Results of genericity concerning ultradifferentiable classes

Céline ESSER
Celine.Esser@uliege.be

University of Liège – Institute of Mathematics

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Introduction

A $\mathcal{C}^\infty$ function f is analytic at $x_0 \in \Omega$ if its Taylor series at x_0 converges to f on an open neighbourhood of x_0. Using Cauchy's estimates, it is equivalent to have the existence of a compact neighborhood $K \subseteq \Omega$ of x_0 and of two constants $C, h > 0$ such that

$$\sup_{x\in K} \left|D^k f(x)\right| \leq C h^k k!\,, \quad \forall k \in \mathbb{N}.$$

We say that a function is nowhere analytic if it is not analytic at any point.

Many examples of $\mathcal{C}^\infty$ nowhere analytic functions exist. An example was given by Cellérier (1890) with the function defined for all $x \in \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{+\infty} \frac{\sin(a^n x)}{n!}$$

where a is a positive integer larger than 1.

Questions:

- How large is the set of nowhere analytic functions in the Fréchet space $\mathcal{C}^\infty(\mathbb{R})$?
- Is it possible to construct large structures of nowhere analytic functions?

The first result in this direction has been obtained using the notion of Baire genericity.

Definition

Let X be a Baire space. A subset M of X is residual (or comeager) in X if M contains a countable intersection of dense open sets in X.

Theorem (Morgenstern, 1954)

The set of nowhere analytic functions in residual in $\mathcal{C}^\infty(\mathbb{R})$.

Other proofs and related results:

- Salzmann and Zeller 1955, Cater 1984, Sarst 1973, Bernal-González 1987, Darji and Swanson 2016.

Large set from a measure point of view

The prevalence is a notion introduced to generalize the concept of "Lebesgue almost everywhere" to infinite dimensional spaces keeping some properties:

- A measure zero set has empty interior (ie."almost every" implies density).
- Every subset of a measure zero set has measure zero.
- A countable union of measure zero sets has measure zero.
- Every translate of a measure zero set has measure zero.

$\longrightarrow$ impossible to define this notion in terms of a specific measure!

Definition (Christensen 1974 / Hunt, Sauer, Yorke, 1992)

Let X be a complete metrizable linear space. A Borel subset B of X is shy (or Haar-null) if there exists a Borel measure μ on X with compact support such that

$$\mu(B + x) = 0\,, \quad \forall x \in X. \tag{1}$$

More generally, a subset M is called shy if it is contained in a shy Borel set. The complement of a shy set is called a prevalent set.

Useful measure to try:
The Lebesgue measure on the unit ball of a finite dimensional subspace V.
Condition (1) becomes

$$\forall x \in X, \quad (x + B) \cap V \text{ is of Lebesgue measure zero.}$$

In this case, V is called a probe for the complement of B.

Large algebraic structure

Definition (Aron, Gurariy, Seoane-Sepúlveda 2005)

Let X be a vector space. A subset M of X is lineable if $M \cup \{0\}$ contains an infinite dimensional vector subspace.

Genericity

The set of nowhere analytic functions in prevalent and lineable in $\mathcal{C}^\infty(\mathbb{R})$.

Proofs and related results:

- Bernal-González 2008
- Bastin, E., Nicolay 2012
- Conejero, Jiménez-Rodríguez, Muñoz-Fernández and Seoane-Sepúlveda 2012
- Bartoszewicz, Bienias, Filipczak and Głąb 2014
- Bastin, Conejero, E. and Seoane 2014

Nowhere Gevrey functions

Gevrey classes

For a real number $s > 0$, a function $f \in C^\infty(\Omega)$ is said to be Gevrey differentiable of order s at $x_0 \in \Omega$ if there exist a compact neighborhood $K \subseteq \Omega$ of x_0 and two constants $C, h > 0$ such that

$$\sup_{x \in K} \left| D^k f(x) \right| \leq C h^k (k!)^s, \quad \forall k \in \mathbb{N}_0.$$

A nowhere Gevrey differentiable function on $\mathbb{R}$ is a function that is not Gevrey differentiable of order s at x_0, for any $x_0 \in \mathbb{R}$ and any $s \geq 1$.

We denote by NG the set of nowhere Gevrey differentiable function.

