Stochastic study of unidirectional composites: from micro-structural statistical information to multi-scale analyzes

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Multi-scale modelling

- **Two-scale modelling**
  - One way: homogenization
  - 2 problems are solved (concurrently)
    - The macro-scale problem
    - The meso-scale problem (on a meso-scale Volume Element)
The problem

- Material uncertainties affect structural behaviors

![Diagram showing probabilistic homogenization process and associated probability distributions for material properties and failure load.](image)
The problem

- Illustration assuming a regular stacking
  - 60%-UD fibers
  - Damage-enhanced matrix behavior

- Question: what does happen for a realistic fibre stacking?
The problem

- Proposed methodology:

- Experimental measurements
- Micro-structure stochastic model
- Copula \((d_{1st}, \Delta d)\)
- SVE realisations
- Stochastic Homogenization
  - Average strength
  - Variance of strength
- SVE size
- Stochastic reduced model

\[ \omega = U_i \cdot \omega_i \]
Experimental measurements

- 2000x and 3000x SEM images

- Fibers detection
Micro-structure stochastic model

• Basic geometric information of fibers' cross sections
  – Fiber radius distribution $p_R(r)$

• Basic spatial information of fibers
  – The distribution of the nearest-neighbor net distance function $p_{d_{1st}}(d)$
  – The distribution of the orientation of the undirected line connecting the center points of a fiber to its nearest neighbor $p_{\theta_{1st}}(\theta)$
  – The distribution of the difference between the net distance to the second and the first nearest-neighbor $p_{\Delta d}(d)$ with $\Delta d = d_{2nd} - d_{1st}$
  – The distribution of the second nearest-neighbor’s location referring to the first nearest-neighbor $p_{\Delta \theta}(\theta)$ with $\Delta \theta = \theta_{2nd} - \theta_{1st}$
Micro-structure stochastic model

- Histograms of random micro-structures’ descriptors

![Histograms of random micro-structures’ descriptors](image)
Micro-structure stochastic model

- Dependency of the four random variables $d_{1st}, \Delta d, \vartheta_{1st}, \Delta \vartheta$
- Correlation matrix

<table>
<thead>
<tr>
<th></th>
<th>$d_{1st}$</th>
<th>$\Delta d$</th>
<th>$\vartheta_{1st}$</th>
<th>$\Delta \vartheta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_{1st}$</td>
<td>1.0</td>
<td>0.21</td>
<td>0.01</td>
<td>0.02</td>
</tr>
<tr>
<td>$\Delta d$</td>
<td>1.0</td>
<td>1.0</td>
<td>0.002</td>
<td>-0.005</td>
</tr>
<tr>
<td>$\vartheta_{1st}$</td>
<td>1.0</td>
<td>0.05</td>
<td>1.0</td>
<td>0.02</td>
</tr>
<tr>
<td>$\Delta \vartheta$</td>
<td></td>
<td></td>
<td></td>
<td>1.0</td>
</tr>
</tbody>
</table>

- Distances correlation matrix

$d_{1st}$ and $\Delta d$ are dependent, they will be generated from their empirical copula
Micro-structure stochastic model

- $d_{1st}$ and $\Delta d$ should be generated using their empirical copula

**SEM sample**

**Generated sample**

Directly from copula generator

Statistic result from generated SVE
• The numerical micro-structure is generated by a fiber additive process
  1) Define $N$ seeds with first and second neighbors distances
• The numerical micro-structure is generated by a fiber additive process
  1) Define $N$ seeds with first and second neighbors distances
  2) Generate first neighbor with its own first and second neighbors distances
• The numerical microstructure is generated by a fiber additive process
  1) Define $N$ seeds with first and second neighbors distances
  2) Generate first neighbor with its own first and second neighbors distances
  3) Generate second neighbor with its own first and second neighbors distances
• The numerical micro-structure is generated by a fiber additive process

1) Define $N$ seeds with first and second neighbors distances
2) Generate first neighbor with its own first and second neighbors distances
3) Generate second neighbor with its own first and second neighbors distances
4) Change seeds & then change central fiber of the seeds

