DETERMINATION OF MASS-LOSS RATES FROM THE FIRST ORDER MOMENT W_1 OF UNSATURATED P CYGNI LINE PROFILES

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Abstract. The escape probability method introduced by Sobolev to treat the transfer of line photons is used in order to derive the expressions of the nth order moment $W_n \propto \int (E(\lambda)/E_c - 1)(\lambda - \lambda_{12})^n \cdot \mathrm{d}\lambda$ of a P Cygni profile formed in rapidly expanding envelopes around a central point-like source under various physical and geometrical conditions.

With the only assumption that there is mass-conservation of the species in the flow, we state for the case of optically thin lines that the relation between the first order moment W_1 and the quantity $\dot{M}n$ (level), first established by Castor *et al.* (1981) under more restrictive conditions, is in fact independent of the type of velocity field $v(r)$ and a fortiori of the distribution adopted for the radial opacity $\tau_{12}^r(X')$. These results also remain unchanged when including collisions ($\varepsilon \neq 0$) and/or an additional rotational velocity field $v^{\perp}(r)$ in the expanding atmosphere. We investigate the presence of an underlying photospheric absorption line and conclude that for realistic cases, neglecting this boundary condition to the radiative transfer leads to an underestimate of the mass-loss rate by a factor of about 20%. By means of a three-level atom model, we demonstrate that all results derived for a single line transition equally apply for an unresolved doublet profile provided W_1 and $\dot{M}n$ (level) are calculated with the weighted wavelength λ_D and total oscillator strength f_D of the doublet.

Considering the occultation and inclination effects caused by the finite size of the central core, we refine the value of the multiplicative constant fixing the ratio of W_1 to $\dot{M}n$ (level). We show that this relation allows a determination of the mass-loss rate with an uncertainty less than 30%, irrespective of the size $L_{\max}$ of the atmosphere and of the limb-darkening affecting the stellar core. Reviewing all possible sources of error, we finally conclude that this method of deriving a mass-loss rate from the analysis of an unsaturated P Cygni profile is very powerful. The total uncertainty affecting the determination of $\dot{M}n$ (level) from the measurement of W_1 should be smaller than 60%.

1. Introduction

Unsaturated P Cygni profiles are known to provide the best means of deriving mass-loss rates from early-type stars. In this context, Castor *et al.* (1981) have developed a theory particularly suitable for the interpretation of IUE and other low resolution spectra. In the framework of the Sobolev approximation (Sobolev, 1947, 1957, 1958; Castor, 1970; Lucy, 1971), these authors have established that for optically thin lines, the first order moment $W_1 \propto \int (E(\lambda)/E_c - 1)(\lambda - \lambda_0)\,\mathrm{d}\lambda$ is proportional to the quantity n (level). $\dot{M}$, where n (level) is the fractional abundance of an element in the lower atomic level associated with the given line transition, and $\dot{M}$ is the mass-loss rate.

Whereas Castor *et al.* (1981) have discussed the negligible dependence of W_1, and consequently of $\dot{M}$, onto errors caused by the location of the continuum level E_c and

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to errors in the wavelength scale λ, it is not clear how much their choice of an ad hoc velocity field $v(r)$, fictitious radial opacity $\tau_{12}^r(X')$, etc. may affect the mass-loss rate they derived for the central star of the planetary nebula NGC 6543.

It is our purpose to investigate in the present paper the relation between W_1 and $\dot{M}$ when taking into account various physical and geometrical effects. Because it is essentially a good approximation to assume that the size of the expanding envelope from which arise the observed P Cygni profiles is large with respect to the central stellar core, we first establish for the case of a point-like source the general expression of the nth order moment W_n in terms of the usual parameters inherent to Sobolev-type theories (see Section 2). In the subsequent sections, we show how these first results are altered when considering separately the possible effects caused by collisions and rotation (Section 3), by the presence of an underlying photospheric absorption line (Section 4), by a doublet line transition (Section 5), by the finite size of the central source and by the limb-darkening of the stellar core (Section 6). A discussion and general conclusions constitute the last section.