Existence of nowhere Gevrey differentiable functions (Bastin, E., Nicolay 2012)

Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of $(0,+\infty)$ such that $\lambda_k \geq (k+1)^{(k+1)^2}$ and $\lambda_{k+1} \geq 2\sum_{j=1}^{k} \lambda_j^{2+k-j}$, $\forall k \in \mathbb{N}$. The function f defined on $\mathbb{R}$ by

$$f(x) = \sum_{k=1}^{+\infty} \lambda_k^{1-k} e^{i\lambda_k x}$$

belongs to $\mathcal{C}^\infty(\mathbb{R})$ and is nowhere Gevrey differentiable.

Proof. Note that

$$\begin{aligned} \left|f^{(n)}(x)\right| &= \left|\sum_{k=1}^{n-1} \lambda_k^{n+1-k} e^{i\lambda_k x} + \lambda_n e^{i\lambda_n x} + \sum_{k>n} \lambda_k^{n+1-k} e^{i\lambda_k x}\right| \\ &\geq \lambda_n - \sum_{k=1}^{n-1} \lambda_k^{n+1-k} - \sum_{k>n} \lambda_k^{n+1-k} \geq \sum_{k=1}^{n-1} \lambda_k^{n+1-k} - \sum_{k>n} \lambda_k^{n+1-k} \\ &\geq \lambda_{n-1}^2 - \sum_{j=0}^{+\infty} \frac{1}{\lambda_j^j} \geq n^{2n^2} - e \geq \frac{1}{2} n^{2n^2} \geq Ch^n (n!)^s. \end{aligned}$$

Genericity (Bastin, E., Nicolay 2012; Bastin, Conejero, E., Seoane 2014)

The set NG of nowhere Gevrey differentiable functions is residual, prevalent and lineable (with maximal dimension) in $C^\infty(\mathbb{R})$.

Proof of the prevalence. We have

$$C^\infty(\mathbb{R}) \setminus NG = \bigcup_{s\in\mathbb{N}} \bigcup_{I\subset\mathbb{R}} \bigcup_{m\in\mathbb{N}} \underbrace{\left\{ f \in C^\infty(\mathbb{R}) : \sup_{x\in I} |f^{(n)}(x)| \leq m^n (n!)^s \ \forall n \in \mathbb{N}_0 \right\}}_{:=A(s,I,m) \text{ closed subset}}$$

where I denotes all rational subintervals. Hence NG is a Borel set.
Let us consider $f \notin \bigcup_m A(s,I,m)$. Then, there is at most one element in the set $\left\{\alpha \in \mathbb{R} : \alpha f + g \in \bigcup_m A(s,I,m)\right\}$; Otherwise, if $\alpha \neq \beta$ belongs to this set, then

$$(\alpha - \beta) f = (\alpha f + g) - (\beta f + g) \in \bigcup_m A(s,I,m)$$

hence $f \in \bigcup_m A(s,I,m)$ (linear subspace). Consequently, the linear hull of f is a probe for the complement of $\bigcup_m A(s,I,m)$. □

Actually, there is an older result which deals with the same kind of questions.

Proposition (Schmets, Valdivia 1991)

The Gevrey class of order $\alpha > 1$ contains a vector subspace which, for every $\beta \in (1, \alpha)$, contains an infinite dimensional vector subspace, non-zero elements of which are nowhere Gevrey differentiable of order β.

$\longrightarrow$ In particular, for every $\beta \in (1, \alpha)$, the set of nowhere Gevrey differentiable function of order β is lineable in the Gevrey class of order α.

Questions:

- Other notions of genericity?
- Similar results in the context of classes of ultradifferentiable functions?

Denjoy-Carleman classes

Definition

Let M be an arbitrary sequence of positive real numbers and let $\Omega \subseteq \mathbb{R}$ be an open set. The M-ultradifferentiable Roumieu type class is defined by

$$\mathcal{E}_{\{M\}}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : \forall\, K \subseteq U \text{ compact } \exists\, h > 0 : \ \|f\|^M_{K,h} < +\infty\},$$

and the M-ultradifferentiable Beurling type class by

$$\mathcal{E}_{(M)}(\Omega) := \{f \in \mathcal{C}^\infty(\Omega) : \forall\, K \subseteq U \text{ compact } \forall\, h > 0 : \ \|f\|^M_{K,h} < +\infty\},$$

where

$$\|f\|^M_{K,h} := \sup_{j \in \mathbb{N}, x \in K} \frac{|f^{(j)}(x)|}{h^j M_j}.$$

Particular case: The weight sequences $(k!)_k$ and $((k!)^\alpha)_k$.