Micro-structure stochastic model
Micro-structure stochastic model

- The numerical micro-structure is generated by a fiber additive process
  - The effect of the initial number of seeds $N$ and
  - The effect of the maximum regenerating times $n_{\text{max}}$ after rejecting a fiber due to overlap

**SEM:** Average $V_f$ of 103 windows;
**Numerical micro-structures:** Average $V_f$ of 104 windows.

![Graph showing the relationship between $n_{\text{max}}$ and Fiber Volume fraction for different initial numbers of seeds $N$.](image)
Micro-structure stochastic model

• Comparisons of fibers spatial information
Micro-structure stochastic model

- Numerical micro-structures are generated by a fiber additive process
  - Arbitrary size
  - Arbitrary number

- Possibility to generate non-homogenous distributions
Stochastic homogenization on the SVEs

- **Stochastic homogenization**
  - Extraction of Stochastic Volume Elements
    - 2 sizes considered: \( l_{SVE} = 10 \mu m \) & \( l_{SVE} = 25 \mu m \)
    - Window technique to capture correlation
      \[
      R_{rs}(\tau) = \frac{\mathbb{E}\left[ (r(x) - \mathbb{E}(r))(s(x + \tau) - \mathbb{E}(s)) \right]}{\sqrt{\mathbb{E}\left[ (r - \mathbb{E}(r))^2 \right]} \sqrt{\mathbb{E}\left[ (s - \mathbb{E}(s))^2 \right]}}
      \]
  - For each SVE
    - Extract apparent homogenized material tensor \( \mathbb{C}_M \)
      \[
      \begin{align*}
      \varepsilon_M &= \frac{1}{V(\omega)} \int_\omega \varepsilon_m d\omega \\
      \sigma_M &= \frac{1}{V(\omega)} \int_\omega \sigma_m d\omega \\
      \mathbb{C}_M &= \frac{\partial \sigma_M}{\partial \mathbf{u}_M} \otimes \nabla_M
      \end{align*}
      \]
- Consistent boundary conditions:
  - Periodic (PBC)
  - Minimum kinematics (SUBC)
  - Kinematic (KUBC)
Stochastic homogenization on the SVEs

- **Apparent properties**

  Increasing $l_{SVE}$

  When $l_{SVE}$ increases
  - Average values for different BCs get closer (to PBC one)
  - Distributions narrow
  - Distributions get closer to normal
Stochastic homogenization on the SVEs

- When $l_{SVE}$ increases: marginal distributions of random properties closer to normal
  - $l_{SVE} = 10 \mu m$
  - $l_{SVE} = 25 \mu m$
Stochastic homogenization on the SVEs

- **Correlation**

Increasing $l_{\text{SVE}}$

1. Auto/cross correlation vanishes at $\tau = l_{\text{SVE}}$
2. When $l_{\text{SVE}}$ increases, distributions get closer to normal

(1)+(2) Apparent properties are independent random variables

However, the distribution depends on
   - $l_{\text{SVE}}$
   - The boundary conditions
Stochastic homogenization on the SVEs

- **Quid larger SVEs?**
  - Computational cost affordable in linear elasticity
  - Computational cost non affordable in failure analyzes

- **How to deduce the stochastic content of larger SVEs?**
  - Take advantages of the fact that the apparent tensors can be considered as random variables
Stochastic homogenization on the SVEs

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Stochastic homogenization on the SVEs

- **Quid larger SVEs?**
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- **How to deduce the stochastic content of larger SVEs?**
  - Take advantages of the fact that the apparent tensors can be considered as random variables
  - Accuracy depends on Small/Large SVE sizes

\[
l_{SSVE} = 10 \, \mu m
\]

\[
l_{SSVE} = 25 \, \mu m
\]
Stochastic homogenization on the SVEs

- Numerical verification of 2-step homogenization
  - Direct homogenization of larger SVE (BSVE) realizations
  - 2-step homogenization using BSVE subdivisions
Stochastic reduced order model

- Stochastic model of the anisotropic elasticity tensor
  - Extract (uncorrelated) tensor realizations $C_M^i$
    - Represent each realization $C_M^i$ by a vector $\mathbf{v}$ of 9 (dependant) $v^{(r)}$ variables
    - Generate random vectors $\mathbf{v}$ using the Copula method

