_1, \ldots, r_m \in \{0, \ldots, k-1\}. \qquad (1.3)$$

Remark 6. Notice that considering a subsequence of the form $(x(k^i n + s))_{n \geq 0}$ with $\mathrm{rep}_k(s) = r_m \cdots r_1$, $m \leq i$, corresponds exactly to extracting the subsequence of indices whose base-k expansions have a suffix of length i of the form $0^{i-m} r_m \cdots r_1$. (We assume that expansions of length less that i may be preceded by leading zeroes.)

Theorem 7 (Eilenberg). *A sequence is k-automatic if and only if its k-kernel is finite.*

A proof can be found in [8, Thm. 6.6.2]. We provide an alternative proof of the fact that automaticity implies finiteness of the kernel.

Proof. We make use of Theorem 5. Let $\mathbf{x}$ be a k-automatic sequence obtained by feeding a DFAO $\mathcal{M}$ with base k-expansions, i.e., for all $n \geq 0$, $\mathbf{x}(n) = \mathcal{M} \cdot \mathrm{rep}_k(n)$. We first assume that there is no extra coding involved. Words of $\{0, \ldots, k-1\}^*$ act on the set Q of states of $\mathcal{M}$. For all words $u \in \{0, \ldots, k-1\}^*$, we have a map

$$f_u : Q \to Q, \quad q \mapsto q \cdot u.$$

The set of such maps endowed with composition is referred to as the *transition monoid* of $\mathcal{M}$. It is finite because the number of such maps is bounded by $\#Q^{\#Q}$. Let $u \in \{0, \ldots, k-1\}^*$. We observe that the subsequence $(\mathbf{x}(k^{|u|}n + \mathrm{val}_k(u)))_{n \geq 0}$ is obtained as follows

$$\mathbf{x}(k^{|u|}n + \mathrm{val}_k(u)) = \mathcal{M} \cdot (\mathrm{rep}_k(n)u) = f_u(\mathcal{M} \cdot \mathrm{rep}_k(n)).$$

Hence the number of subsequences in the k-kernel is equal to the cardinality of the transition monoid of $\mathcal{M}$ (assuming that every state of $\mathcal{M}$ can be reached from the initial state[5]). If there is an extra coding to be applied to $f^\omega(a)$, since we are considering a mapping onto a smaller alphabet, the number of distinct subsequences decreases. $\square$

How to compute the transition monoid of a DFA? Assume that the set of states Q is $\{q_1, \ldots, q_n\}$. Then starting with the n-tuple $(q_1, \ldots, q_n)$, we compute $(q_1 \cdot a, \ldots, q_n \cdot a)$ for all letters a. We list the pairwise distinct and newly created n-tuples. For each such tuple, we iterate the process and build a labeled graph where transitions are of the form

$$(r_1, \ldots, r_n) \xrightarrow{a} (r_1 \cdot a, \ldots, r_n \cdot a).$$

This algorithm explores all the n-tuples of the form $(q_1 \cdot w, \ldots, q_n \cdot w)$ where w is a word. Paths from $(q_1, \ldots, q_n)$, in this graph, correspond to the possible maps f_u. In particular, if two words u and v lead to same vertex, then $f_u = f_v$. We apply this

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[5] This means that every letter of the alphabet appears at least once in $\mathbf{x}$.

algorithm to the automaton depicted in Figure 1.2 and start with the 3-tuple $(\mathtt{a},\mathtt{b},\mathtt{c})$. More details can, for instance, be found in [57].

	ε	0	1	2	01	02	10	20	22	010	020	022	201	202	220	0220	2022
a	a	a	b	c	b	c	c	b	a	c	b	a	b	c	a	a	a
b	b	c	b	c	c	a	c	b	a	b	a	c	b	c	a	b	a
c	c	b	c	a	b	c	b	a	c	c	b	a	b	c	b	a	a

Table 1.4 The 3-kernel of $f^{\omega}(\mathtt{a})$ for the morphism given in Example 6.

Example 8 (Thue–Morse word). The *Thue–Morse word* is the fixed point $f^{\omega}(\mathtt{a})$ of the morphism $f : \mathtt{a} \mapsto \mathtt{ab}, \mathtt{b} \mapsto \mathtt{ba}$,

$$\mathtt{abbabaabbaababbabaababbaabbabaab}\cdots$$

Its 2-kernel is finite and contains 2 elements. This word is ubiquitous in COW [7]. For instance, it is well-known that the Thue–Morse word is overlap-free: it does not contain any factor of the form *cvcvc* where *c* is a letter and *v* belongs to $\{\mathtt{a},\mathtt{b}\}^*$. In particular, the Thue–Morse word is not ultimately periodic.

1.4.4 Changing the base

Example 5 shows that the sequence $(\mathscr{G}(n))_{n \geq 0}$ (when removing $1, 2$ or 4 tokens) is periodic: $(0,1,2)^{\omega}$. It is not difficult to devise a k-automatic sequence equal to $(0,1,2)^{\omega}$. For instance, assume that $k = 2$. We consider the following 2-uniform morphism and coding

$$f : \mathtt{a} \mapsto \mathtt{ab},\ \mathtt{b} \mapsto \mathtt{cd},\ \mathtt{c} \mapsto \mathtt{ef},\ \mathtt{d} \mapsto \mathtt{ab},\ \mathtt{e} \mapsto \mathtt{cd},\ \mathtt{f} \mapsto \mathtt{ef}$$

$$g : \mathtt{a},\mathtt{d} \mapsto 0,\ \mathtt{b},\mathtt{e} \mapsto 1,\ \mathtt{c},\mathtt{f} \mapsto 2.$$

More generally, every ultimately periodic word uv^{ω}, $v \neq \varepsilon$, is k-automatic for all $k \geq 2$.

What could happen when computing the 3-kernel of the Thue–Morse word? It turns out that this 3-kernel is infinite. This can be obtained from a famous result on numeration systems.

Theorem 8 (Cobham's theorem [30]). *Let $k, \ell \geq 2$ be two multiplicatively independent integers, i.e., $\log k / \log \ell$ is irrational. If a sequence is both k-automatic and ℓ-automatic, then it is ultimately periodic.*

Proposition 5. *Let $k \geq 2$ and $n \geq 1$ be integers. A sequence is k-automatic if and only if it is k^n-automatic.*

Cobham's theorem can be considered as difficult. No immediate proof seems[6] to be known [68, 76]. On the other hand, Proposition 5 is an easy exercise. Each digit in base k^n corresponds to n digits in base k and the image by a morphism preserves the regularity of languages.

Cobham's theorem has been extended to various settings (multidimensional and logical frameworks [17], non-constant length morphism, ...). In the non-constant length setting, what replaces the base k is the Perron–Frobenius eigenvalue of the matrix associated with the morphism. See Cobham–Durand's theorem [38].

1.4.5 k-regular sequences

We have seen that k-automatic sequences take values in an alphabet. Nevertheless, when studying unbounded sequences (which is the case of most Grundy functions) we need to replace k-automaticity with a more general concept.

Regular sequences (again the usual terminology could be misleading) can be studied in a more general algebraic setting than the one considered here. Since, we are exclusively dealing with sequences taking integer entries, e.g. $(\mathscr{G}_{NIM}(m,n))_{m,n\geq 0}$, we will limit ourselves a bit and have a simple presentation with Definition 19. In this section, we still consider the case of infinite words. The multidimensional setting will be presented later on.

Definition 19. The sequence $\mathbf{x} \in \mathbb{Z}^{\mathbb{N}}$ is *k-regular*, if the $\mathbb{Z}$-module generated by $\mathrm{Ker}_k(\mathbf{x})$ is finitely generated, i.e., there exists $\mathbf{t}_1, \ldots, \mathbf{t}_\ell \in \mathbb{Z}^{\mathbb{N}}$ such that

$$\langle \mathrm{Ker}_k(\mathbf{x}) \rangle = \langle \mathbf{t}_1, \ldots, \mathbf{t}_\ell \rangle. \tag{1.4}$$

The ring $\mathbb{Z}$ is embedded in fields such as $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Thus sequences we are considering can be seen as elements of $\mathbb{Q}^{\mathbb{N}}$ which is a $\mathbb{Q}$-vector space (instead of considering a $\mathbb{Z}$-module). So another way to describe k-regularity is to say that the orbit of $\mathbf{x}$ under the action of compositions of the operators (1.2) of k-decimation $\partial_{k,r}$ remains in a finite dimensional vector space.

Remark 7. The original definition given by Allouche and Shallit [6] is the following one. Let $\mathfrak{R}$ be a ring containing a commutative Noetherian ring $\mathfrak{R}'$, i.e., every ideal of $\mathfrak{R}'$ is finitely generated. A sequence $\mathbf{x}$ in $\mathfrak{R}^{\mathbb{N}}$ is *($\mathfrak{R}'$,k)-regular*, if there exists $\mathbf{t}_1, \ldots, \mathbf{t}_\ell \in \mathfrak{R}^{\mathbb{N}}$ such that every sequence in $\mathrm{Ker}_k(\mathbf{x})$ is an $\mathfrak{R}'$-linear combination of $\mathbf{t}_1, \ldots, \mathbf{t}_\ell$.

Observe that if $\mathbf{x}$ is $(\mathfrak{R}',k)$-regular, then $\langle \mathrm{Ker}_k(\mathbf{x}) \rangle$ is a submodule of a finitely generated $\mathfrak{R}'$-module. In general, this does not imply that the submodule itself is finitely generated. This means that we only have the inclusion $\langle \mathrm{Ker}_k(\mathbf{x}) \rangle \subset \langle \mathbf{t}_1, \ldots, \mathbf{t}_\ell \rangle$ instead of equality in (1.4). Nevertheless, since $\mathfrak{R}'$ is assumed to be Noetherian, one can show (using more algebra) that every submodule of a finitely

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[6] When writing this chapter, a paper by Thijmen J. P. Krebs appeared on arXiv [54].

generated $\mathfrak{R}'$-module is finitely generated and thus $\langle \mathrm{Ker}_k(\mathbf{x}) \rangle$ is finitely generated. From that (up to taking other generators), we can assume equality in (1.4). Dealing with integer sequences, we can take $\mathfrak{R} = \mathfrak{R}' = \mathbb{Z}$.

Similarly to recognizable formal series, with every k-regular sequence $(s(n))_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$ is associated a *linear representation* (λ, μ, ν). There exist a positive integer r, a row vector $\lambda \in \mathbb{Z}^{1 \times r}$ and a column vector $\nu \in \mathbb{Z}^{r \times 1}$, a matrix-valued morphism $\mu : \{0, \ldots, k-1\} \to \mathbb{Z}^{r \times r}$ such that

$$s(n) = \lambda \ \mu(c_0 \cdots c_\ell) \ \nu = \lambda \ \mu(c_0) \cdots \mu(c_\ell) \ \nu$$

for all $c_\ell, \ldots, c_0 \in \{0, \ldots, k-1\}^*$ such that $\mathrm{val}_k(c_\ell \cdots c_0) = \sum_{i=0}^{\ell} c_i k^i = n$. The converse also holds: if there exists a linear representation associated with the canonical k-ary expansion of integers (one has to take into account the technicality of representations starting with leading zeroes), then the sequence is k-regular. See, for instance, [8, Theorem 16.2.3]. As a corollary, the nth term of a k-regular sequence can be computed with $\lfloor \log_k(n) \rfloor$ matrix multiplications.

Remark 8. These matrix multiplications are a natural extension of those encountered in Remark 5. Instead of a classical DFA, the linear representation can be seen as a *weighted automaton* with r states.

Proof (Existence of a linear representation). Let $\mathbf{s} = (s(n))_{n \geq 0} \in \mathbb{Z}^{\mathbb{N}}$ be a k-regular sequence. By definition, there exists a finite number of sequences $\mathbf{t}_1, \ldots, \mathbf{t}_\ell$ such that $\langle \mathrm{Ker}_k(\mathbf{s}) \rangle = \langle \mathbf{t}_1, \ldots, \mathbf{t}_\ell \rangle$. In particular, each $\mathbf{t}_j$ is a $\mathbb{Z}$-linear combination of elements in the k-kernel of $\mathbf{s}$. We have finitely many $\mathbf{t}_j$'s, so $\mathbf{t}_1, \ldots, \mathbf{t}_\ell$ are $\mathbb{Z}$-linear combinations of finitely many elements in $\mathrm{Ker}_k(\mathbf{s})$. Thus we can assume that $\langle \mathrm{Ker}_k(\mathbf{s}) \rangle$ is generated by finitely many elements from $\mathrm{Ker}_k(\mathbf{s})$ itself. Without loss of generality, we assume from now on that $\mathbf{t}_1, \ldots, \mathbf{t}_\ell$ belong to $\mathrm{Ker}_k(\mathbf{s})$.

From (1.3), for all $r \in \{0, \ldots, k-1\}$ and all $i \in \{1, \ldots, \ell\}$, $\partial_{k,r}(\mathbf{t}_i)$ is a sequence in $\mathrm{Ker}_k(\mathbf{s})$ and thus, there exist coefficients $(A_r)_{1,i}, \ldots, (A_r)_{\ell,i}$ such that

$$\partial_{k,r}(\mathbf{t}_i) = \sum_{j=1}^{\ell} (A_r)_{j,i} \mathbf{t}_j.$$

Notice that A_r is an $\ell \times \ell$ matrix. Roughly, if we were in a vector space setting, this means that the matrices A_r represent the linear operators $\partial_{k,r}$ in the basis $\mathbf{t}_1, \ldots, \mathbf{t}_\ell$. Let $p \geq 0$ be an integer. Notice that if $\mathrm{rep}_k(p) = r_m \cdots r_0$, then $s(p)$ is the first term, i.e., corresponding to the index $n = 0$, of the sequence

$$(s(b^{m+1}n + p))_{n \geq 0} = \partial_{k,r_0} \circ \cdots \circ \partial_{k,r_m}((s(n))_{n \geq 0}).$$

We will use the fact that $\partial_{k,r}$ is linear, i.e., if α, β are coefficients and $\mathbf{v}, \mathbf{w}$ are two sequences, then $\partial_{k,r}(\alpha \mathbf{v} + \beta \mathbf{w}) = \alpha \partial_{k,r}(\mathbf{v}) + \beta \partial_{k,r}(\mathbf{w})$. It is easy to see that

$$\partial_{k,r_0} \circ \cdots \circ \partial_{k,r_m}(\mathbf{t}_i) = \sum_{j=1}^{\ell} (A_{r_0} \cdots \cdot A_{r_m})_{j,i} \mathbf{t}_j.$$

If we have the following decomposition of $\mathbf{s}$ (in a vector space setting, we would have a unique decomposition of $\mathbf{s}$ in the basis $\mathbf{t}_1,\ldots,\mathbf{t}_\ell$)

$$\mathbf{s} = \sum_{i=1}^{\ell} \sigma_i \, \mathbf{t}_i$$

then, by linearity,

$$(s(b^{m+1}n+p))_{n\geq 0} = \sum_{i=1}^{\ell} \sigma_i \sum_{j=1}^{\ell} (A_{r_0} \cdots A_{r_m})_{j,i} \, (\mathbf{t}_j(n))_{n\geq 0} = \sum_{j=1}^{\ell} \tau_j \, (\mathbf{t}_j(n))_{n\geq 0}$$

where

$$\begin{pmatrix} \tau_1 \\ \vdots \\ \tau_\ell \end{pmatrix} = A_{r_0} \cdots A_{r_m} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_\ell \end{pmatrix}.$$

Consequently, $s(p)$ is obtained as

$$s(p) = \sum_{i=1}^{\ell} \tau_i \, \mathbf{t}_i(0) = \underbrace{\left(t_1(0) \;\cdots\; t_\ell(0) \right)}_{=:\lambda} A_{r_0} \cdots A_{r_m} \underbrace{\begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_\ell \end{pmatrix}}_{=:v}$$

and $\mu(c) = A_c$. □

Example 9. For the *sum-of-digits function* given by

$$s_2(n) = \sum_{i=0}^{\ell} n_i \quad \text{whenever } \mathrm{rep}_2(n) = n_\ell \cdots n_0,$$

the sequence $\mathbf{s} = (s_2(n))_{n\geq 0}$ has a (base-2) linear representation given by

$$\lambda = \begin{pmatrix} 0 & 1 \end{pmatrix}, \; \mu(i) = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \; v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad i = 0, 1.$$

We let $\mathbf{1}$ denote the constant sequence $(1)_{n\geq 0}$. It does not belong to the 2-kernel of $\mathbf{s}$ but it belongs to the $\mathbb{Z}$-module generated by it because it is equal to $\partial_{2,1}(\mathbf{s}) - \mathbf{s}$. Nevertheless, it is enough to see that $\partial_{2,0}(\mathbf{1}) = \partial_{2,1}(\mathbf{1}) = \mathbf{1}$ and take $\mathbf{s}$ and $\mathbf{1}$ as generators to proceed as in the proof above. From the following relations, we deduce the two columns of the matrix $\mu(0)$

$$\partial_{2,0}(\mathbf{s}) = 1 \cdot \mathbf{s} + 0 \cdot \mathbf{1}, \quad \partial_{2,0}(\mathbf{1}) = 0 \cdot \mathbf{s} + 1 \cdot \mathbf{1}$$

and, for $\mu(1)$,

$$\partial_{2,1}(\mathbf{s}) = 1 \cdot \mathbf{s} + 1 \cdot \mathbf{1}, \quad \partial_{2,1}(\mathbf{1}) = 0 \cdot \mathbf{s} + 1 \cdot \mathbf{1}.$$

The vector λ is given by $s_2(0) = 0$ and $\mathbf{1}(0) = 1$. The vector ν is obtained from $\mathbf{s} = 1 \cdot \mathbf{s} + 0 \cdot \mathbf{1}$. To compute $s_2(19)$, observe that $\text{rep}_2(19) = 10011$. Thus we compute

$$(0\ 1)\, \mu(1)\mu(1)\mu(0)\mu(0)\mu(1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\,.$$

Remark 9. In [9, Section 6], a practical procedure to guess relations a possibly k-regular sequence will satisfy is described. Consider a sequence $(s(n))_{n \geq 0}$. The idea is to construct a matrix in which the rows represent truncated versions of elements of the k-kernel of $(s(n))_{n \geq 0}$, together with row reduction. Start with a matrix having a single row, let us say corresponding to the first m elements of the sequence. Then, in view of the first paragraph of the proof providing the existence of a linear representation, repeatedly add subsequences of the form $(s(k^{\ell} n + r))_{n \geq 0}$ not linearly dependent of the previous stored sequences. From this, you have candidate relations that remain to be proved.