Questions:

- When do we have $\mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(N)}(\Omega)$?
- In that case, “how small” is $\mathcal{E}_{\{M\}}(\Omega)$ in $\mathcal{E}_{(N)}(\Omega)$?

General assumptions:

- We assume that any weight sequence M is logarithmically convex, i.e.

$$M_k^2 \leq M_{k-1}M_{k+1}, \quad \forall k \in \mathbb{N}\,.$$

- We assume that any weight sequence M is such that $M_0 = 1$.
- We will often work with non-quasianalytic weight sequences M, i.e. such that

$$\sum_{k=1}^{+\infty} (M_k)^{-1/k} < +\infty.$$

By Denjoy-Carleman theorem, it is equivalent to the fact that there exists non-zero functions with compact support in $\mathcal{E}_{\{M\}}(\mathbb{R})$.

Inclusions between Denjoy-Carleman classes

Notation: $M \lhd N \iff \lim_{k\to+\infty} \left(\frac{M_k}{N_k}\right)^{\frac{1}{k}} = 0.$

Proposition

Let M, N be two weight sequences and let Ω be an open subset of $\mathbb{R}$. Then

$$M \lhd N \iff \mathcal{E}_{\{M\}}(\Omega) \subsetneq \mathcal{E}_{(N)}(\Omega).$$

Keys for the strict inclusion:

(1) If $M \lhd N$, then there exists a weight sequence L such that $M \lhd L \lhd N$.

(2) The function

$$f(x) = \sum_{k=1}^{+\infty} \frac{L_k}{2^k} \left(\frac{L_{k-1}}{L_k}\right)^k \exp\left(2i\frac{L_k}{L_{k-1}}x\right)$$

belongs to $\mathcal{E}_{\{L\}}(\mathbb{R})$. Moreover, $|f^{(k)}(0)| \geq L_k \, \forall k \in \mathbb{N}_0$, so that $f \notin \mathcal{E}_{(L)}(\mathbb{R})$.

Construction

Definition

We say that a function is nowhere in $\mathcal{E}_{\{M\}}$ if its restriction to any non-empty open subset Ω of $\mathbb{R}$ does not belong to $\mathcal{E}_{\{M\}}(\Omega)$.

Proposition (E. 2014)

Assume that M and N are two weight sequences such that $M \lhd N$. If M is non-quasianalytic, there exists a function of $\mathcal{E}_{(N)}(\mathbb{R})$ which is nowhere in $\mathcal{E}_{\{M\}}$.

Proof. We know by (1) that there is $N^\star$ such that $M \lhd N^\star \lhd N$. By induction, we get a sequence $(L^{(p)})_{p\in\mathbb{N}}$ of weight sequences such that

$$M \lhd L^{(1)} \lhd L^{(2)} \lhd \cdots \lhd L^{(p)} \lhd \cdots \lhd N^\star \lhd N.$$

For every $p \in \mathbb{N}$, (2) allows us to consider a function $f_p \in \mathcal{E}_{\{L^{(p)}\}}(\mathbb{R})$ such that

$$|f_p^{(j)}(0)| \geq L_j^{(p)}, \quad \forall j \in \mathbb{N}_0 .$$

Since M is non-quasianalytic,

$$\exists \phi \in \mathcal{E}_{\{M\}}(\mathbb{R}) \text{ with compact support, } \phi \equiv 1 \text{ in a nbh of } 0.$$

Let $\{x_p : p \geq 1\}$ be a dense subset of $\mathbb{R}$. For every $p \geq 2$, we can find $k_p > 0$ such that the function

$$\phi_p := \phi\big(k_p(\cdot - x_p)\big)$$

has its support disjoint from $\{x_1, \ldots, x_{p-1}\}$. We define g_p

$$g_p(x) := \underbrace{f_p(x - x_p)}_{\in \mathcal{E}_{\{L(p)\}}(\mathbb{R})} \underbrace{\phi_p(x)}_{\in \mathcal{E}_{\{M\}}(\mathbb{R})} \in \mathcal{E}_{(N^\star)}(\mathbb{R}).$$