- Simulations require two discretizations
  - Random vector discretization
  - Finite element discretization
Stochastic reduced order model

- Ply loading realizations
  - Non-uniform homogenized stress distributions
  - Different realizations yield different solutions
Stochastic Mean-Field Homogenization

- **Mean-Field-homogenization (MFH)**
  - Linear composites
    \[
    \begin{align*}
    \sigma_M &= \bar{\sigma} = \nu_0 \sigma_0 + \nu_I \sigma_I \\
    \varepsilon_M &= \bar{\varepsilon} = \nu_0 \varepsilon_0 + \nu_I \varepsilon_I \\
    \varepsilon_I &= \mathbf{B}^\varepsilon (I, \mathbb{C}_0, \mathbb{C}_I) : \varepsilon_0
    \end{align*}
    \]
  - We use Mori-Tanaka assumption for \( \mathbf{B}^\varepsilon (I, \mathbb{C}_0, \mathbb{C}_I) \)

- **Stochastic MFH**
  - How to define random vectors \( \nu_{MT} \) of \( I, \mathbb{C}_0, \mathbb{C}_I, \nu_I \)?
Mean-Field-homogenization (MFH)

- Linear composites

\[
\begin{align*}
\sigma_M &= \bar{\sigma} = \nu_0 \sigma_0 + \nu_I \sigma_I \\
\varepsilon_M &= \bar{\varepsilon} = \nu_0 \varepsilon_0 + \nu_I \varepsilon_I \\
\varepsilon_I &= B^\varepsilon(I, C_0, C_I): \varepsilon_0
\end{align*}
\]

\[C_M = C_M(I, C_0, C_I, \nu_I)\]

Consider an equivalent system

- For each SVE realization \(i\):
  - \(C_M\) and \(\nu_I\) known
  - Anisotropy from \(C^i_M\)
  - \(\theta\) is evaluated
  - Fibre behaviour uniform
  - \(\hat{C}_I\) for one SVE

- Remaining optimization problem:

\[
\min_{\frac{a}{b}, \hat{E}_0, \hat{\nu}_0} \| C_M - \hat{C}_M(\frac{a}{b}, \hat{E}_0, \hat{\nu}_0; \nu_I, \theta, \hat{C}_I) \|
\]
• Inverse stochastic identification
  – Comparison of homogenized properties from SVE realizations and stochastic MFH

\[ C_M \approx \hat{C}_M(\hat{I}, \hat{C}_0, \hat{C}_I, \nu_I, \theta) \]
Stochastic Mean-Field Homogenization

- **Stochastic MFH model:**
  - Homogenized properties \( \mathbb{C}_M(\frac{a}{b}, E_0, \nu_0; \mathbb{C}_I) \)
  - Random vectors \( \mathcal{V}_{MT} \)
    - Realizations \( v_{MT} = \{ \frac{a}{b}, E_0, \nu_0, \nu_I, \theta \} \)
    - Characterized by the distance correlation matrix
    - Generator using the copula method

<table>
<thead>
<tr>
<th></th>
<th>( \nu_I )</th>
<th>( \theta )</th>
<th>( \frac{a}{b} )</th>
<th>( \hat{E}_0 )</th>
<th>( \hat{\nu}_0 )</th>
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<td>( \nu_I )</td>
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<td>( \frac{a}{b} )</td>
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<td>( \hat{E}_0 )</td>
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<tr>
<td>( \hat{\nu}_0 )</td>
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**Distances correlation matrix**
Stochastic Mean-Field Homogenization

- **Stochastic simulations**
  - 2 discretization: Random field $V_{MT}$ & Stochastic finite-elements

- Realizations to reach a given deflection $\delta$
Conclusions

- Stochastic generator based on SEM measurements of unidirectional fibre reinforced composites
- Computational homogenization on SVEs
- Two-step computational homogenization for Big SVEs
- Definition of a Stochastic MFH method
- In progress: nonlinear and failure analyzes
Thank you for your attention!