Remark 10. There is an intermediate notion of a *k-synchronized* sequence s where a DFA, in the sense discussed in Section 1.6, accepts pairs of base-k expansions corresponding to $(n, s(n))$. See [18].

1.5 Characterizing $\mathscr{P}$-position of Wythoff's game in polynomial time

First let us explain what we mean by "the Fibonacci word is *coding* the $\mathscr{P}$-positions of Wythoff's game". From that result, we will introduce Zeckendorf [88] (or Fibonacci) numeration system and reobtain a well-known result by Fraenkel, Theorem 11, characterizing $\mathscr{P}$-positions of Wythoff's game from Fibonacci expansions of integers. In this section, we will deal with two copies of the Fibonacci word! One is defined as a map whose domain is $\mathbb{N}$ and the other one has domain $\mathbb{N}_{>0}$, we will try to avoid any confusion using notation $\mathbf{f}$ and $\mathbf{f}'$ respectively. The alphabet of $\mathbf{f}$ and $\mathbf{f}'$ can be either $\{\mathtt{a}, \mathtt{b}\}$ or, $\{1, 2\}$. It does not really matter.

Definition 20 (Fibonacci word). The *Fibonacci word* over $\{\mathtt{a}, \mathtt{b}\}$

$$\mathbf{f} = \mathtt{abaababaabaababaababaaba}\cdots$$

is the unique fixed point of the morphism $\mathscr{F} : \mathtt{a} \mapsto \mathtt{ab}, \mathtt{b} \mapsto \mathtt{a}$.

A (homogeneous) *Beatty sequence* is just the sequence of integer part of positive multiples of an irrational number. In our setting, Theorem 2 provides us with a pair of *complementary* Beatty sequences because $\frac{1}{\varphi} + \frac{1}{\varphi^2} = 1$. Thus $(\lfloor n\varphi \rfloor)_{n \geq 1}$ and $(\lfloor n\varphi^2 \rfloor)_{n \geq 1}$ make a partition of $\mathbb{N}_{>0}$. This is a consequence of Beatty theorem (sometimes known as Rayleigh theorem). Regarding this result, papers of interest are [41, 67].

Theorem 9 (Beatty theorem). *Let* $\alpha, \beta > 1$ *be irrational numbers such that* $1/\alpha + 1/\beta = 1$. *Then* $(\lfloor n\alpha \rfloor)_{n \geq 1}$ *and* $(\lfloor n\beta \rfloor)_{n \geq 1}$ *make a partition of* $\mathbb{N}_{>0}$.

There are various characterizations of Sturmian words (the Fibonacci word is Sturmian). One of those is given in terms of irrational mechanical words, i.e., the sequence of first differences of a Beatty sequence. See, for details, [62, Sec. 2.1.2]. Also see [40]. Here we start indexing infinite words with 1, because of Beatty theorem and the first non-zero $\mathscr{P}$-position of Wythoff is $(1, 2)$. In particular, using those mechanical words, one can show that the nth letter in the Fibonacci word

$$\mathbf{f}' = f_1 f_2 f_3 \cdots = 212212122122 \cdots \tag{1.5}$$

over $\{1, 2\}$ can be obtained as

$$f_n = \lfloor (n+1)\varphi \rfloor - \lfloor n\varphi \rfloor, \quad \forall n \geq 1.$$

Wythoff's result, Theorem 2, can therefore be restated as follows.

Proposition 6. *The* n*th* $\mathscr{P}$-*position* $\big(\lfloor n\varphi \rfloor, \lfloor n\varphi^2 \rfloor\big)$ *of Wythoff's game is given by the indices (starting with* 1*) of the* n*th* a *and* n*th* b *in the Fibonacci word* $\mathbf{f}'$.

Proof. Since $f_n = \lfloor (n+1)\varphi \rfloor - \lfloor n\varphi \rfloor$, if $f_n = 1$, this means $\lfloor n\varphi \rfloor$ and $\lfloor (n+1)\varphi \rfloor$ are consecutive positive integers. If $f_n = 2$, this means that there exists m such that $\lfloor n\varphi \rfloor$, $\lfloor m\varphi^2 \rfloor$ and $\lfloor (n+1)\varphi \rfloor$ are three consecutive integers. Indeed, because of Beatty theorem, the gap between two consecutive multiples of φ must be filled with a multiple of φ^2.

Apply the morphism $1 \mapsto$ a and $2 \mapsto$ ab to $\mathbf{f}' = 21221 \cdots$. We get exactly the Fibonacci word over $\{$a, b$\}$ because the Fibonacci word is the fixed point of the morphism $\mathscr{F}$: a $\mapsto$ ab, b $\mapsto$ a and we may replace $1, 2$ by b and a respectively. The effect of $2 \mapsto$ ab is to insert an extra symbol b between two a's (the images of 1 and 2 both start with a). Thus the multiples $\lfloor n\varphi \rfloor$ are exactly given by the indices of the a's in the Fibonacci word $\mathbf{f}'$. The remaining indices (filling the partition of $\mathbb{N}_{>0}$) corresponding to b's are thus given by the multiples $\lfloor n\varphi^2 \rfloor$. $\quad\square$

1.5.1 From morphic words to automatic words

Since the Fibonacci morphism $\mathscr{F}$ is not of constant length (it is not uniform in the sense of Definition 14), it is time to extend Definition 16. Note that the automaton defined below is not necessarily complete: the transition function could be partial.

Definition 21. We associate with a morphism $f : A^* \to A^*$ and a letter $a \in A$, a DFA

$$\mathscr{A}_f = \big(A, a, \{0, \ldots, \max_{b \in A} |f(b)| - 1\}, \delta_f, A\big)$$

where $\delta_f(b, i) = w_{b,i}$ if $f(b) = w_{b,0} \cdots w_{b,|f(b)|-1}$. We assume that the letter a is clear from the context, if it is not the case, one can use a notation such as $\mathscr{A}_{f,a}$.

Definition 22 (genealogical order). Let $(A, <)$ be an ordered alphabet. We can order A^* using the *genealogical ordering* (also called *radix order*). Words are first ordered by increasing length and for words of the same length, one uses the lexicographic ordering induced by the order $<$ on A. The order is denoted by $<_{gen}$.

Compared with the lexicographic order, the genealogical order is a *well-order*, that is, every non-empty subset has a minimal element. For instance, the set $a^+b = \{a^n b \mid n > 0\}$ has no minimal element for the lexicographic order.

Here is the natural generalization of Proposition 3, base-k expansions are replaced with words accepted by $\mathscr{A}_f$. We will proceed to the proof of this result at the end of this section.

Theorem 10. *Let $(A, <)$ be an ordered alphabet. Let $\mathbf{x} \in A^\mathbb{N}$ be an infinite word, fixed point $f^\omega(a)$ of a morphism $f : A^* \to A^*$ prolongable on a. Consider the language L_f of words accepted by $\mathscr{A}_f$ except those starting with 0. If L_f is genealogically ordered: $L_f = \{w_0 <_{gen} w_1 <_{gen} w_2 <_{gen} \cdots\}$, then the nth symbol of $\mathbf{x}$, $n \geq 0$, is the state $\mathscr{A}_f \cdot w_n$.*

Remark 11. Adding an extra coding and considering the word $g(f^\omega(a))$ does not lead to any difficulty. The DFA is replaced with a DFAO where the output function is simply g. Thus, the nth symbol of $g(f^\omega(a))$ is given by $g(\mathscr{A}_f \cdot w_n)$. See Example 10 below.

Let us apply this theorem to the Fibonacci word $\mathbf{f}$. The corresponding automaton is depicted in Figure 1.3. The language accepted by this DFA (excluding words
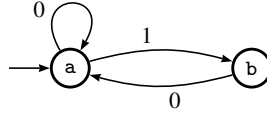
Fig. 1.3 DFA associated with $\mathtt{a} \mapsto \mathtt{ab}$, $\mathtt{b} \mapsto \mathtt{a}$.

starting with 0) is $1\{0, 01\}^* \cup \{\varepsilon\}$. This is the set of words over $\{0, 1\}$ avoiding the factor 11. This is exactly the language of the Fibonacci numeration: the greedy expansions of the non-negative integers in this positional numeration system. Details on numeration systems are presented in the next section.

Remark 12. For a morphism f of constant length k, the language L_f given by Theorem 10 is $\{1, \ldots, k-1\}\{0, \ldots, k-1\}^* \cup \{\varepsilon\}$ which is exactly the language of the base-k numeration system. For the Fibonacci morphism $\mathscr{F}$, the corresponding language is associated with the so-called Fibonacci numeration system.

Nevertheless, the language L_f given by Theorem 10 is not always related to a "well-known" numeration system. The term "well-known" means associated with a positional numeration system defined in Section 1.5.2. In the general situation: genealogically ordering a regular language leads to abstract numeration systems discussed in Section 1.7.1.

Prior to the proof of Theorem 10, we need the following lemma. If $\mathbf{x} = x_0 x_1 x_2 \cdots$ is an infinite word, then the shifted word $\sigma(\mathbf{x})$ is the word $x_1 x_2 x_3 \cdots$.

Lemma 2. *Let $A = \{a_1 < \cdots < a_n\}$ be a totally ordered alphabet. Let $\mathbf{z} \notin Q$. Let $\mathscr{A} = (Q, q_0, A, \delta_{\mathscr{A}}, F)$ be a DFA where $\delta_{\mathscr{A}} : Q \times A \to Q$ is (in general) a partial[7] function. Define the morphism $\psi_{\mathscr{A}} : (Q \cup \{\mathbf{z}\})^* \to (Q \cup \{\mathbf{z}\})^*$ by $\psi_{\mathscr{A}}(\mathbf{z}) = \mathbf{z} q_0$ and, for all $q \in Q$,*

$$\psi_{\mathscr{A}}(q) = \delta_{\mathscr{A}}(q, a_1) \cdots \delta_{\mathscr{A}}(q, a_n).$$

In this latter expression if $\delta_{\mathscr{A}}(q, a_i)$ is not defined for some i, then it is replaced by ε. Let L be the regular language accepted by (Q, q_0, A, δ, Q) where all states of $\mathscr{A}$ are final. Then the shifted word $\sigma(\psi_{\mathscr{A}}^{\omega}(\mathbf{z}))$ is the sequence $(x_n)_{n \in \mathbb{N}}$ of the states reached in $\mathscr{A}$ by the words of L in genealogical order, i.e., for all $n \in \mathbb{N}$,

$$x_n = \mathscr{A} \cdot w_n$$

where w_n is the $(n+1)$th word of the genealogically ordered language L.

Proof. By definition of ψ, first observe that we have the following factorization, see (1.1):

$$\psi_{\mathscr{A}}^{\omega}(\mathbf{z}) = \mathbf{z} x_0 x_1 x_2 \cdots = \mathbf{z} q_0 \, \psi_{\mathscr{A}}(q_0) \, \psi_{\mathscr{A}}^2(q_0) \cdots$$

and $x_0 = q_0 = \delta_{\mathscr{A}}(q_0, \varepsilon)$. Then by the definition of $\psi_{\mathscr{A}}$, if $x_n = \delta_{\mathscr{A}}(q_0, w_n)$, $n \geq 0$, then the factor

$$u_n = \psi_{\mathscr{A}}(x_n) = \delta_{\mathscr{A}}(q_0, w_n a_1) \cdots \delta_{\mathscr{A}}(q_0, w_n a_n) \tag{1.6}$$

appears in $\psi_{\mathscr{A}}^{\omega}(\mathbf{z})$ with the usual convention of replacing the undefined transitions with ε. Indeed, $\mathbf{z} x_0 x_1 x_2 \cdots$ is a fixed point of $\psi_{\mathscr{A}}$ and each x_n produces a factor $\psi_{\mathscr{A}}(x_n) = u_n$ appearing later on in the infinite word, similar to the situation depicted in Figure 1.1. Moreover this factor is preceded by

$$\delta_{\mathscr{A}}(q_0, w_{n-1} a_1) \cdots \delta_{\mathscr{A}}(q_0, w_{n-1} a_n)$$

and followed by

$$\delta_{\mathscr{A}}(q_0, w_{n+1} a_1) \cdots \delta_{\mathscr{A}}(q_0, w_{n+1} a_n).$$

It is therefore clear that we get all states reached from the initial state when considering the labels of all the paths in $\mathscr{A}$ in increasing genealogical order. $\quad\square$

Proof (Proof of Theorem 10). From Cobham's theorem on morphic words (Theorem 4), we assume that f is non-erasing. Let $C = \{0, \ldots, \max_{b \in A} |f(b)| - 1\}$ and consider the DFA $\mathscr{A}_f = (A, a, C, \delta, A)$ from Definition 21 having a as initial state.

Let $L \subseteq C^*$ be the language recognized by $\mathscr{A}_f$. Since $f(a) = au$ for some non-empty word u, it is clear that if $w \in L$ then $0w \in L$. Indeed by definition of $\mathscr{A}_f$, its initial state a has a loop labeled by 0, the first letter in C. If we apply Lemma 2 to this automaton $\mathscr{A}_f$, we obtain a morphism $\psi_{\mathscr{A}_f}$ generating the sequence of the states

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[7] Compared with complete functions in Definition 16.

reached by the words of L. This morphism is defined as follows. Let $\mathtt{z} \notin A$. We have $\psi_{\mathscr{A}_f}(\mathtt{z}) = \mathtt{z}a$ and, for all $b \in A$, $\psi_{\mathscr{A}_f}(b) = f(b)$.