Let $\gamma_p > 0$ be such that

$$\sup_{x \in \mathbb{R}} |g_p^{(j)}(x)| \leq \gamma_p N_j^\star, \quad \forall j \in \mathbb{N}$$

and define the function g by

$$g := \sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} g_p.$$

1. $g \in \mathcal{E}_{(N)}(\mathbb{R})$: For every $j \in \mathbb{N}$ and every $x \in \mathbb{R}$, we have

$$\sum_{p=1}^{+\infty} \frac{1}{\gamma_p 2^p} |g_p^{(j)}(x)| \leq \sum_{p=1}^{+\infty} \frac{1}{2^p} N_j^\star \leq N_j^\star$$

which implies that g belongs to $\mathcal{E}_{\{N^\star\}}(\mathbb{R}) \subseteq \mathcal{E}_{(N)}(\mathbb{R})$.

2. g is nowhere in $\mathcal{E}_{\{M\}}$: By contradiction, assume that there exists an open subset Ω of $\mathbb{R}$ such that $g \in \mathcal{E}_{\{M\}}(\Omega)$. Let $p_0 \in \mathbb{N}$ such that $x_{p_0} \in \Omega$. Remark that

$$\sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} g_p = \underbrace{g}_{\in \mathcal{E}_{\{M\}}(\Omega) \subseteq \mathcal{E}_{(L^{(p_0)})}(\Omega)} - \underbrace{\sum_{p=1}^{p_0-1} \frac{1}{\gamma_p 2^p} g_p}_{\in \mathcal{E}_{(L^{(p_0)})}(\Omega)} \in \mathcal{E}_{(L^{(p_0)})}(\Omega).$$

But, since the support of g_p is disjoint of x_{p_0} for every $p > p_0$, we also have

$$\left| \sum_{p=p_0}^{+\infty} \frac{1}{\gamma_p 2^p} g_p^{(j)}(x_{p_0}) \right| = \frac{1}{\gamma_{p_0} 2^{p_0}} \left| g_{p_0}^{(j)}(x_{p_0}) \right| = \frac{1}{\gamma_{p_0} 2^{p_0}} \left| f_{p_0}^{(j)}(0) \right| \geq \frac{1}{\gamma_{p_0} 2^{p_0}} L_j^{(p_0)}$$

for every $j \in \mathbb{N}$, hence a contradiction.

Remark. Given a sequence $(L^{(p)})_{p\geq 1}$ such that

$$M \lhd L^{(1)} \lhd L^{(2)} \lhd \cdots \lhd L^{(p)} \lhd \cdots \lhd N^* \lhd N$$

and a dense subset $\{x_p : p \in \mathbb{N}\}$ of $\mathbb{R}$, we have constructed a function g such that

- $g \in \mathcal{E}_{\{N^*\}}(\mathbb{R})$
- $\notin \mathcal{E}_{(L^{(p)})}(\Omega)$ for every neighbourhood Ω of x_p.

$\longrightarrow$ Main tool for the lineability!

Lineability

Proposition (E. 2014)

Assume that $M \lhd N$. If M is non quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{M\}}$ is lineable (with maximal dimension).

Idea of the proof. For every $t \in (0,1)$, we set

$$L_k^{(t)} := (M_k)^{1-t}(N_k)^t \quad \forall k \in \mathbb{N}_0 .$$

Then $M \lhd L^{(t)} \lhd N$ for all $t \in (0,1)$ and $L^{(t)} \lhd L^{(s)}$ if $t < s$.
Remark that

$$M \lhd L^{(\frac{t}{2})} \lhd L^{(\frac{2t}{3})} \lhd L^{(\frac{3t}{4})} \lhd \cdots \lhd L^{(t)} \lhd N, \quad \forall t \in (0,1).$$

and we can consider $g_t \in \mathcal{E}_{\{L^{(t)}\}}(\mathbb{R})$ which is not in $\mathcal{E}_{\left(L^{\left((1-\frac{1}{p})t\right)}\right)}(\Omega)$, for any open neighbourhood Ω of x_p and for any $p \geq 2$. Then take