2. General Hypotheses and Expression of the Moment W_n for the Case of a Point-Like Star

In the frame of a two-level atom model and using Sobolev-type approximations for the transfer of line radiation (conservative scattering) in a spherical envelope accelerated radially outward around a point-like source, the expression $E(X)/E_c$ of a P Cygni line profile as a function of the dimensionless frequency $X \in [-1, 1]$ is given by (see Surdej, 1979; referred to below as Paper I)

$$\frac{E(X)}{E_c} = \int_{\mathrm{Max}(|X|,\, -X_{\min})}^{1} \frac{P(X')P(X', X)\,\mathrm{d}X'}{2X'} + \\ + \begin{cases} \exp(-\tau_{12}^r(X))\,, & \text{if} \quad X \in [-1, X_{\min}] \\ 1 \,, & \text{if} \quad X \in [X_{\min}, 1] \end{cases}, \tag{1}$$

with

$$X_{\min} = -v_0/v_{\max}\,, \tag{2}$$

$$P(X') = 1 - \exp(-\tau_{12}^r(X'))\,, \tag{3}$$

and

$$P(X', X) = \frac{(1 - \exp(-\tau_{12}(X', X)))}{\tau_{12}(X', X)} \Big/ \beta_{12}^1(X')\,. \tag{4}$$

We briefly recall hereafter the physical meaning of the different quantities appearing in the previous relations. v_0 (resp. $v_{\max}$) denotes the initial (resp. maximal) radial velocity $v(r)$ of the flow. $\tau_{12}(X', X)$ (resp. $\tau_{12}^r(X')$) represents the fictive opacity evaluated at the distance $L(X')$ along the direction making an angle θ (resp. $\theta = 0$) with respect to the radial direction. $\beta^1_{12}(X')$ stands for the usual escape probability of a line photon emitted along any direction in the expanding medium. We still have the relation

$$X = X' \cos(\theta)\,, \tag{5}$$

with

$$X' = -v(L)/v_{\max}\,, \tag{6}$$

where $L(X')$ expresses the radial distance r to the central star in stellar radii R^* units.

Adopting the definition of Castor *et al.* (1981; hereafter referred to as CLS)

$$W_n = \left(\frac{c}{\lambda_{12} v_{\max}}\right)^{n+1} \int \left(\frac{E(\lambda)}{E_c}\right) - 1(\lambda - \lambda_{12})^n \,\mathrm{d}\lambda\,, \tag{7}$$

for the nth order moment of a line profile, and making use of relation (5) combined with the classical Doppler relation, the former expression reduces to

$$W_n = \int_{-1}^{1} X^n \left(\frac{E(X)}{E_c} - 1\right) \mathrm{d}X\,. \tag{8}$$

Inserting expression (1) for the line profile function into relation (8), we easily obtain for the moments (see Appendix)

$$\begin{aligned} W_n = (1 + (-1)^n) \int_{-X_{\min}}^{1} \frac{P(X')X'^n}{2\beta^1_{12}(X')} \int_0^1 \mu^n \frac{(1 - \exp(-\tau_{12}(X', \mu)))}{\tau_{12}(X', \mu)} \mathrm{d}\mu\, \mathrm{d}X' - \\ -(-1)^n \int_{-X_{\min}}^{1} X'^n (1 - \exp(-\tau^r_{12}(X')))\, \mathrm{d}X'\,, \end{aligned} \tag{9}$$

where $\mu = \cos(\theta)$. (10)

Recalling (cf. Surdej, 1977) the expression of the escape probability

$$\beta^1_{12}(X') = \int_0^1 \frac{(1 - \exp(-\tau_{12}(X', \mu)))}{\tau_{12}(X', \mu)} \mathrm{d}\mu\,, \tag{11}$$

we find it straightforward to establish that

$$W_0 = 0\,, \tag{12}$$

and

$$W_1 = \int_{-X_{\min}}^{1} X'(1 - \exp(-\tau_{12}^r(X')))\, \mathrm{d}X' \,. \tag{13}$$

Within our assumptions of conservative scattering and of a point-like source, the first of these results simply indicates that the equivalent width of a P Cygni profile is equal to zero in accordance with the principle of energy conservation. If a given line is optically thick everywhere in the expanding medium, relation (13) becomes

$$W_1 = \int_{-X_{\min}}^{1} X'\, \mathrm{d}X' \sim \tfrac{1}{2}\,. \tag{14}$$

For optically thin lines (i.e., $\tau_{12}^r(X') < 1$), relation (13) becomes

$$W_1 = \int_{-X_{\min}}^{1} X'\, \tau_{12}^r(X')\, \mathrm{d}X' \,, \tag{15}$$

and only in this particular case will it be possible to infer a relation between the first order moment W_1 and the mass-loss rate M of the central source. In order to do so, we shall follow CLS in defining the fractional abundance n (level) of an element in the lower atomic level 1 for the given line transition as