The main point leading to the conclusion is to compare $\psi_{\mathscr{A}_f}^{\omega}(\mathtt{z})$ and $f^{\omega}(a)$. Since $f(a) = au$, using (1.1) we have the following factorizations

$$f^{\omega}(a) = au\, f(u)\, f^2(u)\, f^3(u) \cdots$$

and

$$\psi_{\mathscr{A}_f}^{\omega}(\mathtt{z}) = \mathtt{z}a\, a\, u\, f(a)\, f(u)\, f^2(a)\, f^2(u)\, f^3(a)\, f^3(u) \cdots.$$

If we erase the factors $\mathtt{z}$, a, $f(a)$, $f^2(a),\dots$ occurring in that order in the above factorization of $\psi_{\mathscr{A}_f}^{\omega}(\mathtt{z})$, we recover $f^{\omega}(a)$. Recall that $\psi_{\mathscr{A}_f}^{\omega}(\mathtt{z})$ is, except for $\mathtt{z}$, the sequence of states reached in $\mathscr{A}_f$ by considering all the possible paths in genealogical order. The second occurrence of a in $\psi_{\mathscr{A}_f}^{\omega}(\mathtt{z})$ is the state reached in $\mathscr{A}_f$ when reading $0 \in L$. By the property (1.6) of $\psi_{\mathscr{A}_f}$, the factor $f^n(a)$ in the above factorization corresponds to the states reached in $\mathscr{A}_f$ when reading the words in L of length $n+1$ starting with 0. Consequently, when giving to $\mathscr{A}_f$ the words of $L_f := L \setminus 0C^*$ in increasing genealogical order, we build exactly the sequence $f^{\omega}(a) = (y_n)_{n \geq 0}$, i.e., if $w_0 <_{gen} w_1 <_{gen} w_2 <_{gen} \cdots$ are the words of L_f in genealogical order, then $y_n = \mathscr{A} \cdot w_n$. $\square$

Example 10. Consider the alphabets $A = \{\mathtt{a},\mathtt{b},\mathtt{c}\}$, $B = \{\mathtt{d},\mathtt{e}\}$ and the morphisms

$$f : A^* \to A^*, \begin{cases} \mathtt{a} \mapsto \mathtt{abc} \\ \mathtt{b} \mapsto \mathtt{bc} \\ \mathtt{c} \mapsto \mathtt{aac} \end{cases} \quad \text{and} \quad g : A^* \to B^*, \begin{cases} \mathtt{a} \mapsto \mathtt{d} \\ \mathtt{b},\mathtt{c} \mapsto \mathtt{e}. \end{cases}$$

The corresponding automaton $\mathscr{A}_f$ is given in Figure 1.4 and the output function is represented on the outgoing arrows. The infinite word generated by the morphism f

$w_0 = \varepsilon$	$w_8 = 100$	$w_{16} = 210$
$w_1 = 1$	$w_9 = 101$	$w_{17} = 211$
$w_2 = 2$	$w_{10} = 110$	$w_{18} = 212$
$w_3 = 10$	$w_{11} = 111$	$w_{19} = 220$
$w_4 = 11$	$w_{12} = 112$	$w_{20} = 221$
$w_5 = 20$	$w_{13} = 200$	$w_{21} = 222$
$w_6 = 21$	$w_{14} = 201$	$\cdots$
$w_7 = 22$	$w_{15} = 202$	
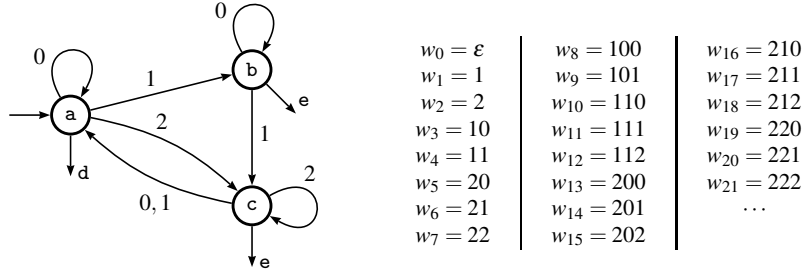
Fig. 1.4 The automaton $\mathscr{A}_f$ associated with $\mathtt{a} \mapsto \mathtt{abc}$, $\mathtt{b} \mapsto \mathtt{bc}$, $\mathtt{c} \mapsto \mathtt{aac}$ and the first few words in the associated language L_f.

is $f^{\omega}(\mathtt{a}) = (x_n)_{n \geq 0} = \mathtt{abcbcaacbcaacabcabcaac}\cdots$.

Corollary 1. *Let $(A, <)$ be an ordered alphabet. Let $\mathbf{x} \in A^{\mathbb{N}}$ be an infinite word, fixed point $f^{\omega}(a)$ of a morphism $f : A^* \to A^*$ prolongable on a. Consider the language*

$L_f = \{w_0 <_{gen} w_1 <_{gen} w_2 <_{gen} \cdots\}$ *of words accepted by $\mathscr{A}_f$ except those starting with* 0.

Let $n \geq 0$. If $\mathscr{A}_f \cdot w_n = $ b *and* $|f(\mathtt{b})| = r_b$, *then the factor* $f(\mathtt{b})$ *occurs in* **x** *in position corresponding to* $w_n 0, \ldots, w_n(r_b - 1)$.

In Example 10, the fifth b occurring in $f^{\omega}(\mathtt{a})$ in position 17 corresponds to the word 211 in L_f. Deleting the last one gives the word 21 which corresponds to the third a occurring in $f^{\omega}(\mathtt{a})$. In particular, this symbol a gives the factor $f(\mathtt{a}) = \mathtt{abc}$ corresponding to the words $210, 211, 212$. Finally, the first c corresponds to the word 2, so the second and third a occur in $f(\mathtt{c})$.

1.5.2 Positional numeration systems

A *numeration system* is a sequence $U = (U_n)_{n \geq 0}$ of integers such that U is increasing, $U_0 = 1$ and that the set $\{U_{i+1}/U_i \mid i \geq 0\}$ is bounded. This latter condition ensures that there exists an alphabet of digits used to represent integers. If $w = w_\ell \cdots w_0$ is a word over an alphabet $A \subset \mathbb{Z}$ then the numerical value of w is

$$\mathrm{val}_U(w) = \sum_{i=0}^{\ell} w_i U_i.$$

Using a greedy algorithm [43], every integer n has a unique *(normal) U-representation* or *U-expansion* $\mathrm{rep}_U(n) = w_\ell \cdots w_0$ which is a finite word over a minimal alphabet called the *canonical alphabet* of U and denoted by A_U. The normal U-representation satisfies

$$\mathrm{val}_U(\mathrm{rep}_U(n)) = n \text{ and for all } i \in \{0, \ldots, \ell-1\}, \ \mathrm{val}_U(w_i \cdots w_0) < U_{i+1}.$$

Remark 13. We call these systems *positional* because the position of a digit within an expansion is relevant. A digit in ith position ($i = 0$ corresponding to the rightmost digit) is multiplied by U_i to get the numerical value of the expansion.

For some general references on numeration systems, see Frougny's chapter in [62] or [72, 73]. The greediness of the expansion has the following consequence.

Proposition 7 (Order preserving map). *Let m, n be two non-negative integers. We have $m < n$ if and only if $\mathrm{rep}_U(m)$ is genealogically less than $\mathrm{rep}_U(n)$.*

Definition 23. Let $B \subset \mathbb{Z}$ be an alphabet. If $w \in B^*$ is such that $\mathrm{val}_U(w) \geq 0$, then the function that maps w to $\mathrm{rep}_U(\mathrm{val}_U(w))$ is called *normalization*.

Definition 24. A numeration system U is said to be *linear* if there exist $k \in \mathbb{N} \setminus \{0\}$, $d_1, \ldots, d_k \in \mathbb{Z}$, $d_k \neq 0$, such that, for all $n \geq k$, $U_n = d_1 U_{n-1} + \cdots + d_k U_{n-k}$. The polynomial $P_U(X) = X^k - d_1 X^{k-1} - \cdots - d_{k-1} X - d_k$ is called the *characteristic polynomial* of U.

Definition 25. Recall that a *Pisot-(Vijayaraghavan) number* is an algebraic integer $\beta > 1$ whose Galois conjugates have modulus less than 1. We say that $U = (U_n)_{n \geq 0}$ is a *Pisot numeration system* if the numeration system U is linear and $P_U(X)$ is the minimal polynomial of a Pisot number β. Integer[8] base numeration systems are particular cases of Pisot systems. For instance, see [16] where it is shown that most properties related to integer base systems, can be extended to Pisot systems. For a Pisot system β, there exists some $c > 0$ such that $|U_n - c\beta^n| \to 0$, as n tends to infinity.

Example 11. With the Fibonacci sequence $F = 1, 2, 3, 5, 8, 13, \ldots$, we have a Pisot numeration system associated with the Golden ratio. Thanks to the greediness of the expansions, we have $\mathrm{rep}_F(\mathbb{N}) = 1\{0, 01\}^* \cup \{\varepsilon\}$.

Thanks to the observation stated in Proposition 7, we can directly reformulate Theorem 10.

Corollary 2. *Let $(A, <)$ be an ordered alphabet. Let $\mathbf{w} \in A^{\mathbb{N}}$ be an infinite word, fixed point $f^\omega(a)$ of a morphism $f : A^* \to A^*$ prolongable on a. Let L_f be the language of words accepted by $\mathscr{A}_f$ except those starting with 0. If there exists a numeration system U such that $\mathrm{rep}_U(\mathbb{N}) = L_f$, then the nth symbol of $\mathbf{w}$, $n \geq 0$, is $\mathscr{A}_f \cdot \mathrm{rep}_U(n)$.*

Remark 14. The above corollary implies that if the state $a \in A$ is reached in $\mathscr{A}_f$ when reading a word w from the initial state, then the symbol with index $\mathrm{val}_U(w)$ occurring in the fixed point $\mathbf{w}$ is a. Moreover, the ordered outgoing transitions from state a with respective labels $0, \ldots, |f(a)| - 1$ are such that $f(a) = (a \cdot 0) \cdots (a \cdot (|f(a)| - 1))$.

Let us apply this result to the Fibonacci word $\mathbf{f} = f_0 f_1 f_2 \cdots = \mathtt{abaab} \cdots$ starting with index 0. It is important to note that in Theorem 10 and the above corollary, indices start with 0 (which is not the case in Proposition 6 and this difference has to be dealt with). Applying the above corollary with the DFA depicted in Figure 1.3, we get that

- $f_j = \mathtt{a}$ if and only if $\mathrm{rep}_F(j)$ ends with 0;
- $f_j = \mathtt{b}$ if and only if $\mathrm{rep}_F(j)$ ends with 1.

As for k-automatic sequences and Lemma 1, we can still "keep track of the past": since $\mathscr{F} : \mathtt{a} \mapsto \mathtt{ab}, \mathtt{b} \mapsto \mathtt{a}$, we have

$$f_j = \mathtt{a} \text{ for some } \mathrm{rep}_F(j) = u \text{ if and only if } f_{\mathrm{val}_F(u0)} = \mathtt{a} \text{ and } f_{\mathrm{val}_F(u1)} = \mathtt{b}$$

and similarly,

$$f_j = \mathtt{b} \text{ for some } \mathrm{rep}_F(j) = v \text{ if and only if } f_{\mathrm{val}_F(v0)} = \mathtt{a} \text{ and } f_{\mathrm{val}_F(v1)} = \mathtt{a}.$$

---

[8] Integer are the only rational numbers that are Pisot numbers.

In particular, the nth symbol $\mathtt{b}$ occurring in $\mathbf{f}$ belongs to the image by the morphism $\mathscr{F}$ of the nth $\mathtt{a}$ in $\mathbf{f}$. Otherwise stated, the nth $\mathtt{a}$ and the nth $\mathtt{b}$ in the Fibonacci word occur at indices (we recall, starting with 0) of the form:

$$\mathrm{val}_F(u0) \text{ and } \mathrm{val}_F(u01) \text{ with } u \in \mathrm{rep}_F(\mathbb{N}) . \tag{1.7}$$

If $\mathrm{rep}_F(j) = c_i \cdots c_2 c_1 c_0$, then f_j belongs to $\mathscr{F}(x_{\mathrm{val}_F(c_i \cdots c_1)})$. The symbol $x_{\mathrm{val}_F(c_i \cdots c_1)}$ appears itself in the image by the morphism $\mathscr{F}$ of the letter $x_{\mathrm{val}_F(c_i \cdots c_2)}$; x_j appears in $\mathscr{F}^2(x_{\mathrm{val}_F(c_i \cdots c_2)})$, and so on and so forth. For k-automatic sequences, we were simply using DIV k iteratively, here the result is similar: at each step, we remove the last digit of the F-representation of j.

1.5.3 Syntactic characterization of $\mathscr{P}$-positions of Wythoff's game

We can now put together all the results and material of this section to obtain the following result. It provides us with a polynomial time algorithm to decide whether or not a pair is a $\mathscr{P}$-position. Note that (x,x) is a $\mathscr{N}$-position because one can directly play to $(0,0)$.

Theorem 11 (Fraenkel [42]). *A pair (x,y), with $x < y$, is a $\mathscr{P}$-position of Wythoff's game if and only if $\mathrm{rep}_F(x)$ ends with an even number of zeroes and $\mathrm{rep}_F(y) = \mathrm{rep}_F(x)0$.*

Proof. From Proposition 6, we know that a pair (x,y), with $0 < x < y$, is a $\mathscr{P}$-position if and only if there exists n such that x is the index (starting with 1) of the nth $\mathtt{a}$ (resp. nth $\mathtt{b}$) occurring in the Fibonacci word $\mathbf{f}'$.

We now make use of Corollary 2 and observation (1.7): (x,y) is a $\mathscr{P}$-position if and only if there exists a (valid) F-representation u such that

$$(\mathrm{rep}_F(x-1), \mathrm{rep}_F(y-1)) = (u0, u01). \tag{1.8}$$

We subtract 1 because Corollary 2 deals with the word $\mathbf{f}$ whose indices start with 0 and not 1 as in $\mathbf{f}'$. From (1.8) what can be said about the form of $\mathrm{rep}_F(x)$ and $\mathrm{rep}_F(y)$? First, we have

$$\mathrm{rep}_F(x) = \mathrm{rep}_F(\mathrm{val}_F(u0) + 1) \text{ and } \mathrm{rep}_F(y) = \mathrm{rep}_F(\mathrm{val}_F(u01) + 1)$$

- First case[9]: Assume that u is of the form $u'0$. We easily get valid F-representations:

$$\mathrm{rep}_F(x) = \mathrm{rep}_F(\mathrm{val}_F(u'00) + 1) = u'01 \text{ ends with no zero,}$$

$$\mathrm{rep}_F(y) = \mathrm{rep}_F(\mathrm{val}_F(u'001) + 1) = u'010 \text{ i.e., previous word shifted by one zero.}$$

---

[9] The first few values may be checked by hand.

- Second case: Assume that u is of the form $u'1$ (where u' ends with 0 to have a valid F-representation). We get

$$\text{rep}_F(x) = \text{rep}_F(\text{val}_F(u'10)+1) = \text{rep}_F(\text{val}_F(u'11)),$$

$$\text{rep}_F(y) = \text{rep}_F(\text{val}_F(u'101)+1) = \text{rep}_F(\text{val}_F(u'110)).$$

This means that we need to normalize $u'11$ and $u'110$ (in the sense of Definition 23) or equivalently, to compute the successor of $u'10$ and $u'101$. Hopefully, a transducer computing the successor function from right to left, i.e., least significant digit first, for the Fibonacci system is well-known. See [47]. This transducer is depicted in Figure 1.5. If we feed the transducer with a valid F-representation
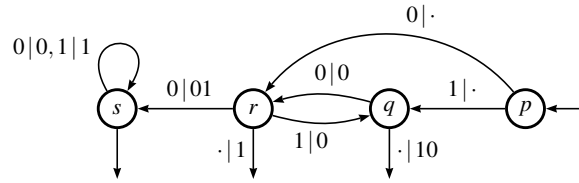
Fig. 1.5 Transducer computing the successor function from right to left (Fibonacci system).

$u'10$, it is either of the form $w100(10)^n10$ or, $10(10)^n10$, for some $n \geq 0$ (because we are considering large enough numbers). A computation for the first form is depicted below:

$$s \xleftarrow{\binom{w1}{w1}} s \xleftarrow{\binom{0}{01}} r \left[\xleftarrow{\binom{0}{0}} q \xleftarrow{\binom{1}{0}} r \right]^n \xleftarrow{\binom{0}{0}} q \xleftarrow{\binom{1}{0}} r \xleftarrow{\binom{0}{\cdot}} p \text{ :from right to left.}$$

So in the first situation, the successor of $u'10$ is

$$\underbrace{w10(01)^n0}_{u'}10 \to w101(00)^n00 \quad \text{ends with } 2n+2 \text{ zeroes, } n \geq 0$$

and for the successor of $u'101$, we get

$$\underbrace{w10(01)^n0}_{u'}101 \to w101(00)^n000 \quad \text{i.e., the previous word shifted by one zero.}$$

In the second situation, we have for the successor of $u'10$

$$\underbrace{1(01)^n0}_{u'}10 \to 100(00)^n00 \quad 2n+4 \text{ zeroes, } n \geq 0$$

and the successor of $u'101$

$$\underbrace{1(01)^n 0}_{u'} 101 \to 100(00)^n 000 \quad \text{i.e., the previous word shifted by one zero.}$$

Putting together the different cases gives the expected result. □

To conclude with this section, let us present the bidimensional word $\mathbf{P}_W$ coding the $\mathscr{P}$-positions of Wythoff's game. We have $\mathbf{P}_W(m,n) = 1$ if and only if (m,n) is a $\mathscr{P}$-position. Thanks to Proposition 2, it is a projection of $(\mathscr{G}(m,n))_{m,n\geq 0}$ given in Table 1.3.

	0	1	2	3	4	5	6	7	8	9	10
10	0	0	0	0	0	0	**1**	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	**1**	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	**1**
5	0	0	0	**1**	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	**1**	0	0	0
3	0	0	0	0	0	**1**	0	0	0	0	0
2	0	**1**	0	0	0	0	0	0	0	0	0
1	0	0	**1**	0	0	0	0	0	0	0	0
0	**1**	0	0	0	0	0	0	0	0	0	0

Table 1.5 The bidimensional word $\mathbf{w}(m,n)$ coding $\mathscr{P}$-positions of Wythoff.