$$\mathcal{D} = \operatorname{span}\{g_t : t \in (0,1)\}.$$

Let $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ with $\alpha_N \neq 0$ and $t_1 < \cdots < t_N$ in $(0,1)$. Assume that there exists an open subset Ω of $\mathbb{R}$ such that

$$G = \sum_{n=1}^{N} \alpha_n g_{t_n} \in \mathcal{E}_{\{M\}}(\Omega).$$

Let $p_0 \in \mathbb{N}$ such that $x_{p_0} \in \Omega$ and $t_{N-1} < \left(1 - \frac{1}{p_0}\right) t_N$. Hence,

$$g_{t_N} = \frac{1}{\alpha_N} \Bigg(G - \underbrace{\sum_{n=1}^{N-1} \alpha_n g_{t_n}}_{\in \mathcal{E}_{\{L^{(t_{N-1})}\}}(\mathbb{R})} \Bigg) \in \mathcal{E}_{\{L^{(t_{N-1})}\}}(\Omega).$$

From the choice of p_0, we have $\mathcal{E}_{\{L^{(t_{N-1})}\}}(\Omega) \subset \mathcal{E}_{\left(L^{\left((1-\frac{1}{p_0})t_N\right)}\right)}(\Omega)$ and this leads to a contradiction with the construction of g_{t_N}. $\square$

Dense-lineability

Note that we can modify the above construction of $\mathcal{D} = \mathrm{span}\{g_t : t \in (0,1)\}$ to get the density of the linear subspace.

Let us consider

- $(P_m)_{m\in\mathbb{N}}$ a dense sequence of polynomials in $\mathcal{E}_{(N)}(\mathbb{R})$,
- $(t_m)_{m\in\mathbb{N}}$ a sequence of different elements of $(0,1)$,
- $\{U_m : m \in \mathbb{N}\}$ a 0-nbh basis in $\mathcal{E}_{(N)}(\mathbb{R})$,
- for every $m \in \mathbb{N}$, $k_m > 0$ such that $k_m g_{t_m} \in U_m$.

Then, the linear subspace

$$\mathcal{D}_d = \mathrm{span}\left(\left\{P_m + k_m g_{t_m} : m \in \mathbb{N}\right\} \cap \left\{g_t : t \in (0,1), t \neq t_m\right\}\right)$$

has the desired property.

More results:

- Same results using the notion of Baire residuality and of prevalence : The set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are somewhere in $\mathcal{E}_{\{M\}}$ is given by

$$\bigcup_{I\subset\mathbb{R}}\bigcup_{m\in\mathbb{N}}\bigcup_{s\in\mathbb{N}} E(I,m,s),$$

  where I denotes all rational subintervals of $\mathbb{R}$ and

$$E(I,m,s) := \left\{ f\in\mathcal{E}_{(N)}(\mathbb{R}) : \sup_{x\in I}|f^{(j)}(x)| \leq s\, m^j M_j \;\forall j\in\mathbb{N}_0 \right\}.$$

- Generalization using weight functions (without exhibiting a particular function) and using weight matrices.

Case of countable unions

Let N be a weight sequence and let $(M^{(n)})_{n\in\mathbb{N}}$ be a sequence of weight sequences such that $M^{(n)} \lhd N$ for every $n \in \mathbb{N}$.

Question.
What about the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$?

Proposition E. 2014

If there is $n_0 \in \mathbb{N}$ such that the weight sequence $M^{(n_0)}$ is non quasianalytic, the set of functions of $\mathcal{E}_{(N)}(\mathbb{R})$ which are nowhere in $\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}}$ is (dense) lineable (with maximal dimension), prevalent and residual in $\mathcal{E}_{(N)}(\mathbb{R})$.

Idea. Construct a weight sequence P such that

$$\bigcup_{n\in\mathbb{N}} \mathcal{E}_{\{M^{(n)}\}} \subseteq \mathcal{E}_{\{P\}} \subsetneq \mathcal{E}_{(N)}.$$

Particular case

Recall that Gevrey classes correspond to Roumieu classes given by the weight sequence

$$M_k := (k!)^\alpha, \quad k \in \mathbb{N}\,.$$

Corollary

Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$, is (dense) lineable (with maximal dimension), prevalent and residual in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

Proof. It suffices to apply the previous result with $M_k^{(n)} := (k!)^{\beta_n}$, with $1 < \beta_n \nearrow \alpha$.