$$n\,(\text{level}) = n_1/N\,(\text{element})\,, \tag{16}$$

where N (element) is the total number density for the element concerned. The abundance A (element) of this same element can then be expressed by

$$A\,(\text{element}) = N\,(\text{element})/N_{\text{tot}}\,, \tag{17}$$

N_{tot} being the total nucleon density. Combining the expression for the mass-loss rate

$$\dot{M} = -4\Pi L^2 R^{*2} v(L) N_{\text{tot}} \bar{\mu} M_{\text{amu}}\,, \tag{18}$$

where $\bar{\mu}$ and M_{amu} denote, respectively, the mean atomic weight of the nuclei and the unit of atomic mass, with relation (4.7) in CLS, we derive a useful relation for the fictitious radial opacity

$$\tau_{12}^r(X') = \frac{\Pi e^2}{mc} f_{12}\lambda_{12} \frac{A\,(\text{element})\dot{M}}{2\Pi\bar{\mu}M_{\text{amu}}v_{\max}^2 R^*} \frac{\mathrm{d}(1/L)}{\mathrm{d}(X'^2)}\, n\,(\text{level})\,. \tag{19}$$

The general expression for the fictitious opacity $\tau_{12}(X', \mu)$ is similarly found to be

$$\tau_{12}(X', \mu) = \tau_{12}^r(X') \Big/ \left(\mu^2\left(1 - \frac{\mathrm{d}\ln L}{\mathrm{d}\ln X'}\right) + \frac{\mathrm{d}\ln L}{\mathrm{d}\ln X'}\right). \tag{20}$$

Making use of the variable transformation $L = L(X')$ and inserting relation (19) into

(15), we obtain

$$W_1 = \frac{\Pi e^2}{mc} f_{12}\lambda_{12} \frac{A\,(\text{element})\dot{M}}{4\Pi\bar{\mu}M_{\text{amu}}v_{\max}^2 R^*} \int_1^{L_{\max}} n\,(\text{level})\,\mathrm{d}(1/L)\,. \tag{21}$$

We conclude that unless the ionization equilibrium is specified throughout the expanding atmosphere, it will not be possible to evaluate the integral factor in the previous relation. However, a very reasonable approach simply consists in assuming that the fractional abundance n (level) is a constant across the medium, i.e., there is mass conservation of the species under study. This is certainly a good approximation for the resonance lines of dominant ions in the flow. In that case, relation (21) reduces to

$$W_1 = \frac{\Pi e^2}{mc} f_{12}\lambda_{12} \frac{A\,(\text{element})\dot{M}n\,(\text{level})}{4\Pi\bar{\mu}M_{\text{amu}}v_{\max}^2 R^*} (1/L_{\max} - 1)\,, \tag{22}$$

and we note that this result is totally independent of any assumption concerning the type of velocity field, and a fortiori of any ad hoc distribution for the fictive radial opacity. Despite the fact that CLS have derived a very similar relation – only differing by a multiplicative constant – between W_1 and $\dot{M}$ (see their relations (4.15) and (4.16)), one should be aware that these authors have adopted specific relations for the velocity field and for the fictitious radial opacity.

3. Effects of Collisions and Rotation

Assuming as previously a central point-like source, let us first establish the expression of the nth order moment of a P Cygni profile when taking into account the effects due to collisions.

A. Collisions

In the framework of the Sobolev approximation, the expression of the source function S_{12} for a two-level atom model is (see Castor, 1970)

$$S_{12} = \frac{(1-\varepsilon)I_c\beta_{12}^3 + \varepsilon B_{\nu_{12}}(T_e)}{\varepsilon + \beta_{12}^1(1-\varepsilon)}\,, \tag{23}$$

where ε is the probability of de-excitation of the upper atomic level by collisions, $B_{\nu_{12}}(T_e)$ the Planck function calculated at the electron temperature T_e and $I_c\beta_{12}^3$ the mean intensity of the stellar radiation field. Because of the collisions, it is easy to visualize that every time a 'stellar' photon is scattered in the line transition $1 \rightleftharpoons 2$, a fraction $\varepsilon B_{\nu_{12}}(T_e)/(1-\varepsilon)I_c\beta_{12}^3$ of 'thermal' line photons is simultaneously created at the same place. Considering these radiative processes at a distance $L(X')$ from the stellar core, we may state that these emitted line photons will escape the expanding atmosphere along a direction θ after any number n ($n = 0, 1, 2, \ldots$) of local scatterings with a frequency $X \in [X', -X']$ and a probability $P(X', X)/|2X'|$, where