We ask the same kind of (not yet answered) questions as before for the tables $(\mathscr{G}_{NIM}(m,n))_{m,n\geq 0}$ or $(\mathscr{G}_W(m,n))_{m,n\geq 0}$. We would like to have insight about the structure of the table, find some patterns, etc. The main difference between Table 1.5 and $(\mathscr{G}_{NIM}(m,n))_{m,n\geq 0}$ or $(\mathscr{G}_W(m,n))_{m,n\geq 0}$ is that we are with $\mathbf{P}_W$ over an alphabet $\{0,1\}$.

In Section 1.6.1, we prove that $(\mathscr{G}_{NIM}(m,n))_{m,n\geq 0}$ is 2-regular. It is well-known that the mapping on a finite alphabet of a 2-regular word is 2-automatic [6]. In Section 1.7, we prove that $(\mathbf{P}_W(m,n))_{m,n\geq 0}$ is a morphic shape-symmetric word.

1.6 Extension to a multidimensional setting

This extension is pretty straightforward. The only technical part is that one has to pad shorter expansions to deal with words of the same length. In the first sections of this chapter, we have considered DFA over alphabets such as $\{0,\ldots,k-1\}$. But there is no objection to consider other finite sets as alphabets such as

$$\{0,1\}^2 = \{0,1\} \times \{0,1\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}. \tag{1.9}$$

With this alphabet, the corresponding languages are subsets of $(\{0,1\}^2)^*$. We make no distinction between pairs written horizontally or vertically, but it seems more natural to write them as column vectors because that is what the machine should read simultaneously. Examples of DFA over $(\{0,1\}^2)^*$ are given in Figures 1.8 and 1.20. There is no objection to take d-tuples instead of pairs and also, we can have alphabets with more than two symbols. A DFA over $(\{0,1\}^3)^*$ is given in Figure 1.9.

Definition 26. Let $d \geq 1$, $k \geq 2$ be integers. The base-k expansion of a d-tuple of non-negative integers is defined by

$$\text{rep}_k : \mathbb{N}^d \to \left(\{0,\ldots,k-1\}^d\right)^*, \quad (n_1,\ldots,n_d) \mapsto \begin{pmatrix} 0^{\ell-|\text{rep}_k(n_1)|} \text{ rep}_k(n_1) \\ \vdots \\ 0^{\ell-|\text{rep}_k(n_d)|} \text{ rep}_k(n_d) \end{pmatrix}$$

where $\ell = \max\{|\text{rep}_k(n_1)|,\ldots,|\text{rep}_k(n_d)|\}$.

Let $d \geq 1$, $k \geq 2$ be integers. A map $f : A \to A^{k^d}$ where the image of a letter in A is a d-dimensional cube of size k, can be extended to a k-uniform morphism over the set of d-dimensional cubes. The image of a cube of size ℓ is a cube of size $k \cdot \ell$. The morphism is *prolongable* on the letter $a \in A$ if the "lower-left" corner of $f(a)$ is equal to a.

Example 12 (Pascal triangle mod 2). Consider the morphism

$$\psi : 0 \mapsto \begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}, \quad 1 \mapsto \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 1 & 1 \\ \hline \end{array}.$$
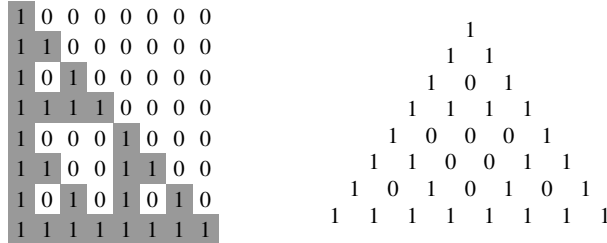
Fig. 1.6 Third iterate of ψ and the corresponding Pascal triangle modulo 2.

Iterating n times the morphism ψ from 1 gives (explanations will follow) the first 2^n rows of the binomial coefficients modulo 2 (up to a rotation of $3\pi/4$). The binomial coefficient $\binom{m}{n}$ (mod 2) corresponds to the point of coordinates $(m-n,n)$ in the picture on the left of Figure 1.6. It is important to keep in mind this "change of variables". Note that the sequence of normalized sets with ratio $1/2^n$ where 1's

(resp. 0's) are full (resp. empty) unit squares, is a sequence of compact sets converging, for the Hausdorff distance, to the Sierpiński gasket [3, 11]. On fractal patterns of the Pascal triangle and generalizations, see, e.g., [5, 4, 49, 60].

Lemma 1 and thus Theorem 5 are translated verbatim.

Lemma 3. *Let f be a k-uniform d-dimensional morphism prolongable on a and $\mathbf{x} = f^{\omega}(a) = (x(n_1, \ldots, n_d))_{n_1, \ldots, n_d \geq 0}$. Let $(j_1, \ldots, j_d)$ such that $k^{m_i} \leq j_i < k^{m_i+1}$, for some $m_i \geq 0$, for all i. Then $j_i = kq_i + r_i$ with $k^{m_i-1} \leq q_i < k^{m_i}$ and $0 \leq r_i < k$ and the symbol $x_{(j_1, \ldots, j_d)}$ occurs in the d-cube of size k $f(x_{(q_1, \ldots, q_d)})$ in position $(r_1, \ldots, r_d)$.*

Example 13. With notation of the previous lemma, $k = d = 2$, consider Figure 1.7. The symbol in position $(12, 10)$ where $\text{rep}_2(12) = 1100$ and $\text{rep}_2(10) = 1010$ is in the image of the symbol in position $(\text{val}_2(110), \text{val}_2(101)) = (6, 5)$. Moreover, since the last letter of 1100 and 1010 are both 0, $x(12, 10)$ is the lower-left corner of $f(x(6, 5))$. Now, $x(6, 5)$ appears in the image of $x((\text{val}_2(11), \text{val}_2(10)) = x(3, 2)$. Since the last digits in the base-2 expansion of 6 and 5 are respectively 0 and 1, $x(6, 5)$ is the upper-left corner of $f(x(3, 2))$. We can continue this way until we reach $x(1, 1)$.
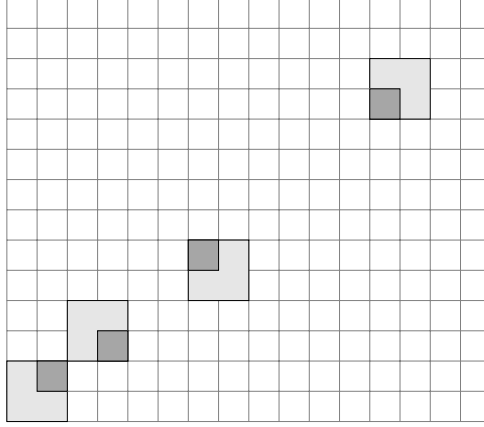
Fig. 1.7 Tracking the past of a symbol in a bidimensional 2-automatic word.

Theorem 12. *[82, 81] Let $\mathbf{w} : \mathbb{N}^d \to B$ be an infinite d-dimensional word over an alphabet B. It is of the form $g(f^{\omega}(a))$ where $f : A \to A^{k^d}$ is a k-uniform morphism, prolongable on $a \in A$ and $g : A \to B$ is a coding if and only if there exists a DFAO*

$$(A, a, \{0, \ldots, k-1\}^d, \delta, \mu : A \to B)$$

such that $\delta(a, (0, \ldots, 0)) = a$ and, for all $(j_1, \ldots, j_d) \in \mathbb{N}^d$,

$$\mathbf{w}(j_1, \ldots, j_d) = \mu(\delta(a, \text{rep}_k(j_1, \ldots, j_d)))$$

with the base-k expansion given in Definition 26.

The next example should suffice to explain how to derive a suitable DFAO from a morphism. If images of letters are d-dimensional cubes of size k, then transitions are labeled by d-tuples of digits in $\{0, \ldots, k-1\}$.

Example 14 (Pascal triangle modulo 2). We associate with the morphism ψ an automaton with input alphabet (1.9). Let $r, s, t, u, v \in \{0, 1\}$. If

$$\psi(r) = \begin{array}{|c|c|} \hline u & v \\ \hline s & t \\ \hline \end{array}$$

then, we have the transitions

$$r \xrightarrow{\binom{0}{0}} s, \quad r \xrightarrow{\binom{1}{0}} t, \quad r \xrightarrow{\binom{0}{1}} u, \quad r \xrightarrow{\binom{1}{1}} v.$$

The morphism from Example 12 corresponds to the DFA depicted in Figure 1.8.
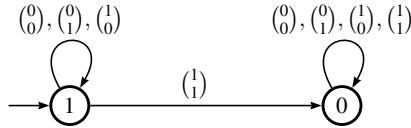
Fig. 1.8 An automaton generating the Pascal triangle modulo 2 (left of Figure 1.6).

We can make use of Lucas's theorem (recalled below) for $p = 2$. From (1.10), the binomial coefficient $\binom{m}{n}$, with $m \geq n$, is even if and only if $\text{rep}_2(m, n)$ contains the pair of digits $\binom{0}{1}$ (because the other three pairs give binomial coefficients equal to 1). To prove that the morphism ψ generates the Pascal triangle modulo 2, recall that we have some change of variables. It suffices to observe that $\text{rep}_2(m, n)$ contains the pair of digits $\binom{0}{1}$ if and only if $\text{rep}_2(m - n, n)$ contains the pair of digits $\binom{1}{1}$. The reader can probably carry on a proof of this necessary and sufficient condition[10]. We take the opportunity of Lemma 4 to give a proof involving automata. The latter condition is recognized by the DFA depicted in Figure 1.8 and associated with the morphism ψ.

Theorem 13 (Lucas's theorem). *Let p be a prime. Let m and n be two non-negative integers. If*

$$\text{rep}_p(m, n) = \begin{pmatrix} m_k & m_{k-1} & \cdots & m_1 & m_0 \\ n_k & n_{k-1} & \cdots & n_1 & n_0 \end{pmatrix}$$

then the following congruence relation holds

---

[10] One can relate this result to a theorem of Kummer. The p-adic valuation of $\binom{m}{n}$ is the number of carries when adding n to $m - n$ in base p. See, e.g., [77] and the references therein.

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \bmod p, \tag{1.10}$$

using the following convention: $\binom{m}{n} = 0$ *if* $m < n$.

Lemma 4. *Let* $m \geq n$. *The pair* $\mathrm{rep}_2(m,n)$ *contains the pair of digits* $\binom{0}{1}$ *if and only if* $\mathrm{rep}_2(m-n,n)$ *contains the pair of digits* $\binom{1}{1}$.

Proof. The DFA depicted in Figure 1.9 is mimicking base-2 addition. It recognizes 3-tuples of words (u,v,w) such that u, v and w have the same length and $\mathrm{val}_2(u) + \mathrm{val}_2(v) = \mathrm{val}_2(w)$. Successful paths are those starting and ending in state 0. The DFA reads least significant digits first. State 1 corresponds to the situation where there is a carry to take into account. The undefined transitions lead to some sink state.
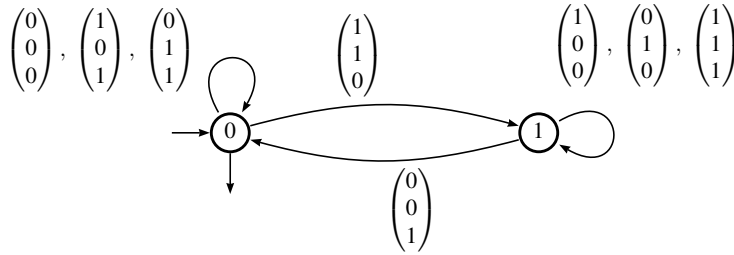
Fig. 1.9 A DFA accepting $\mathrm{rep}_2(x,y,x+y) \subset \{0,1\}^3$.

We can make use of this DFA to recognize $\mathrm{rep}_2(m-n,n,m)$. Notice that whenever $\mathrm{rep}_2(n,m)$ contains a pair $\binom{1}{0}$ (the last two of the three components), in a successful path, we must use the unique transition from state 0 to state 1. This means that $\mathrm{rep}_2(m-n,n)$ contains the pair $\binom{1}{1}$ (the first two of the three components) and conversely. □

Definition 27. For the sake of simplicity, consider the case $d = 2$. Consider a bidimensional sequence $\mathbf{x} = (x(m,n))_{m,n \geq 0}$. The *k-kernel* of $\mathbf{x}$ is the set of bidimensional subsequences

$$\mathrm{Ker}_k(\mathbf{x}) = \left\{ (x(k^i m + r, k^i n + s))_{m,n \geq 0} \mid i \geq 0, 0 \leq r,s < k^i \right\}.$$

Note that we have the same multiplicative factor k^i for both components. One element of the *k*-kernel corresponds to selecting two suffixes

$$(0^{i-p} r_p \cdots r_1, 0^{i-q} s_q \cdots s_1)$$

where $\mathrm{rep}_k(r) = r_p \cdots r_1$ and $\mathrm{rep}_k(s) = s_q \cdots s_1$. Theorem 7 can be extended to this setting: a *d*-dimensional word satisfies the conditions of Theorem 12 if and only if its *k*-kernel is finite. See, for instance, [82, 81, 8].

Example 15 (Pascal triangle modulo 2). Take the fixed point $(\mathbf{p}_2(m,n))_{m,n\geq 0}$ of the morphism ψ given in Example 12. We compute its 2-kernel as follows. From Lemma 3 and the form of ψ, we derive that, for all $m,n \geq 0$,

$$
\begin{aligned}
\mathbf{p}_2(2m,2n) &= \mathbf{p}_2(m,n),\\
\mathbf{p}_2(2m+1,2n) &= \mathbf{p}_2(m,n),\\
\mathbf{p}_2(2m,2n+1) &= \mathbf{p}_2(m,n),\\
\mathbf{p}_2(2m+1,2n+1) &= 0.
\end{aligned}
$$

These relations can also be deduced from the DFA in Figure 1.8: Reading the pair of digits $\binom{1}{1}$ leads to state 0, reading another pair does not change the state. Hence, the 2-kernel only contains the sequence itself and the null sequence. An alternative is to proceed as in the Proof of Theorem 7 and compute the transition monoid of the DFA in Figure 1.8.

1.6.1 2-*Regularity for Grundy values of the game of Nim*

Compared with Example 15 where the 2-kernel of the Pascal triangle modulo 2 is finite, we can define a multidimensional k-regular sequence: the $\mathbb{Z}$-module generated by $\mathrm{Ker}_k(\mathbf{x})$ is finitely generated. It is now time to reconsider Table 1.2.

Proposition 8 (Exercises 21 and 22, Section 16.6, p. 451 [8]). *For the game of Nim on two piles of tokens, the bidimensional sequence $(\mathscr{G}_N(m,n))_{m,n\geq 0}$ is 2-regular.*

Proof. From Bouton's theorem and simple base-2 manipulations, we get

$$
\begin{aligned}
\mathscr{G}_N(2m,2n) &= & 2m \oplus 2n &= 2\mathscr{G}_N(m,n),\\
\mathscr{G}_N(2m+1,2n) &= & (2m+1)\oplus 2n &= 2\mathscr{G}_N(m,n)+1,\\
\mathscr{G}_N(2m,2n+1) &= & 2m\oplus(2n+1) &= 2\mathscr{G}_N(m,n)+1,\\
\mathscr{G}_N(2m+1,2n+1) &= & (2m+1)\oplus(2n+1) &= 2\mathscr{G}_N(m,n).
\end{aligned}
$$

Hence the 2-kernel is generated by $(\mathscr{G}_N(m,n))_{m,n\geq 0}$ and the constant sequence $(1)_{m,n\geq 0}$. $\quad\square$

Remark 15. To prove that a sequence $(s(n))_{n\geq 0}$ is k-regular, it is enough to find some $j \geq 1$ and k^j linear relations expressing, for all $r < k^j$, $s(k^j n + r)$ in terms of elements of the form $s(k^i n + t)$ with $i < j$ and $t < k^i$. This observation is similar in a multidimensional setting.