Proposition (Schmets, Valdivia 1991)

Let $\alpha > 1$. The set of functions of $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$ which are nowhere in $\mathcal{E}_{\{(k!)^\beta\}}$ for every $\beta \in (1, \alpha)$ is residual in $\mathcal{E}_{((k!)^\alpha)}(\mathbb{R})$.

Non-surjectivity of the Borel map

Definition

The spaces of germs at 0 of the M-ultradifferentiable functions of Roumieu and Beurling types are defined respectively by

$$\mathcal{E}^0_{\{M\}} := \varinjlim_{k\in\mathbb{N}_{>0}} \mathcal{E}_{\{M\}}\left(\left(-\frac{1}{k},\frac{1}{k}\right)\right),$$

and

$$\mathcal{E}^0_{(M)} := \varinjlim_{k\in\mathbb{N}_{>0}} \mathcal{E}_{(M)}\left(\left(-\frac{1}{k},\frac{1}{k}\right)\right).$$

Particular case : The sequence $M_p = p!$ in the Roumieu case gives $\mathcal{O}^0$, the germs of analytic function at 0.

Definition

We define the sequence spaces $\Lambda^r_{\{M\}}$ and $\Lambda^r_{(M)}$ by

$$\Lambda_{\{M\}} = \left\{ \mathbf{b} \in \mathbb{C}^{\mathbb{N}} : \exists h > 0, |\mathbf{b}|_h^M < +\infty \right\}$$

and

$$\Lambda_{(M)} = \left\{ \mathbf{b} \in \mathbb{C}^{\mathbb{N}} : \forall h > 0, |\mathbf{b}|_h^M < +\infty \right\},$$

where for any $h > 0$,

$$|\mathbf{b}|_h^M := \sup_{j \in \mathbb{N}} \frac{|b_j|\, j!}{h^j M_j}.$$

The Borel map j^∞ is defined by

$$j^\infty : \mathcal{E}^0_{[M]} \longrightarrow \Lambda_{[M]}, \quad j^\infty(f) = \left(\frac{f^{(j)}(0)}{j!} \right)_{\alpha \in \mathbb{N}}$$

Theorem (Carleman 1920, Thilliez 2008, Rainer and Schindl 2017)

Let M be a quasianalytic weight sequence. Assume that $\mathcal{O}^0 \subsetneq \mathcal{E}^0_{[M]}$. Then the Borel map $j^\infty : \mathcal{E}^0_{[M]} \longrightarrow \Lambda_{[M]}$ is not surjective.

$\longrightarrow$ In this case, "how far away" is j^∞ to be surjective?

- Together with Gerhard, in the quasianalytic setting, we have studied the size of $\Lambda_{(M)} \setminus j^\infty(\mathcal{E}^0_{(M)})$ using the different notions of genericity.
- Our approach is based on the proof of the previous theorem due to Thilliez.

Representation formula (Thilliez 2008)

Let M be a quasianalytic weight sequence. There exist numbers $(\omega_{j,k}^M)_{j,k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty} \omega_{j,k}^M = 1, \quad \forall j \in$$

and such that, given any function $f \in \mathcal{E}^0_{\{M\}}$, one has

$$\boxed{f(x) = \lim_{k\to+\infty} \sum_{j=0}^{k-1} \omega_{j,k}^M \frac{f^{(j)}(0)}{j!} x^j}$$

for every $x > 0$ small enough.

$\longrightarrow$ If $\mathbf{F} = (F_j)_{j\in\mathbb{N}}$ is a sequence for which there exists $(x_n)_{n\in\mathbb{N}} \searrow 0$ such that

$$\limsup_{k\to+\infty} \left| \sum_{j=0}^{k-1} \omega_{j,k}^M F_j x_n^j \right| = +\infty\,, \quad \forall n \in \mathbb{N},$$

then $\mathbf{F} \notin j^\infty(\mathcal{E}^0_{\{M\}})$.