$$P(X', X) = (1 - \varepsilon)\left(\frac{1 - \exp(-\tau_{12})}{\tau_{12}}\right) + (1 - \varepsilon)(1 - \beta_{12}^1) \times$$

$$\times (1 - \varepsilon)\left(\frac{1 - \exp(-\tau_{12})}{\tau_{12}}\right) + \ldots$$

or

$$P(X', X) = (1 - \varepsilon)\left(\frac{1 - \exp(-\tau_{12})}{\tau_{12}}\right) \frac{1}{(\varepsilon + \beta_{12}^1(1 - \varepsilon))}. \tag{24}$$

If we combine all previous results, the expression (1) for the line profile function becomes

$$\frac{E(X)}{E_c} = \int_{\mathrm{Max}(|X|, -X_{\min})}^{1} g(X') \frac{P(X')}{2X'} \left(\frac{1 - \exp(-\tau_{12})}{\tau_{12}\beta_{12}^1}\right) \mathrm{d}X' +$$

$$+ \begin{cases} \exp(-\tau_{12}^r(X)), & \text{if } X \in [-1, X_{\min}], \\ 1, & \text{if } X \in [X_{\min}, 1], \end{cases} \tag{25}$$

where

$$g(X') = \frac{((1 - \varepsilon)I_c\beta_{12}^3 + \varepsilon B_{\nu_{12}}(T_e))\beta_{12}^1}{I_c\beta_{12}^3(\varepsilon + \beta_{12}^1(1 - \varepsilon))} \tag{26}$$

i.e., it expresses the ratio between the source functions calculated with ($\varepsilon \neq 0$) and without ($\varepsilon = 0$) collisions.

If we follow a similar procedure as in the Appendix, the nth order moment W_n is now found to be

$$W_n = (1 + (-1)^n) \int_{-X_{\min}}^{1} \frac{g(X')P(X')X'^n}{2\beta_{12}^1(X')} \times$$

$$\times \int_0^1 \mu^n \frac{(1 - \exp(-\tau_{12}(X', \mu)))}{\tau_{12}(X', \mu)} \,\mathrm{d}\mu\, \mathrm{d}X' - (-1)^n \times$$

$$\times \int_{-X_{min}}^{1} X'^n(1 - \exp(-\tau_{12}^r(X'))) \,\mathrm{d}X'. \tag{27}$$

For $n = 0$, we obtain

$$W_0 = \int_{-X_{\min}}^{1} (1 - \exp(-\tau_{12}^r(X')))(g(X') - 1)\,\mathrm{d}X' , \tag{28}$$

with the expected results that $W_0 \gtrless 0$, if $B_{\nu_{12}}(T_e) \gtrless I_c\beta_{12}^3/\beta_{12}^1$ everywhere in the medium. Due to the symmetry of the expanding atmosphere with respect to the central core, expression (13) for the first order moment W_1 remains unchanged and relation (22) between the mass-loss rate and W_1 equally applies for the case of unsaturated P Cygni profiles affected by collisions.

B. ROTATION

If one superimposes a rotational velocity field $v^{\perp}(r)$ to the outward-accelerated motion of the atmosphere, it is straightforward to demonstrate that expression (13) for W_1 and relation (22) for the case of optically thin lines are still valid. Indeed, if there is rotation, the part of the line profile which accounts for the emissivity of the atmosphere remains a symmetric function of the frequency $X \in [-1, 1]$. Furthermore, as the expression of the velocity gradient

$$\frac{\partial v_s}{\partial s} = \frac{\mathrm{d}v}{\mathrm{d}r}\cos^2(\theta) + \frac{v(r)}{r}\sin^2(\theta) + \left(\frac{\mathrm{d}v^{\perp}}{\mathrm{d}r} - \frac{v^{\perp}}{r}\right)\sin(\theta)\cos(\theta) , \tag{29}$$

evaluated along the radial direction is unaffected by the presence of a transverse velocity field, we conclude that the first order moment W_1 still reduces to its expression (13).