From the relations given in the proof of Proposition 8, we can express any element $(\mathscr{G}_N(2^j m + r, 2^j n + t))_{m,n\geq 0}$ of the 2-kernel as a linear combination of $(\mathscr{G}_N(m,n))_{m,n\geq 0}$ and $(1)_{m,n\geq 0}$. An example should be enough to understand the reasoning. Can $(\mathscr{G}_N(8m+5,8n+2))_{m,n\geq 0}$ be expressed as a $\mathbb{Z}$-linear combination of these two sequences?

$$\begin{aligned}
\mathscr{G}_N(8m+5,8n+2) &= \mathscr{G}_N\left(2(4m+2)+1,2(4n+1)\right) \\
&= 2\mathscr{G}_N(4m+2,4n+1)+1 \\
&= 2\mathscr{G}_N(2(2m+1),2.2n+1)+1 \\
&= 2\left[2\mathscr{G}_N(2m+1,2n)+1\right]+1 \\
&= 4\mathscr{G}_N(2m+1,2n)+3 \\
&= 4\left[2\mathscr{G}_N(m,n)+1\right]+3 \\
&= 8\mathscr{G}_N(m,n)+7.
\end{aligned}$$

We can discuss a bit further the meaning of the relations given in the proof of Proposition 8. The situation is similar to the one of Lemma 3 and Figure 1.7. The only difference is that we have a function whose domain is not necessarily bounded. A 2×2 block in the bidimensional word $(\mathscr{G}_N(i,j))_{i,j \geq 0}$ whose lower-left corner has coordinates $(2m,2n)$ is completely determined from the value of $\mathscr{G}_N(m,n)$:

$$\mathscr{G}_N(m,n) \mapsto \begin{array}{|c|c|} \hline 2\mathscr{G}_N(m,n)+1 & 2\mathscr{G}_N(m,n) \\ \hline 2\mathscr{G}_N(m,n) & 2\mathscr{G}_N(m,n)+1 \\ \hline \end{array} \qquad (1.11)$$

The situation is depicted within Table 1.6. The value of $\mathscr{G}_N(2,1)$ determines the 2×2 block with lower-left corner of coordinates $(4,2)$. The four elements of that block give the 4×4 block whose lower-left corner has coordinates $(8,4)$ and so on and so forth.

$$
\begin{array}{cccccccccc}
9 & 8 & 11 & 10 & 13 & 12 & 15 & 14 & 1 & 0 \\
8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 0 & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 15 & 14 & 13 & 12 \\
6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 & 14 & 15 & 12 & 13 \\
5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 & 13 & 12 & 15 & 14 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 12 & 13 & 14 & 15 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 & 11 & 10 \\
2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 & 10 & 11 \\
1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 & 9 & 8 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
$$

Table 1.6 First few values of $\mathscr{G}_N(m,n)$ highlighting the action of the map (1.11).

1.6.2 Grundy values of the game of Wythoff

Even though it looks quite similar to the situation encountered with the game of Nim, Table 1.3 is challenging! We quote the book [85, p. 200]: "*No general for-*

mula is known for computing arbitrary $\mathscr{G}$-values of Wythoff. In general, they appear chaotic, though they exhibit a striking fractal-like pattern... Despite this apparent chaos, the $\mathscr{G}$-values nonetheless have a high degree of geometric regularity."

We collect some results from Blass and Fraenkel about $\mathscr{G}_W(m,n)$ [14]:

- On every parallel to the main diagonal, $(\mathscr{G}_W(n, n+j))_{n\geq 0}$ takes every possible value.
- Points with Grundy value 1 are "close" to those with value 0.
- Recursive algorithms to determine the points with Grundy value 1 are provided.

Definition 28. A sequence $(a_j)_{j\geq 0}$ is *additively periodic* if

$$\exists p,q, \ \forall j \geq q : a_{j+p} = a_j + p.$$

Note that $(a_j)_{j\geq 0}$ is additively periodic if and only if $(a_j - j)_{j\geq 0}$ is ultimately periodic.

Example 16. As an example, the row $\mathscr{G}_W(5,n)$ is such that for all $n \geq 27$, $\mathscr{G}_W(5, n+24) = \mathscr{G}_W(5,n) + 24$.
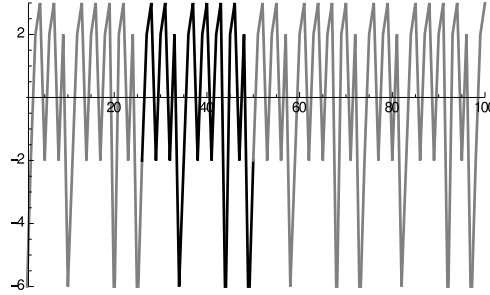
Fig. 1.10 Graph of $\mathscr{G}_W(5,n) - n$ for $50 \leq n \leq 150$.

Every row and column of $\mathscr{G}_W(m,n)$ is additively periodic, see [32, 55].

1.7 Shape-symmetry

Table 1.5 will permit us to introduce the notion of shape-symmetry. Our aim in this section is to prove Theorem 15 stating that Table 1.5 is the fixed point of a morphism with the shape-symmetry property. A picture is the analogue in a multidimensional setting of a finite word. It is a bit more intricate to define the concatenation of pictures. The results and material presented in this section come from [63] and [22]. We first start with formal definitions, Example 17 is presented afterwards.

Definition 29 (Picture). Let A be an alphabet. Let $s_1, \ldots, s_d \geq 1$ be in $\mathbb{N}_{>0} \cup \{\infty\}$. A *d-dimensional picture* over A is a map

$$x : \{0, \ldots, s_1 - 1\} \times \cdots \times \{0, \ldots, s_d - 1\} \to A$$

and $(s_1, \ldots, s_d)$ is the *shape* of x. It is denoted by $|x|$ and $|x|_i = s_i$. If $s_i < \infty$, for all i, x is *bounded*. The set of bounded pictures over A is denoted by $\mathscr{B}_d(A)$.

Definition 30 (Factor). Let $x \in \mathscr{B}_d(A)$ be a bounded picture of shape $(s_1, \ldots, s_d)$. Let $i_1, j_1, \ldots, i_d, j_d$ be integers such that $0 \leq i_k \leq j_k < s_k$ for all $k \in \{1, \ldots, d\}$. We let

$$x[(i_1, \ldots, i_d), (j_1, \ldots, j_d)]$$

denote the picture y of shape $(j_1 - i_1 + 1, \ldots, j_d - i_d + 1)$ defined by

$$y(n_1, \ldots, n_d) = x(i_1 + n_1, \ldots, i_d + n_d)$$

for all $n_1 < j_1 - i_1, \ldots, n_d < j_d - i_d$. In 2 dimensions, this just means that we specify the lower-left and upper-right corner of a sub-picture.

We let $\mathbf{0}$ (resp. $\mathbf{1}$) denote the row vector (of convenient dimension) whose entries are all equal to zero (resp. one). Let $i \in \{1, \ldots, s\}$. If z is a d-tuple, we let $z_{\hat{i}}$ denote the $(d-1)$-tuple where the ith coordinate has been removed.

Definition 31 (Concatenation). Let $x, y \in \mathscr{B}_d(A)$. If for some $i \in \{1, \ldots, d\}$, $|x|_{\hat{i}} = |y|_{\hat{i}} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_d)$, then we define the *concatenation* of x and y *along the ith direction* to be the d-dimensional picture $x \odot^i y$ of shape

$$(s_1, \ldots, s_{i-1}, |x|_i + |y|_i, s_{i+1}, \ldots, s_d).$$

satisfying

1. $x = (x \odot^i y)[\mathbf{0}, |x| - \mathbf{1}]$ and
2. $y = (x \odot^i y)[(0, \ldots, 0, |x|_i, 0, \ldots, 0), (0, \ldots, 0, |x|_i, 0, \ldots, 0) + |y| - \mathbf{1}]$.

In our examples, taking the usual convention for matrices, we first count the number of rows, then the number of columns. Concatenation along the direction 1 (resp. 2) follows the vertical (resp. horizontal) axis.

Example 17. Consider the two pictures

$$x = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \quad \text{and} \quad y = \begin{array}{|c|c|c|} \hline a & a & b \\ \hline b & c & d \\ \hline \end{array}$$

of respective shape $|x| = (2,2)$ and $|y| = (2,3)$. Since $|x|_{\hat{2}} = |y|_{\hat{2}} = 2$, we get

$$x \odot^2 y = \begin{array}{|c|c|c|c|c|} \hline a & b & a & a & b \\ \hline c & d & b & c & d \\ \hline \end{array} \quad \text{and} \quad y \odot^2 x = \begin{array}{|c|c|c|c|c|} \hline a & a & b & a & b \\ \hline b & c & d & c & d \\ \hline \end{array}.$$

However $x \odot^1 y$ is not defined because $2 = |x|_{\widehat{1}} \neq |y|_{\widehat{1}} = 3$. The pictures $x \odot^2 y$ and $y \odot^2 x$ both have shape $(2,5)$. Thus we can, for instance, define

$$z = (x \odot^2 y) \odot^1 (y \odot^2 x) = \begin{array}{|c|c|c|c|c|} \hline a & a & b & a & b \\ \hline b & c & d & c & d \\ \hline a & b & a & a & b \\ \hline c & d & b & c & d \\ \hline \end{array}.$$

Remark 16. A map $\gamma\colon A \to \mathscr{B}_d(A)$ cannot necessarily be extended to a morphism $\gamma\colon \mathscr{B}_d(A) \to \mathscr{B}_d(A)$. As an example, consider the map defined by

$$\gamma\colon a \mapsto \begin{array}{|c|c|} \hline b & d \\ \hline a & a \\ \hline \end{array}, \quad b \mapsto \begin{array}{|c|} \hline b \\ \hline c \\ \hline \end{array}, \quad c \mapsto \begin{array}{|c|c|} \hline a & a \\ \hline \end{array}, \quad d \mapsto \begin{array}{|c|} \hline d \\ \hline \end{array}.$$

Take the following bounded picture

$$x = \begin{array}{|c|c|} \hline c & d \\ \hline a & b \\ \hline \end{array}.$$

Considering the two rows of x, we may apply $\odot^2$ because $|\gamma(a)|_{\widehat{2}} = |\gamma(b)|_{\widehat{2}} = 2$ and $|\gamma(c)|_{\widehat{2}} = |\gamma(d)|_{\widehat{2}} = 1$. Considering the two columns of x, we may apply $\odot^1$ because $|\gamma(a)|_{\widehat{1}} = |\gamma(c)|_{\widehat{1}} = 2$ and $|\gamma(b)|_{\widehat{1}} = |\gamma(d)|_{\widehat{1}} = 1$. Hence the image of x by γ is well-defined:

$$\gamma(x) = \begin{array}{|c|c|c|} \hline a & a & d \\ \hline b & d & b \\ \hline a & a & c \\ \hline \end{array}$$

but $\gamma^2(x)$ is not well-defined! Indeed, see Figure 1.11, if we try to put together the images by γ of the different letters, we do not get a picture: Conditions to have a

Fig. 1.11 A map cannot necessarily be extended to a morphism.

well-defined morphism are given in (1.12) and Theorem 14.

Consider the section of a picture by a hyperplane. In this discrete setting, all the hyperplanes that we consider have equation of the form $x_i = k$ for some i.

Definition 32 (Section). Let x be a d-dimensional picture of shape $|x| = (s_1, \ldots, s_d)$. For all $i \in \{1, \ldots, d\}$ and $k < s_i$, we let $x_{|i,k}$ denote the $(d-1)$-dimensional picture

of shape

$$|x|_{\hat{i}} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_d)$$

defined by setting the ith coordinate equal to k in x, that is,

$$x_{|i,k}(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d) = x(n_1, \ldots, n_{i-1}, k, n_{i+1}, \ldots, n_d)$$

for all $n_j < s_j$ with $j \in \{1, \ldots, d\} \setminus \{i\}$.

What is the exact meaning for a morphism to be well-defined on a given picture? Let $\gamma \colon A \to \mathscr{B}_d(A)$ be a map and x be a bounded d-dimensional picture such that

$$\forall i \in \{1, \ldots, d\}, \forall k < |x|_i, \forall a, b \in \mathrm{Alph}(x_{|i,k}) : |\gamma(a)|_i = |\gamma(b)|_i. \qquad (1.12)$$

We let $\mathrm{Alph}(x_{|i,k})$ denote the set of letters occurring in $x_{|i,k}$. Then the *image* of x by γ is the d-dimensional picture defined as

$$\gamma(x) = \odot^1_{0 \le n_1 < |x|_1} \left(\cdots \left(\odot^d_{0 \le n_d < |x|_d} \gamma(x(n_1, \ldots, n_d)) \right) \cdots \right).$$

Note that the ordering of the products in the different directions is unimportant. If a bounded picture x does not satisfy (1.12), then $\gamma(x)$ is *undefined*. This means that the map γ can be extended to a subset of $\mathscr{B}_d(A)$. It is a quite restrictive requirement.

In 2 dimensions, condition (1.12) simply means that the images by γ of all the elements belonging to the same column (resp. rows) are pictures having the same number of columns (resp. row), see Figure 1.12. With d dimensions, (1.12) means that, for all sections by a hyperplane $x_i = k$, the images of all letters in this section have a shape with the same component along the direction orthogonal to that hyperplane. This ensures that building $\gamma(x)$ by concatenating the images of letters, we will not obtain "holes" or "overlaps".
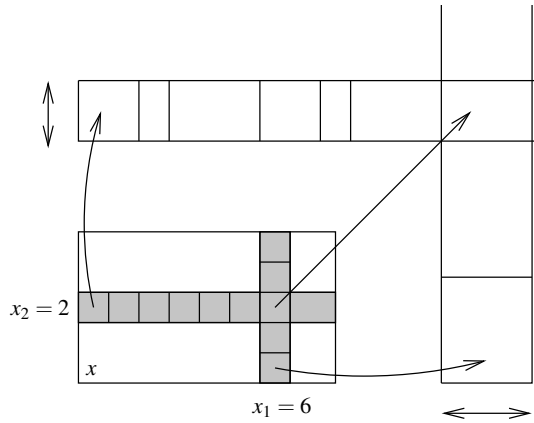
Fig. 1.12 Illustration of the condition (1.12).

Definition 33 (Multidimensional morphism). If for all $a \in A$ and all $n \geq 1$, $\gamma^n(a)$ is well-defined from $\gamma^{n-1}(a)$, then γ is said to be a *d-dimensional morphism*. If there exists a letter b such that $\gamma(b)_{0,\ldots,0} = b$ and, for all i, $|\gamma^n(b)|_i \to +\infty$ when $n \to +\infty$, then γ is said to be *prolongable* on b. We assume that the picture grows in every direction. It is not a strong assumption. If this is not the case and one of the direction remains constant, then we have a finite union of hyperplanes and words of lower dimension.

Remark 17. Let x, y be two bounded pictures such that $|x|_{\widehat{i}} = |y|_{\widehat{i}}$ and a morphism γ. Assume that $\gamma(x)$ and $\gamma(y)$ are well-defined pictures. If $\gamma(x \odot^i y)$ is well-defined, then it is equal to $\gamma(x) \odot^i \gamma(y)$ which is thus well-defined. On the other hand, if $\gamma(x) \odot^i \gamma(y)$ is defined, there is no reason for $\gamma(x \odot^i y)$ to be well-defined. As an example, take

$$\gamma : a \mapsto \begin{array}{|c|c|} \hline b & b \\ \hline a & a \\ \hline \end{array}, \quad b \mapsto \begin{array}{|c|c|} \hline b & b \\ \hline \end{array} \quad \text{and} \quad x = \begin{array}{|c|} \hline a \\ \hline b \\ \hline \end{array} \quad y = \begin{array}{|c|} \hline b \\ \hline a \\ \hline \end{array}$$

We have

$$x \odot^2 y = \begin{array}{|c|c|} \hline a & b \\ \hline b & a \\ \hline \end{array}, \quad \gamma(x) \odot^2 \gamma(y) = \begin{array}{|c|c|c|c|} \hline b & b & b & b \\ \hline a & a & b & b \\ \hline b & b & a & a \\ \hline \end{array}$$

but $\gamma(x \odot^2 y)$ is undefined because (1.12) is not satisfied. The images by γ of letters on each row of $x \odot^2 y$ have different number of rows.

Definition 34 (Projection). Let $\gamma : A \to \mathscr{B}_d(A)$ be a d-dimensional morphism prolongable on a. The *ith projection* of γ is a unidimensional morphism γ_i defined as follows. For all b, $\gamma_i(b)$ is the intersection of the picture $\gamma(b)$ with the hyperplane $x_i = 0$.

Definition 35 (Shape-symmetry). Let $\gamma : \mathscr{B}_d(A) \to \mathscr{B}_d(A)$ be a d-dimensional morphism having the d-dimensional infinite word $\mathbf{x}$ as a fixed point. This word is *shape-symmetric with respect to γ* if, for all permutations ν of $\{1, \ldots, d\}$, we have, for all $n_1, \ldots, n_d \geq 0$,

$$|\gamma(\mathbf{x}(n_1, \ldots, n_d))| = (s_1, \ldots, s_d) \Rightarrow |\gamma(\mathbf{x}(n_{\nu(1)}, \ldots, n_{\nu(d)}))| = (s_{\nu(1)}, \ldots, s_{\nu(d)}).$$

Note that, on the diagonal, $\gamma(\mathbf{x}(j, \ldots, j))$ must be a cube.

Example 18. Multidimensional k-automatic words are trivially shape-symmetric with respect to a morphism that sends letters to hypercubes of size k.