Theorem (E., Schindl 2018)

Let M ba a quasianalytic weight sequence. Let us assume that

$$\lim_{k\to\infty}\left(\frac{M_k}{k!}\right)^{1/k}=+\infty\,,\quad \text{i.e.}\quad \mathcal{O}^0\subsetneq\mathcal{E}^0_{(M)}.$$

Then, the set $\Lambda_{(M)}\setminus j^\infty(\mathcal{E}^0_{\{M\}})$ is residual, i.e. it contains a countable intersection of dense open sets (hence $\Lambda_{(M)}\setminus j^\infty(\mathcal{E}^0_{(M)})$ also).

Proof.

1. Construction of $\mathbf{F}\in\Lambda_{(M)}$ such that there exists $a_0\in(0,1]$ with

$$\limsup_{k\to+\infty}\left|\sum_{j=0}^{k-1}\omega^M_{j,k}F_j a^j\right|=+\infty$$

for any $0<a\leq a_0$ (using the assumptions on M).

2. Fix a sequence $(a_n)_{n\in\mathbb{N}} \searrow 0$ smaller than a_0. If we define

$$\mathcal{G} := \bigcap_{n\in\mathbb{N}} \left\{ \mathbf{b} \in \Lambda_{(M)} : \limsup_{k\to+\infty} \left| \sum_{j=0}^{k-1} \omega_{j,k}^M b_j a_n^j \right| = +\infty \right\},$$

we know that

$$\mathcal{G} \subseteq \Lambda_{(M)} \setminus j^\infty(\mathcal{E}_{\{M\}}^0)\,.$$

It suffices then to prove that $\mathcal{G}$ can be written as a countable intersection of dense open sets of $\Lambda_{(M)}$. One has

$$\mathcal{G} = \bigcap_{n\in\mathbb{N}} \bigcap_{N\in\mathbb{N}} \bigcap_{K\in\mathbb{N}_0} \bigcup_{k\geq K} \underbrace{\left\{ \mathbf{b} \in \Lambda_{(M)} : \left| \sum_{j=0}^{k-1} \omega_{j,k}^M b_j a_n^j \right| > N \right\}}_{:=G(n,N,k)}$$

- $G(n,N,k)$ is open :

Let $(\mathbf{b}^{(l)})_{l\in\mathbb{N}}$ be a sequence of $\Lambda_{(M)}$ which does not belong to $G(n,N,k)$ and which converges to $\mathbf{b}$ in $\Lambda_{(M)}$. Let $\delta>0$ and fix $\varepsilon>0$ such that

$$\varepsilon < \frac{\delta}{\sum_{j=0}^{k-1}|\omega_{j,k}^M|\frac{M_j}{j!}a_n^j}.$$

Let $L\in\mathbb{N}$ be such that $\left|\mathbf{b}^{(l)}-\mathbf{b}\right|_1^M\leq\varepsilon$ for all $l\geq L$. Then, for all $l\geq L$, one has

$$\begin{aligned}\left|\sum_{j=0}^{k-1}\omega_{j,k}^M b_j a_n^j\right| &\leq \left|\sum_{j=0}^{k-1}\omega_{j,k}^M(b_j^{(l)}-b_j)a_n^j\right| + \left|\sum_{j=0}^{k-1}\omega_{j,k}^M b_j^{(l)} a_n^j\right| \\ &\leq \sum_{j=0}^{k-1}|\omega_{j,k}^M|\left|b_j^{(l)}-b_j\right|a_n^j+N \\ &\leq \varepsilon\sum_{j=0}^{k-1}|\omega_{j,k}^M|\frac{M_j}{j!}a_n^j+N\leq\delta+N,\end{aligned}$$

hence $\mathbf{b}\notin G(n,N,k)$ since $\delta>0$ is arbitrary.