Example 19. The following morphism will be extensively used

$$\varphi_W : a \mapsto \begin{array}{|c|c|} \hline c & d \\ \hline a & b \\ \hline \end{array} \quad b \mapsto \begin{array}{|c|} \hline e \\ \hline i \\ \hline \end{array} \quad c \mapsto \begin{array}{|c|c|} \hline i & j \\ \hline \end{array} \quad d \mapsto \begin{array}{|c|} \hline i \\ \hline \end{array} \quad e \mapsto \begin{array}{|c|c|} \hline f & b \\ \hline \end{array}$$

$$f \mapsto \begin{array}{|c|c|} \hline h & d \\ \hline g & b \\ \hline \end{array} \quad g \mapsto \begin{array}{|c|c|} \hline h & d \\ \hline f & b \\ \hline \end{array} \quad h \mapsto \begin{array}{|c|c|} \hline i & m \\ \hline \end{array} \quad i \mapsto \begin{array}{|c|c|} \hline h & d \\ \hline i & m \\ \hline \end{array}$$

$$j \mapsto \begin{array}{|c|} \hline c \\ \hline k \\ \hline \end{array} \quad k \mapsto \begin{array}{|c|c|} \hline c & d \\ \hline l & m \\ \hline \end{array} \quad l \mapsto \begin{array}{|c|c|} \hline c & d \\ \hline k & m \\ \hline \end{array} \quad m \mapsto \begin{array}{|c|} \hline h \\ \hline i \\ \hline \end{array}$$

with the coding

$$\mu_W : a,e,g,j,l \mapsto 1, \quad b,c,d,f,h,i,k,m \mapsto 0.$$

The fourth iterate of $\varphi_W^4(a)$ is given in Table 1.7.

i	m	i	i	**j**	i	m	i
h	d	h	c	d	h	d	h
i	m	i	**l**	m	i	m	i
h	d	c	h	d	h	d	**e**
i	m	k	i	m	**g**	b	i
i	**j**	i	f	b	i	m	i
c	d	**e**	h	d	h	d	h
a	b	i	i	m	i	m	i

Table 1.7 The fourth iterate of $\varphi_W^4(a)$.

It can be shown that φ_W is a 2-dimensional morphism, see the next theorem. The size of the nth iterate from a is given by the nth Fibonacci number. The infinite word with a in position $(0,\ldots,0)$ and which is a fixed point of φ_W, is shape-symmetric with respect to φ_W. For instance, the image of e (second row, third column) has shape $(1,2)$ and the image of j (second column, third row) has a transposed shape $(2,1)$.

The two projections of φ_W are given by

$$[\varphi_W]_1 : a \mapsto ac,\ c \mapsto i,\ i \mapsto ih,\ h \mapsto i \tag{1.13}$$

and

$$[\varphi_W]_2 : a \mapsto ab,\ b \mapsto i,\ i \mapsto im,\ m \mapsto i . \tag{1.14}$$

Up to some renaming of letters, these two morphisms are equal.

The following result can be found in Maes's thesis [63]. In this thesis, Maes provides two different proofs. We only present the one using automata.

Theorem 14 (A. Maes). *Let $d \geq 1$. Let A be an alphabet.*

- *Determining whether or not a map $\mu : A \to \mathscr{B}_d(A)$ can be extended to a d-dimensional morphism, prolongable on a letter a, is a decidable problem.*
- *If μ is prolongable on the letter a, then it is decidable whether or not the fixed point $\mu^\omega(a)$ is shape-symmetric with respect to μ.*

Proof (Sketch of the proof). The reader should be used to the construction. We associate a DFA $\mathscr{A}_\mu$, just as in Example 14, with the map $\mu \colon A \to \mathscr{B}_d(A)$ and a specified initial symbol a. The set of states is A and transitions are labeled by d-tuples of digits: if $\mu(b)$ has shape $|\mu(b)| = (s_1, \ldots, s_d)$, we have a transition $(c_1, \ldots, c_d)$ for $0 \le c_i < s_i$ and for all $i \in \{1, \ldots, d\}$. So the number of outgoing transitions from $b \in A$ is $\prod_i |\mu(b)|_i$. The DFA associated with the map γ of Remark 16 is depicted in Figure 1.13. From this DFA, we build d new NFAs. Let $i \in \{1, \ldots, d\}$. We define
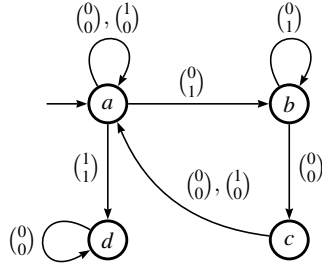
Fig. 1.13 The DFA associated with the map γ of Remark 16.

a NFA $\mathscr{N}_i$ where, for every transition, we only keep the ith component of the label (we proceed to a projection that explains the non-determinism). Moreover, the set I_i of initial states is made of all the states that can be reached from a when reading a word in 0^*.
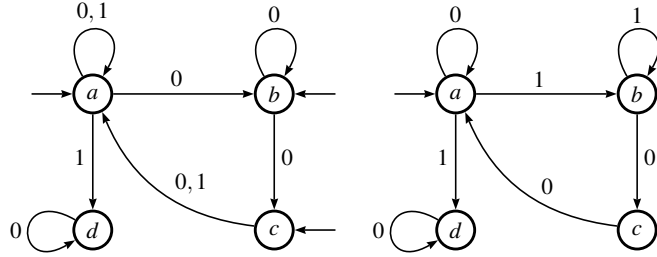
Fig. 1.14 The NFAs $\mathscr{N}_1$ and $\mathscr{N}_2$ derived from $\mathscr{A}_\gamma$.

We apply the subset construction (Rabin–Scott theorem) to these NFAs. The corresponding DFAs are denoted by $\mathscr{D}_i$, $i = 1, \ldots, d$. Recall that states of $\mathscr{D}_i$ are subsets of states of $\mathscr{N}_i$ and thus of $\mathscr{A}_\mu$, i.e., a state of $\mathscr{D}_i$ is a subset of the alphabet A. Observe that the initial state I_i of $\mathscr{D}_i$ is made of the states that can be reached in $\mathscr{A}_\mu$ by a word in $(\mathbb{N}^d)^*$ whose ith component is in 0^*. If μ is a morphism prolongable on a, then I_i is exactly the set of letters occurring in the intersection of $\mu^\omega(a)$ with the hyperplane $x_i = 0$. Therefore, in view of (1.12), for all $b \in I_i$, the quantities $|\mu(b)|_i$ must all be the same. Let Q be a state of $\mathscr{D}_i$. There exists a word $w \in \mathbb{N}^*$ such that,

for all b in Q, there exists a word in $(\mathbb{N}^d)^*$ which is the label of a path from a to b in $\mathscr{A}_\mu$ and whose ith component is in 0^*w. If μ is a morphism prolongable on a, then w can be chosen in L_{μ_i} where μ_i is the ith projection of μ (see Definition 34) and L_{μ_i} is the language given by Theorem 10 where words do not start with 0. If w is the $(n+1)$st word, $n \geq 0$, in the genealogically ordered language L_{μ_i}, then Q is exactly the set of letters occurring in the intersection of $\mu^\omega(a)$ with the hyperplane $x_i = n$. In view of (1.12), for all $b \in Q$, the quantities $|\mu(b)|_i$ must all be the same. We conclude that, if μ is a morphism prolongable on a, then:

- For all $i \in \{1, \ldots, d\}$, for any two letters b, c belonging to the same state of $\mathscr{D}_i$, we have $|\mu(b)|_i = |\mu(c)|_i$.

Conversely, if the above condition holds, then proceed by induction on the iterate $j \geq 0$. Assume that $\mu^j(a)$ exists. Let $n \geq 0$. Then the set of letters occurring in the intersection of $\mu^j(a)$ with $x_i = n$ is a subset of some state in $\mathscr{D}_i$. Therefore, the above condition and (1.12) permit us to define $\mu^{j+1}(a)$. In Table 1.8, we have applied the subset construction to $\mathscr{N}_1$. This shows that, starting with a, the map γ cannot be extended to a morphism. The last column contains the first component of the shape of $\gamma(e)$ for the letters e in the corresponding subset.

	state of $\mathscr{D}_1$	$|\gamma(\cdot)|_1$
$\mathscr{D}_1 \cdot \varepsilon = I_1$	$\{a, b, c\}$	$2, 2, 1$
$\mathscr{D}_1 \cdot 1$	$\{a, d\}$	$2, 1$
$\mathscr{D}_1 \cdot 10$	$\{a, b, d\}$	$2, 2, 1$
$\mathscr{D}_1 \cdot 100$	$\{a, b, c, d\}$	$2, 2, 1, 1$

Table 1.8 Subset construction applied to $\mathscr{N}_1$.

Let us now turn to the second decision problem. From Definition 35, it should be clear that:

- Given a morphism μ prolongable on a letter a, the fixed point $\mu^\omega(a)$ is shape-symmetric with respect to μ if and only if the languages $L_{\mu_1}, \ldots, L_{\mu_d}$ are equal.

It is well-known that testing equality of regular languages is decidable (see, e.g., [86]). For instance, from (1.13) and (1.14), we directly see that if $\varphi_W^\omega(a)$ exists then it is shape-symmetric with respect to φ_W. $\quad\square$

In Figure 1.15, we have depicted the first iteration (two viewpoints) of what a shape-symmetric morphism looks like in three dimensions.

We can already state the main result related to Wythoff's game because it is related to shape-symmetric morphisms.

Theorem 15. *The morphism φ_W and the coding μ_W give the 2-dimensional infinite word coding the $\mathscr{P}$-positions of Wythoff.*
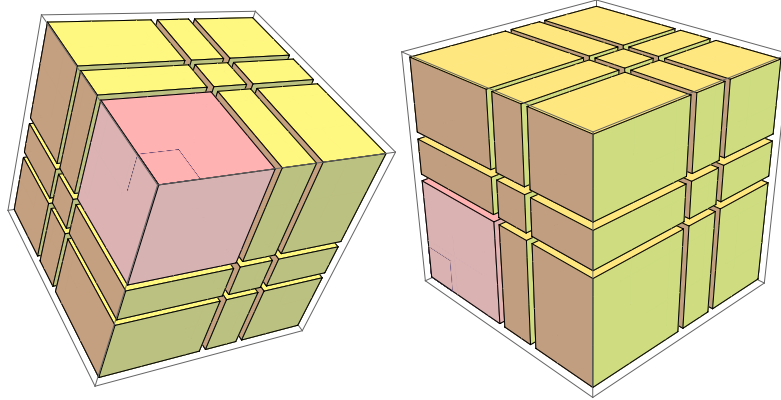
Fig. 1.15 Initial blocks of some 3-dimensional shape-symmetric morphism.

The proof will be given in Section 1.7.3. In particular, the DFA associated with φ_W is depicted in Figure 1.20. If we apply the procedure described in the proof of Theorem 14, when determinizing the two NFAs we get the following Table 1.9 showing that φ_W is a prolongable morphism.

	state of $\mathscr{D}_1$	$\|\gamma(\cdot)\|_1$		state of $\mathscr{D}_2$	$\|\gamma(\cdot)\|_2$
$\mathscr{D}_1 \cdot \varepsilon = I_1$	$\{a,b,i,m\}$	$2,2,2,2$	$\mathscr{D}_2 \cdot \varepsilon = I_2$	$\{a,c,h,i\}$	$2,2,2,2$
$\mathscr{D}_1 \cdot 1$	$\{c,d,e,h\}$	$1,1,1,1$	$\mathscr{D}_2 \cdot 1$	$\{b,d,j,m\}$	$1,1,1,1$
$\mathscr{D}_1 \cdot 10$	$\{b,f,i,j,m\}$	$2,2,2,2,2$	$\mathscr{D}_2 \cdot 10$	$\{c,e,h,i,k\}$	$2,2,2,2,2$
$\mathscr{D}_1 \cdot 11$	$\{\}$		$\mathscr{D}_2 \cdot 11$	$\{\}$	
$\mathscr{D}_1 \cdot 100$	$\{b,g,i,k,m\}$	$2,2,2,2,2$	$\mathscr{D}_2 \cdot 100$	$\{c,f,h,i,l\}$	$2,2,2,2,2$
$\mathscr{D}_1 \cdot 1000$	$\{b,f,i,l,m\}$	$2,2,2,2,2$	$\mathscr{D}_2 \cdot 1000$	$\{c,g,h,i,k\}$	$2,2,2,2,2$

Table 1.9 Applying Maes's procedure to φ_W.

Example 20 (Product of substitutions). Let $\mu : A^* \to A^*$ and $\nu : B^* \to B^*$ be two morphisms prolongable respectively on $a \in A$ and $b \in B$. We define the *product* (called *direct product* in Priebe Frank's chapter (this volume)) $\mu \times \nu : A \times B \to \mathscr{B}_d(A \times B)$ where $(\mu \times \nu)(c,d)$ is the picture obtained as the cross product of the two finite words $\mu(c)$ and $\nu(d)$. It is easy to check that the map $\mu \times \nu$ is a morphism prolongable on (a,b). The corresponding fixed point is the cross product of $\mu^\omega(a)$ and $\nu^\omega(b)$. Take the Thue–Morse morphism $f : 0 \mapsto 01, 1 \mapsto 10$ of Example 8 and the Fibonacci morphism $\mathscr{F} : \mathtt{a} \mapsto \mathtt{ab}, \mathtt{b} \mapsto \mathtt{a}$ of Definition 20.

$$f \times \mathscr{F} : (0,\mathtt{a}) \mapsto \begin{array}{|c|c|} \hline (0,\mathtt{b}) & (1,\mathtt{b}) \\ \hline (0,\mathtt{a}) & (1,\mathtt{a}) \\ \hline \end{array}, \quad (0,\mathtt{b}) \mapsto \begin{array}{|c|c|} \hline (0,\mathtt{a}) & (1,\mathtt{a}) \\ \hline \end{array},$$

$$(1,\mathtt{a}) \mapsto \begin{array}{|c|c|} \hline (1,\mathtt{b}) & (0,\mathtt{b}) \\ \hline (1,\mathtt{a}) & (0,\mathtt{a}) \\ \hline \end{array}, \quad (1,\mathtt{b}) \mapsto \begin{array}{|c|c|} \hline (1,\mathtt{a}) & (0,\mathtt{a}) \\ \hline \end{array}.$$

Note that we get a word shape-symmetric with respect to $\mu \times \nu$ if and only if L_μ and L_ν are equal (where the languages are given by Theorem 10). In particular, the product of a morphism by itself gives shape-symmetry. See also [80].

1.7.1 Abstract numeration systems

Abstract numeration systems are based on an infinite (regular) language over a totally ordered alphabet. They are natural generalizations of classical systems such as integer base systems or Pisot systems [16]. Recall that the genealogical ordering was introduced in Definition 22. For a survey chapter introducing abstract numeration systems, see [58].

Definition 36. An *abstract numeration system* (or *ANS* for short) is a triple $\mathscr{S} = (L,A,<)$ where L is an infinite regular[11] language over a totally ordered alphabet $(A,<)$. The map $\mathrm{rep}_\mathscr{S} : \mathbb{N} \to L$ is the one-to-one correspondence mapping $n \in \mathbb{N}$ to the $(n+1)$st word in the genealogically ordered language L, which is called the $\mathscr{S}$-*representation* of n. The $\mathscr{S}$-representation of 0 is the first word in L. The inverse map is denoted by $\mathrm{val}_\mathscr{S} : L \to \mathbb{N}$. If w is a word in L, $\mathrm{val}_\mathscr{S}(w)$ is its $\mathscr{S}$-*numerical value*.

ANS were introduced in [59]. Note that $\mathrm{val}_\mathscr{S}(w)$ is sometimes called the *rank* of w. See, for instance, [26].

Remark 18. A motivation for studying abstract numeration systems is that it is quite convenient to have a regular language of admissible representations. Given a finite word, one can decide in linear time with respect to the length of the entry, using a DFA, whether or not this word is a valid representation.

Another motivation comes from Cobham's theorem about base dependence. See Theorem 8 for its statement in terms of k-recognizable[12] sets of integers. In view of this theorem of Cobham, if a set of integers is recognizable within two "sufficiently different" systems, then this set is ultimately periodic. Moreover, every ultimately periodic set is always k-recognizable for every integer base $k \geq 2$. Therefore, if one

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[11] One could relax the assumption about regularity of the language on which the numeration system is built to encompass a larger framework. Nevertheless, most of the nice properties that we shall present (in particular, the equivalence with morphic words) do not hold without the regularity assumption.

[12] A set $X \subseteq \mathbb{N}$ is k-*recognizable* if $\mathrm{rep}_k(X) \subseteq \{0,\ldots,k-1\}^*$ is recognized by a DFA or, equivalently, if the characteristic sequence of X is k-automatic.

thinks about a possible generalization of this theorem of Cobham, then a minimal requirement is that ultimately periodic sets — in particular $\mathbb{N}$ — should have a set of $\mathscr{S}$-representations which is a regular language.

Example 21 (Integer base system). Let $k \geq 2$ be an integer. Consider the language

$$L = \{\varepsilon\} \cup \{1,\ldots,k-1\}\{0,\ldots,k-1\}^*.$$

The ANS built on L using the natural ordering of the digits in $\{0,\ldots,k-1\}$ is the usual base-k numeration system. Note that we do *not* allow leading zeroes in representations. Indeed, adding leading zeroes would change the length of the word and therefore the ordering (and thus the value) of this word. Recall that, in the genealogical ordering, words are first ordered with respect to their length.

Example 22 (Unambiguous integer base system). Let $k \geq 2$ be an integer. Consider the language
$$U = \{1,\ldots,k\}^*.$$

As the reader may observe, in this system, the digit set is $\{1,\ldots,k\}$ instead of $\{0,\ldots,k-1\}$. Therefore, we avoid any discussion about possible leading zeroes. For $k = 2$, the first few words in the ordered language U, using the natural ordering of $\{1,\ldots,k\}$, are

$$\varepsilon <_{gen} 1 <_{gen} 2 <_{gen} 11 <_{gen} 12 <_{gen} 21 <_{gen} 22$$

$$<_{gen} 111 <_{gen} 112 <_{gen} 121 <_{gen} 122 <_{gen} \cdots .$$

Let $\mathscr{U}$ be the ANS built on U. Note that if $c_\ell \cdots c_0$ is a word over $\{1,\ldots,k\}$, then

$$\mathrm{val}_{\mathscr{U}}(c_\ell \cdots c_0) = \sum_{i=0}^{\ell} c_i k^i .$$

Let $n \geq 0$. In particular, note that $\mathrm{val}_{\mathscr{U}}(k^n) = k\frac{k^n-1}{k-1}$ and the next word in the genealogical ordering, i.e., the first word of the next length, gives $\mathrm{val}_{\mathscr{U}}(1^{n+1}) = \frac{k^{n+1}-1}{k-1}$. For more about unambiguous systems, see [50, 51].

Example 23. We can reconsider Example 11. We have $\mathrm{rep}_F(\mathbb{N}) = 1\{0,01\}^* \cup \{\varepsilon\}$. Hence, ordering the words of this regular language gives an abstract numeration system. Because of Proposition 7, this remark can be applied to every positional numeration system whose language of U-representations is regular.

Example 24. Consider $L = \mathtt{a}^*\mathtt{b}^*$ with $\mathtt{a} < \mathtt{b}$ and the ANS $\mathscr{S} = (L, \{\mathtt{a},\mathtt{b}\}, <)$. The first few words in L in increasing genealogical order are

$$\varepsilon <_{gen} \mathtt{a} <_{gen} \mathtt{b} <_{gen} \mathtt{aa} <_{gen} \mathtt{ab} <_{gen} \mathtt{bb} <_{gen} \mathtt{aaa} <_{gen} \mathtt{aab} <_{gen} \mathtt{abb} <_{gen} \mathtt{bbb} <_{gen} \cdots .$$

For example, $\mathrm{val}_{\mathscr{S}}(\mathtt{abb}) = 8$ and $\mathrm{rep}_{\mathscr{S}}(3) = \mathtt{aa}$. If we consider the bijection from L to $\mathbb{N}^2$ mapping the word $\mathtt{a}^i\mathtt{b}^j$ to the pair (i,j), $i,j \geq 0$, it is not difficult to see

that the genealogical ordering of L corresponds to the primitive recursive *Peano enumeration* of $\mathbb{N}^2$, that is

$$\mathrm{val}_{\mathscr{S}}(\mathtt{a}^i\mathtt{b}^j) = \frac{1}{2}(i+j)(i+j+1) + j = \binom{i+j+1}{2} + \binom{j}{1}. \tag{1.15}$$

Many papers are dedicated to numeration systems satisfying conditions of the form (1.15). For more on these combinatorial numeration systems, see [25] and the references therein, in particular [53].

Example 25 (Prefix-closed language). Consider the prefix-closed language

$$\{\mathtt{a},\mathtt{ba}\}^*\{\varepsilon,\mathtt{b}\}.$$

When considering such a language ordered by genealogical order, the nth level of the trie contains all words of L of length n in lexicographic order from left to right assuming that the children of a node are also ordered with respect to the ordering of the alphabet. To enumerate the words in the language: proceed one level at a time, from left to right. In Figure 1.16 we represent the first four levels of the corresponding *trie*, i.e., a rooted tree where the edges are labeled by letters from A, and the nodes are labeled by prefixes of words in the considered language L. Let $u \in A^*$, $a \in A$. If ua is (a prefix of) a word in L, then there is an edge between u and ua. Note that for a prefix-closed language L, all prefixes of words in L belong to L. In the nodes, we have written the $\mathscr{S}$-numerical value of the corresponding words in L. The root is associated with ε. See also [64, 65].
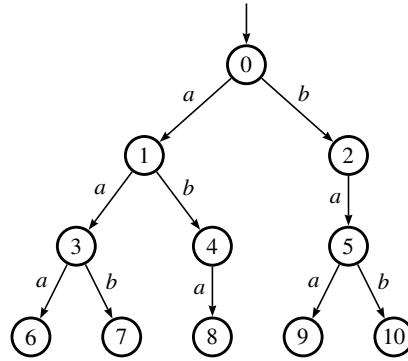
Fig. 1.16 A trie for words of length ≤ 3 in $\{\mathtt{a},\mathtt{ba}\}^*\{\varepsilon,\mathtt{b}\}$.

In Section 1.4 and in particular, with Theorem 5, we have seen that k-automatic sequences can be obtained by feeding a DFAO with base-k expansions of integers. Now that we have generalized numeration systems, we can feed a DFAO with $\mathscr{S}$-representations of integers.

Definition 37. Let $\mathscr{S} = (L, A, <)$ be an ANS. We say that an infinite word $\mathbf{x} = x_0 x_1 x_2 \cdots \in B^{\mathbb{N}}$ is $\mathscr{S}$*-automatic*, if there exists a DFAO $(Q, q_0, A, \delta, \mu : Q \to B)$ such that $x_n = \mu(\delta(q_0, \mathrm{rep}_{\mathscr{S}}(n)))$ for all $n \geq 0$.

This notion was introduced in [70, 75]. For an intermediate notion, see [84].

Example 26. Let $k \geq 2$. Every k-automatic sequence is $\mathscr{S}$-automatic for the ANS introduced in Example 21.

Example 27. We consider the alphabets $A = \{\mathtt{a}, \mathtt{b}\}$, $B = \{0, 1, 2, 3\}$, the ANS $\mathscr{S} = (\mathtt{a}^* \mathtt{b}^*, A, \mathtt{a} < \mathtt{b})$ of Example 24 and the DFAO depicted in Figure 1.17. We obtain
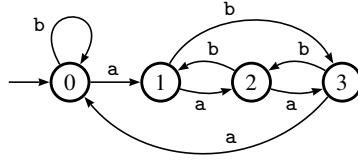
Fig. 1.17 A DFAO with output alphabet $\{0, 1, 2, 3\}$.

the first few terms of the corresponding $\mathscr{S}$-automatic sequence

$$\mathbf{x} = 0102303120023101012302303120312023100231012\cdots.$$

Notice that taking another ANS such as $\mathscr{R} = (\{\mathtt{a}, \mathtt{ba}\}^* \{\varepsilon, \mathtt{b}\}, \{\mathtt{a}, \mathtt{b}\}, \mathtt{a} < \mathtt{b})$, we obtain with the same DFAO another infinite word $\mathbf{y} = 01023\underline{1}31\underline{023}\cdots$ which is $\mathscr{R}$-automatic (underlined letters indicate the differences between $\mathbf{x}$ and $\mathbf{y}$). This stresses the fact that a $\mathscr{S}$-automatic sequence really depends on two ingredients: an ANS and a DFAO.

Theorem 10 can directly be restated in terms of ANS: morphic words are $\mathscr{S}$-automatic for some ANS $\mathscr{S}$. This is just a question of terminology. We state it in the case of a pure morphic word. It is not difficult to add an extra coding taking into account in the output function of the corresponding DFAO.

Theorem 16. *Let $(A, <)$ be an ordered alphabet. Let $\mathbf{w} \in A^{\mathbb{N}}$ be an infinite word, fixed point $f^{\omega}(a)$ of a morphism $f : A^* \to A^*$ prolongable on a. Consider the language L_f of words accepted by the automaton $\mathscr{A}_f$ associated with f, except those starting with 0. Then the word $\mathbf{w}$ is $\mathscr{S}$-automatic for the ANS $\mathscr{S} = (L_f, A, <)$ and the DFAO $\mathscr{A}_F$.*

Now we turn to the converse of Theorem 16.

Proposition 9. *Let $\mathscr{S}$ be an ANS. Every $\mathscr{S}$-automatic sequence is a morphic word.*

Proof. Let $\mathscr{S} = (L, A, <)$ be an ANS. Let $\mathscr{A} = (Q, q_0, A, \delta_{\mathscr{A}}, F)$ be a complete DFA accepting L. Let $\mathscr{B} = (R, r_0, A, \delta_{\mathscr{B}}, \mu : R \to B)$ be a DFAO generating an $\mathscr{S}$-automatic sequence $\mathbf{x} = (x_n)_{n \geq 0}$ over B, i.e., for all $n \geq 0$, $x_n = \mu(\delta_{\mathscr{B}}(r_0, \mathrm{rep}_{\mathscr{S}}(n)))$.

Consider the Cartesian product automaton $\mathscr{P} = \mathscr{A} \times \mathscr{B}$ defined as follows. The set of states of $\mathscr{P}$ is $Q \times R$. The initial state is (q_0, r_0) and the alphabet is A. For any word $w \in A^*$, the transition function $\Delta : (Q \times R) \times A^* \to Q \times R$ is given by

$$\Delta((q, r), w) = (\delta_{\mathscr{A}}(q, w), \delta_{\mathscr{B}}(r, w)).$$

This means that the product automaton mimics the behaviors of both $\mathscr{A}$ and $\mathscr{B}$ in a single automaton. In particular, after reading w in $\mathscr{P}$, $\Delta((q_0, r_0), w)$ belongs to $F \times R$ if and only if w belongs to L. Moreover if $\mathrm{rep}_{\mathscr{S}}(n) = w$ and $\Delta((q_0, r_0), w) = (q, r)$, then $x_n = \mu(r)$.

Now we can apply Lemma 2 to $\mathscr{P}$ and define a morphism $\psi_{\mathscr{P}}$ prolongable on a letter $\mathbf{z}$ which does not belong to $Q \times R$. In view of the previous paragraph, we define $\nu : ((Q \times R) \cup \{\mathbf{z}\})^* \to B^*$ by

$$\nu(q, r) = \begin{cases} \mu(r), & \text{if } q \in F; \\ \varepsilon, & \text{otherwise;} \end{cases}$$

and $\nu(\mathbf{z}) = \varepsilon$. As Lemma 2 can be used to describe the sequence of reached states, $\nu(\psi_{\mathscr{P}}(\mathbf{z}))$ is exactly the sequence $(x_n)_{n \geq 0}$. This proves that $\mathbf{x}$ is morphic. $\square$

Note that the morphisms obtained at the end of this proof are erasing. This is not a problem thanks to Theorem 4.

1.7.2 Multidimensional $\mathscr{S}$-automatic sequences

We have something similar to Definition 26 but since the digit 0 modifies the length of a word (see Examples 21 and 22), we use an extra padding symbol not in the original alphabet.

Definition 38. If $w_1, \ldots, w_d$ are finite words over the alphabet A, the *padding map* $(\cdot)^{\#} : (A^*)^d \to ((A \cup \{\#\})^d)^*$ is defined as

$$(w_1, \ldots, w_d)^{\#} := (\#^{m - |w_1|} w_1, \ldots, \#^{m - |w_d|} w_d)$$

where $m = \max\{|w_1|, \ldots, |w_d|\}$. For all $n_1, \ldots, n_d \geq 0$, we set

$$\mathrm{rep}_{\mathscr{S}}(n_1, \ldots, n_d) := (\mathrm{rep}_{\mathscr{S}}(n_1), \ldots, \mathrm{rep}_{\mathscr{S}}(n_d))^{\#}.$$

As an example, $(ab, bbaa)^{\#} = (\#\#ab, bbaa)$.

Definition 39. An infinite d-dimensional word $\mathbf{x} \in B^{\mathbb{N}}$ is $\mathscr{S}$-*automatic* for an abstract numeration system $\mathscr{S} = (L, A, <)$, if there exists a DFAO $\mathscr{A} = (Q, q_0, (A \cup \#)^d, \delta, \tau : Q \to B)$ such that, for all $n_1, \ldots, n_d \geq 0$,

$$\tau(\delta(q_0, \mathrm{rep}_{\mathscr{S}}(n_1, \ldots, n_d))) = \mathbf{x}_{n_1, \ldots, n_d}.$$

In this case, we say that the DFAO $\mathscr{A}$ generates the infinite word $\mathbf{x}$.

Example 28. Consider the ANS $\mathscr{S} = (\{a, ba\}^*\{\varepsilon, b\}, \{a, b\}, a < b)$ and the DFAO depicted in Figure 1.18. Since this automaton is fed with entries of the form $(\mathrm{rep}_{\mathscr{S}}(n_1), \mathrm{rep}_{\mathscr{S}}(n_2))^{\#}$, we do not consider the transitions on input $(\#, \#)$. If the
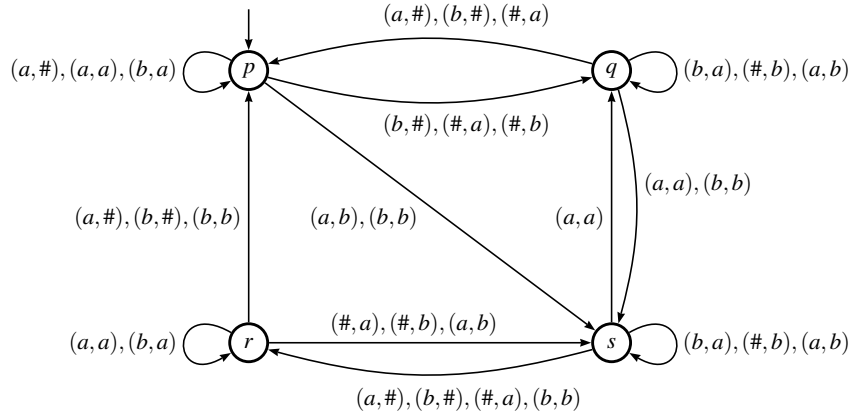
Fig. 1.18 A deterministic finite automaton with output.

outputs of the DFAO are considered to be the states themselves, then the DFAO generates the bidimensional infinite $\mathscr{S}$-automatic word given in Figure 1.19.

	ε	a	b	aa	ab	ba	aaa	aab	
aab	q	p	s	p	s	s	p	s	
aaa	p	p	s	p	s	q	p	s	
ba	p	s	q	p	s	q	s	q	
ab	q	p	s	p	s	s	s	r	
aa	p	p	s	p	s	q	q	s	
b	q	p	s	q	s	q	p	s	
a	p	p	s	s	q	s	p	s	
ε	p	q	q	p	q	p	q	q	$\cdots$

Fig. 1.19 A bidimensional infinite $\mathscr{S}$-automatic word.

The main goal of [22] is to prove the following result, an extension of Salon's theorem.

Theorem 17. *Let $d \geq 1$. The d-dimensional infinite word $\mathbf{x}$ is $\mathscr{S}$-automatic for some abstract numeration system $\mathscr{S} = (L, A, <)$ where $\varepsilon \in L$ if and only if $\mathbf{x}$ is the image by a coding of a shape-symmetric infinite d-dimensional word.*

This result is not surprising: we have seen in the proof of Theorem 14 that an infinite word is shape-symmetric with respect to a morphism μ if and only if the languages L_{μ_i} are all equal. But to define a $\mathscr{S}$-automatic word, we use the same ANS for the representation of every component.

1.7.3 Proof of Theorem 15

Recall that φ_W was defined in Example 19.

Proof (Proof of Theorem 15). We associate with φ_W the DFA with input alphabet (1.9) depicted in Figure 1.20. For the sake of readability, some labels are omitted. Ingoing transitions to state i (resp. m, d, h) all have label $\binom{0}{0}$ (resp. $\binom{1}{0}$, $\binom{1}{1}$, $\binom{0}{1}$). For r, s, t, u, v belonging to the 13-letter alphabet $\{a, b, \ldots, m\}$, if

$$\varphi_W(r) = \begin{array}{|c|c|} \hline u & v \\ \hline s & t \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline s & t \\ \hline \end{array}, \quad \begin{array}{|c|} \hline u \\ \hline s \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|} \hline s \\ \hline \end{array}$$

we have transitions like

$$r \xrightarrow{\binom{0}{0}} s, \quad r \xrightarrow{\binom{1}{0}} t, \quad r \xrightarrow{\binom{0}{1}} u, \quad r \xrightarrow{\binom{1}{1}} v.$$

From Table 1.9 and Theorem 14, we already know that φ_W is a morphism prolongable on a.