- $\bigcup_{k\geq K} G(n,N,k)$ is dense :
  Let $\mathbf{b}\in\Lambda_{(M)}$ and $\varepsilon>0$. From the construction of $\mathbf{F}$, there is $k\geq K$ such that

$$\left|\sum_{j=0}^{k-1}\omega_{j,k}^{M}F_j a_n^j\right|\geq\frac{N}{\varepsilon}.$$

Then, either $\mathbf{b}+\varepsilon\mathbf{F}$ or $\mathbf{b}-\varepsilon\mathbf{F}$ belongs to $G(n,N,k)$: Otherwise, one would have

$$2\varepsilon\left|\sum_{j=0}^{k-1}\omega_{j,k}^{M}F_j a_n^j\right|\leq\left|\sum_{j=0}^{k-1}\omega_{j,k}^{M}(b_j-\varepsilon F_j)a_n^j\right|+\left|\sum_{j=0}^{k-1}\omega_{j,k}^{M}(b_j+\varepsilon F_j)a_n^j\right|\leq 2N.$$

Moreover, for any $h>0$, one has

$$|\mathbf{b}-(\mathbf{b}\pm\varepsilon\mathbf{F})|_h^M=\varepsilon|\mathbf{F}|_h^M,$$

and the density of $\bigcup_{k\geq K} G(n,N,k)$ in $\Lambda_{(M)}$ follows.

□

Other results :

- Equivalent results with the notions of prevalence and lineability
- Mixed setting $\Lambda_{[M^2]} \setminus j^\infty(\mathcal{E}^0_{\{M^1\}})$
- Higher dimensions
- Generalization using weight matrices
- Corollary for classes defined via weight functions

References I

- R.M. Aron, V.I. Gurariy and J.B. Seoane-Sepúlveda, *Lineability and spaceability of sets of functions on* $\mathbb{R}$, Proc. Amer. Math. Soc., **133** (3), 795–803, 2005.
- F. Bastin, J.A. Conejero, C. Esser and J.B. Seoane-Sepúlveda, *Algebrability and nowhere Gevrey differentiability*, Israel J. Math., **205**, 127–143, 2015.
- F. Bastin and C. Esser and S. Nicolay, *Prevalence of "nowhere analyticity"*, Studia Math., **210**, 239–246, 2012.
- L. Bernal-González, *Funciones con derivadas sucesivas grandes y pequeñas por doquier*, Collect. Math., **38**, 117–122, 1987.
- L. Bernal-González, *Lineability of sets of nowhere analytic functions*, J. Math. Anal. Appl., **340**, 1284–1295, 2008.
- Carleman, T., *Sur le calcul effectif d'une fonction quasi analytique dont on donne les dérivées en un point*, C. R. Acad. Sci. Paris, **176**, 59–68, 1923.
- F.S. Cater, *Differentiable, nowhere analytic functions*, Amer. Math. Monthly, **291**, 618–624, 1984
- J.A. Conejero, P. Jiménez-Rodríguez, G.A. Muñoz-Fernández and J.B. Seoane-Sepúlveda, *When the Identity Theorem "seems" to fail*, Amer. Math. Monthly, **121**, 2014.
- A. Bartoszewicz, M. Bienias, M. Filipczak and S. Głąb, *Strong c-algebrability of strong Sierpinski-Zygmund, smooth nowhere analytic and other sets of functions*, J. Math. Anal. Appl., **412**, 620–630, 2014.

References II

U.B. Darji and D. Swanson, *A note on the generic nature of Pringsheim functions*, Adv. Pure Appl. Math., **7**, 231–236, 2016.

C. Esser, *Generic results in classes of ultradifferentiable functions*, J. Math. Anal. Appl., **413**, 378–391, 2014.

C. Esser and G. Schindl, *How far is the Borel map from being surjective in the quasianalytic setting?*

Morgenstern, D., *Unendlich oft differenzierbare nicht-analytische Funktionen*,Math. Nachr., **12**, 74, 1954.

H. Salzmann and K. Zeller, *Singularitäten unendlich oft differenzierbarer Funktionen*, Math. Z., **62**, 354–367, 1955.

J. Schmets and M. Valdivia, *On the extent of the (non) quasi-analytic classes*, Arch. Math., **56**, 593–600, 1991.

H. Komatsu, *Ultradistributions, I: Structure theorems and a characterization*, J.Fac. Sc. Tokyo, Ser. I A, **20**, 25–105, 1973.

A. Rainer and G. Schindl, *Composition in ultradifferentiable classes*, Studia Math., **224**, 97–131, 2014.

A. Rainer and G. Schindl, *On the Borel mapping in the quasianalytic setting*, Math. Scand., **121**, 293–310, 2017.

V. Thilliez, *On quasianalytic local rings*, Expo. Math., **26**, 1–23, 2008.