First observe that, if all states are assumed to be final, this automaton accepts the words

$$\binom{u}{v}$$

where $|u| = |v|$ and u, v are both valid F-representation (possibly padded with leading zeroes to get two words of the same length).

Second, if we restrict to the "black" part (still assuming $a, b, c, e, f, g, j, k, l$ to be final), the automaton accepts exactly the words

$$\binom{0w_1 \cdots w_\ell}{w_1 \cdots w_\ell 0} \quad \text{and} \quad \binom{w_1 \cdots w_\ell 0}{0w_1 \cdots w_\ell}$$

where $w_1 \cdots w_\ell$ is a valid F-representation.

Finally, taking into account the coding given in Example 19, the set of final states is $\{a, e, g, j, l\}$. We have the extra acceptance condition that $w_1 \cdots w_\ell$ ends with an
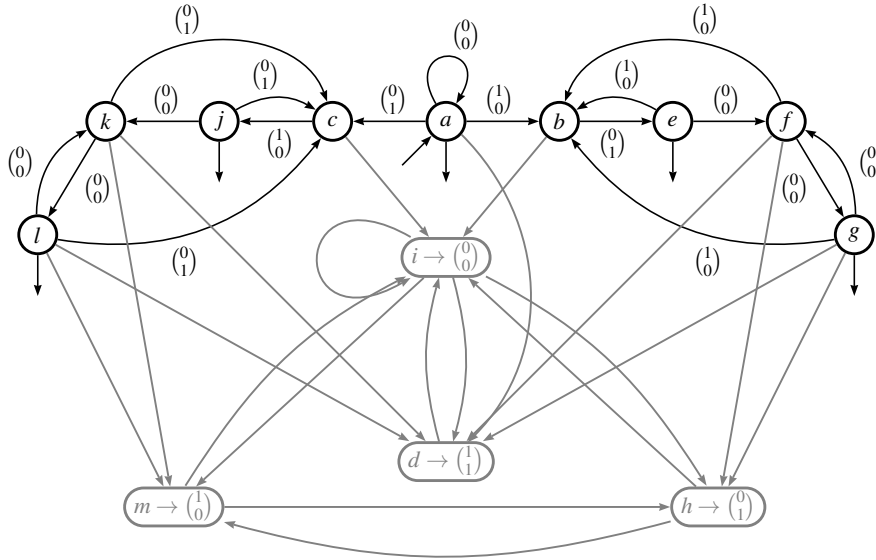
Fig. 1.20 The DFA associated with the morphism φ_W.

even number of zeroes. With our previous characterization of $\mathscr{P}$-positions given by Theorem 11, this concludes the proof. $\quad\square$

As a concluding remark, we try to answer the following question. The reader may wonder how we got the morphism φ_W having only access to Table 1.5. We considered some kind of "reverse engineering" strategy. We first conjectured that the Fibonacci word is playing a role (clear from Theorem 11). Hence, we cut the bidimensional word using, on both axis, the directive sequence $2, 1, 2, 2, 1, 2, 1, 2, 2, 1, \ldots$ deduced from the Fibonacci word $\mathbf{f}'$ (1.5), see Table 1.10. It produces pictures of shape $(2,2)$, $(2,1)$, $(1,2)$ and $(1,1)$. If a morphism φ_W exists, then every time we see a given symbol, its image must have a constant shape (same observation for iterates). Thus by looking at the future of a symbol, we may distinguish several types of 0's and 1's. With this heuristic, we get a finite number of candidates for symbols and images providing the morphism φ_W. For instance the first three 1's in positions $(0,0)$, $(1,2)$ and $(2,1)$ must correspond to different symbols. Indeed, they should give rise to images of respective shape $(2,2)$, $(2,1)$ and $(1,2)$.

1.8 Games with a finite set of moves

In the very first Example 1, we were considering a single pile of tokens and subtraction games where only finitely many moves are available. Note that Nim does not

	0	1	2	3	4	5	6	7	8	9	10
10	0	0	0	0	0	0	1	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	1	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	1
5	0	0	0	1	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	1	0	0	0
3	0	0	0	0	0	1	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9	10

Table 1.10 The bidimensional word $\mathbf{w}(m,n)$ coding $\mathscr{P}$-positions of Wythoff.

belong to this category, one can remove an unbounded number of tokens (whenever available). In this short section, we introduce finite subtraction games on several piles. For one pile, the situation is completely understood [85].

Proposition 10. *Every finite subtraction game on one pile, i.e., the set $I \subset \mathbb{N}$ of moves is finite, has an ultimately periodic Sprague–Grundy function.*

Proof. This is a classical pigeonhole principle argument. Let $m = \#I$ be the maximal number of options for any position. Hence $\mathscr{G}(n) \leq m$ for all n. Let $k = \max I$. Hence, from position $n \geq k$, the options are in $\{n-k, \ldots, n-1\}$. There are $(m+1)^k$ possible k-tuples taking values in $\{0, \ldots, m\}$. Since $\mathscr{G}(n)$ depends only on $\mathscr{G}(n-\ell)$ for $1 \leq \ell \leq k$, hence, by pigeonhole principle, there exist $i < j$ such that

$$\mathscr{G}(i+n) = \mathscr{G}(j+n) \text{ for all } n \in \{0, \ldots, k-1\}.$$

Thus $j - i$ is a period of $\mathscr{G}$ with preperiod i. □

Another similar result is given in [85, p. 188].

Proposition 11. *Consider a finite subtraction game on one pile with $I \subset \mathbb{N}$ as set of moves. If there exist $N \geq 0$ and $p \geq 1$ such that*

$$\mathscr{G}(n+p) = \mathscr{G}(n), \quad \forall n \in \{N, \ldots, N-1+\max I\}$$

then $\mathscr{G}(n+p) = \mathscr{G}(n)$ for all $n \geq N$.

If we may optionally split a pile, the situation is more intricate. It gives rise to the notion of an *octal game* which is played with tokens divided into piles. Two players take turns moving until no moves are possible. The name came from the fact that the rules are coded by words over $\{0, \ldots, 7\}$.

Definition 40 (Octal game). Every move consists of selecting just one of the piles, and either

- removing all of the tokens in the pile, leaving no pile,
- removing some but not all of the tokens, leaving one smaller pile, or
- removing some of the tokens and dividing the remaining tokens into two nonempty piles.

Piles other than the selected pile remain unchanged. The last player to move wins in a normal play convention.

The coding of an octal game (known as *Conway code*) is an infinite word

$$d_0 \bullet d_1 d_2 d_3 \cdots \quad d_i \in \{0, \ldots, 7\}$$

where d_i written in base 2 has a fixed 3-digit length, $e_2^{(i)} e_1^{(i)} e_0^{(i)} \in \{0,1\}^3$. It gives the conditions under which i tokens may be removed.

- if $e_0^{(i)} = 1$, then a (full) pile with i tokens can be suppressed;
- if $e_1^{(i)} = 1$, then a pile with $n > i$ tokens can be replaced with a pile with $n - i$ tokens left;
- if $e_2^{(i)} = 1$, then a pile with $n > i + 1$ tokens can be replaced with two piles containing respectively a and b tokens, $a, b \geq 1$, $a + b = n - i$.

Example 29. The game of Nim on an arbitrary number of piles is coded by $0 \bullet 3333 \cdots$. Indeed, with a leading zero $\mathrm{rep}_2(3) = 011$. In general, classical subtraction games where a pile cannot be split into two piles is coded by a word over $\{0, 1, 2, 3\}$.

A subtraction game is finite if and only if it is coded by a finite word (i.e., an infinite word with only finitely many non-zero digits), e.g. with a set of moves $I = \{3, 5, 6\}$, the game is coded by $0 \bullet 003033$. The game of Example 1 is coded by $0 \bullet 3303$.

Theorem 18 (Octal game periodicity [85]). *Consider a finite octal game coded by* $d_0 \bullet d_1 d_2 \cdots d_k$ *with* $d_k \neq 0$. *If there exist* $N \geq 0$ *and* $p \geq 1$ *such that*

$$\mathscr{G}(n + p) = \mathscr{G}(n), \quad \forall n \text{ with } N \leq n < 2N + p + \max I$$

then $\mathscr{G}(n + p) = \mathscr{G}(n)$ *for all* $n \geq N$.

A general open problem is to determine whether all finite octal games have an ultimately periodic Grundy function. For instance, $0 \bullet 07$ has period 34 and preperiod 53; $0 \bullet 165$ has period 1550 and preperiod 5181; $0 \bullet 106$ has period $\simeq 3.10^{11}$ and preperiod $\simeq 4.10^{11}$. Up to our knowledge, $0 \bullet 007$ has no known periodicity. See [85, 69, 10].

In view of Proposition 10, one can conjecture that for a finite subtraction game on two (or more) piles of tokens, the Sprague–Grundy function should be definable in the Presburger arithmetic $\langle \mathbb{N}, + \rangle$ or, equivalently, each value of $\mathscr{G}$ should correspond to a semilinear set in $\mathbb{N}^2$. Indeed, this is exactly the situation encountered when considering generalizations to $\mathbb{N}^d$ of Cobham's theorem (Theorem 8) to Cobham–Semenov theorem [17]. See the latter survey for precise definitions.

Definition 41. A set X of $\mathbb{N}^n$ is *linear* if there exist $v_0, v_1, \ldots, v_k \in \mathbb{N}^n$ such that

$$X = v_0 + \mathbb{N}v_1 + \cdots + \mathbb{N}v_k.$$

The vectors $v_1, \ldots, v_k$ are usually called the *periods* of X. A set X of $\mathbb{N}^n$ is *semilinear* if it is a finite union of linear sets.

The next facts follow from a work in progress with X. Badin De Montjoye, V. Gledel, V. Marsault and A. Massuir.

Proposition 12. *Every finite subtraction game on two piles with at most two moves, has a Sprague–Grundy function definable in $\langle \mathbb{N}, + \rangle$.*

In Figure 1.21, we have represented $\mathscr{G}$-value 0 in white. Darker points correspond to higher Grundy values. In this figure, we have considered two piles of tokens with the moves $\alpha = (2,1)$ and $\beta = (3,5)$. The reader may notice that $\alpha + \beta$ is a vector of period. On the right of Figure 1.21, we have depicted an example with 3 moves $(1,3)$, $(3,1)$ and $(4,4)$.
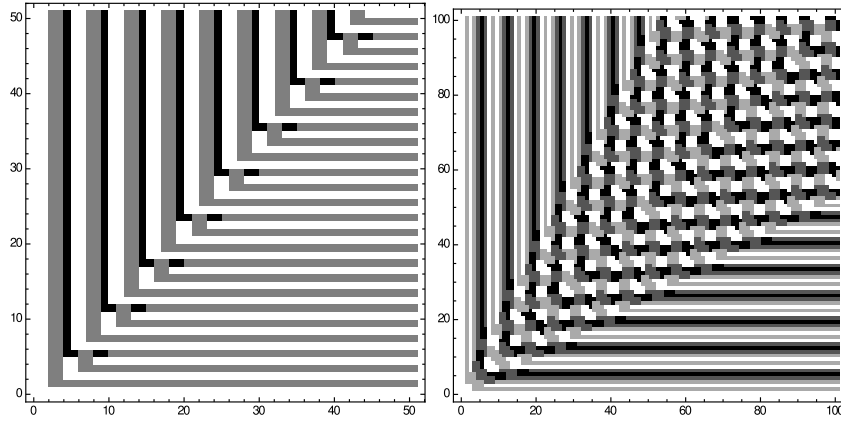
Fig. 1.21 First values of $\mathscr{G}$ with $I = \{(2,1),(3,5)\}$ (left) and $I = \{(1,3),(3,1),(4,4)\}$ (right).

Proof. If there is a single move, the result is clear. Assume that we have the moves $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. The first $2\max(\alpha_1, \beta_1)$ columns of $(\mathscr{G}(m,n))_{m,n \geq 0}$ are ultimately periodic. Similarly, the first $2\max(\alpha_2, \beta_2)$ rows are ultimately periodic. We show that $\alpha + \beta$ is a vector of period. Let $x \in \mathbb{N}^2$ be a position such that $x - \alpha$ and $x - \beta$ belong to $\mathbb{N}^2$. We will prove that

$$\mathscr{G}(x) = \mathscr{G}[x + \alpha + \beta].$$

Assume that $\mathscr{G}(x) = 0$ and $\mathscr{G}[x + \alpha + \beta] = 2$. From the position $x + \alpha + \beta$, we can either subtract α or β. By definition of the Grundy function, $\{\mathscr{G}[x+\alpha], \mathscr{G}[x+\beta]\} =$

$\{0,1\}$. But this contradicts the fact that $\mathscr{G}(x) = 0$; we would have two consecutive positions x and either $x+\alpha$ or $x+\beta$ with the same $\mathscr{G}$-value. The reasoning is similar if we assume that $\mathscr{G}[x+\alpha+\beta] = 1$.

Assume that $\mathscr{G}(x) = 1$ and $\mathscr{G}[x+\alpha+\beta] = 2$. From the last equality, we have $\{\mathscr{G}[x+\alpha], \mathscr{G}[x+\beta]\} = \{0,1\}$. One of these two positions leads directly to x. This is a contradiction.

Assume that $\mathscr{G}(x) = 1$ and $\mathscr{G}[x+\alpha+\beta] = 0$. These equalities imply that $\mathscr{G}[x+\alpha] = \mathscr{G}[x+\beta] = 2$. Then, considering the options of these last two positions, we deduce that $\mathscr{G}[x+\alpha-\beta] = \mathscr{G}[x+\beta-\alpha] = 0$. Since, $\mathscr{G}(x) = 1$, either $\mathscr{G}[x-\beta] = 0$ or, $\mathscr{G}[x-\alpha] = 0$ contradicting the previous equality.

Assume that $\mathscr{G}(x) = 2$ and $\mathscr{G}[x+\alpha+\beta] = 0$. This implies that $\mathscr{G}[x+\alpha] = \mathscr{G}[x+\beta] = 1$. Then, considering the options of these two positions, we deduce that $\mathscr{G}[x+\alpha-\beta] = \mathscr{G}[x+\beta-\alpha] = 0$. Since, $\mathscr{G}(x) = 2$, then $\{\mathscr{G}[x-\beta], \mathscr{G}[x-\alpha]\} = \{0,1\}$ contradicting the previous equality. The reasoning is similar if we assume that $\mathscr{G}[x+\alpha+\beta] = 1$. $\square$

We conjecture that with two piles of tokens and the set of moves

$$I = \{(1,2), (2,1), (3,5), (5,3), (2,2)\},$$

the corresponding word $(\mathscr{G}(m,n))_{m,n\geq 0}$ is not definable in $\langle \mathbb{N}, + \rangle$. We can therefore wonder how Proposition 10 could be generalized to two or more piles of tokens.

1.9 Bibliographic notes

The link between morphisms and automata which is a cornerstone of this chapter can already be found in the fundamental work of Cobham [31]. In this chapter, we did not present the Dumont–Thomas approach relating substitutions to numeration systems. See [37]. The papers [64, 65] also develop similar constructions. Several surveys are of interest, see [17] for integer base systems, [71] where connections with games are also mentioned.

A few papers are dealing with combinatorial games linked with morphic words. See [34, 35, 36, 46, 56]. In particular, Theorem 15 was proved in [33]. For instance, we have also considered alterations (adding/removing moves) of the set of moves in order to keep the same set of $\mathscr{P}$-positions as the original game. We may characterize moves that can be adjoined without changing the $\mathscr{P}$-positions of Wythoff's game. No move is redundant. The notion of *invariant game* is introduced in [36]. About Wythoff's game, see [42] and [45]. For connections between games and Beatty sequences, see [36] and then [56, 20].

We did not discuss much about the logical characterization of k-automatic sequences (and generalizations to Pisot systems). See again [17], [24] and also the last chapter of [73] for a comprehensive introduction. Maes's motivations were primarily set on decidability of arithmetic theories: which expansions of $\langle \mathbb{N}, < \rangle$ by mor-

phic predicates or automata are decidable? See also [19] where it is shown that for a morphic predicate P the associated monadic second-order theory $MTh\langle\mathbb{N}, <, P\rangle$ is decidable. A trace of abstract numeration system (not mentioned in these terms at that time) can already be found in Maes's thesis [63, Rem. 6.9, p. 134]: "*The set of codes of* $\mathbb{N}$ *given by the above automaton is of course a regular language... The language read by A is* 0^*L. *However, the above coding is* not *a numeration system in the sense of [16]. Indeed, the representation of a natural number is not obtained using a 'Euclidean division' algorithm.*". In some sense, Maes was conjecturing Theorem 16 and Proposition 9: the set of morphic words is equal to the set of $\mathscr{S}$-automatic words, for some ANS $\mathscr{S}$.

Acknowledgments

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