4. Underlying Photospheric Absorption Line

With some direct calculations, Castor and Lamers (1979) have shown the importance of treating rigorously the interplay between photospheric and envelope lines in the calculation of P Cygni profiles. Since the latter cannot be considered as a linear superposition of the photospheric and envelope profiles, we naturally expect the expression of W_1 to be dependent on the presence of an underlying photospheric absorption line. If $\zeta(X')$ denotes the photospheric absorption profile in the frequency interval $X' \in [-1, 1]$, the line profile function (see relation (1)) is easily found to be

$$\frac{E(X)}{E_c} = \int_{\mathrm{Max}(|X|, -X_{\min})}^{1} \frac{\zeta(X')P(X')P(X', X)}{2X'}\,\mathrm{d}X' + \\ + \begin{cases} \zeta(X)\exp(-\tau_{12}^r(X)), & \text{if} \quad X \in [-1, X_{\min}] , \\ \zeta(X) , & \text{if} \quad X \in [X_{\min}, 1] , \end{cases} \tag{30}$$

assuming that $\zeta(-X) = \zeta(X)$.

If we combine relations (8) and (30), the expression for W_n now takes the form

$$W_n = (1 + (-1)^n) \int_{-X_{\min}}^{1} \frac{\zeta(X')P(X')X'^n}{2\beta_{12}^1(X')} \int_0^1 \mu^n \left(\frac{1 - \exp(-\tau_{12}(X', \mu))}{\tau_{12}(X', \mu)}\right) \times$$

$$\times \,d\mu \,dX' - (1 + (-1)^n) \int_0^1 X'^n (1 - \zeta(X'))\,dX' -$$

$$- (-1)^n \int_{-X_{\min}}^{1} \zeta(X') X'^n (1 - \exp(-\tau_{12}^r(X')))\,dX' . \tag{31}$$

For $n = 0$,

$$W_0 = \int_{-1}^{+1} (\zeta(X') - 1)\,dX' , \tag{32}$$

i.e., the zero order moment gives a measure of the equivalent width of the photospheric absorption line. For $n = 1$, we obtain

$$W_1 = \int_{-X_{\min}}^{1} \zeta(X') X' (1 - \exp(-\tau_{12}^r(X')))\,dX' , \tag{33}$$

and for the particular case of optically thin lines,

$$W_1 = \int_{-X_{\min}}^{1} \zeta(X') X' \tau_{12}^r(X')\,dX' . \tag{34}$$

Inserting results (16)–(20) in (34) and assuming that there is mass conservation of the species in the flow, we derive a new relation between W_1 and $\dot{M}n$ (level): namely,

$$W_1 = \frac{\Pi e^2}{mc} f_{12} \lambda_{12} \frac{A\,(\text{element})\dot{M}n\,(\text{level})}{4\Pi \bar{\mu} M_{\text{amu}} v_{\max}^2 R^*} q , \tag{35}$$

where

$$q = \int_{-X_{\min}}^{1} \zeta(X') \frac{d(1/L)}{dX'}\,dX' . \tag{36}$$

With these and previous results, we can now evaluate quantitatively the effect of an underlying photospheric absorption line when deriving the quantity $\dot{M}n$ (level) from the measurement of the first order moment W_1 of a P Cygni profile. By means of relations (22) and (35) we may directly infer that the relative error, expressed in %, which results between the determinations of $\dot{M}n$ (level) when including or neglecting the presence of a photospheric line is equal to $1 + q$. Calculations of this last quantity prompt us to adopt realistic functions for $\zeta(X')$ and $X'(L)$. In the following, we assume $\zeta(X')$ to have a Gaussian shape (cf. Olson and Castor, 1981) represented by

$$\zeta(X') = 1 - k \exp\left(-\frac{X'^2}{\sigma^2}\right), \tag{37}$$

where k is the line center absorption depth and $\sigma = X(\frac{1}{2})/\sqrt{\ln 2}$ is a width chosen such that the half width at half maximum is $X(\frac{1}{2})$. We also adopt the velocity laws

$$\left.\begin{aligned} X' &= -X_{\min} + (1 + X_{\min})(1 - \sqrt{1/L}), && \text{(a)} \\ X' &= -X_{\min} + (1 + X_{\min})(1 - 1/L), && \text{(b)} \\ X' &= \sqrt{1 + (X_{\min}^2 - 1)/L}, && \text{(c)} \end{aligned}\right\} \tag{38}$$

which are thought to encompass most of observed velocity fields prevailing in the winds of early-type stars (Castor *et al.*, 1975; Castor and Lamers, 1979; Olson, 1981; Olson and Ebbets, 1981).

Combining relations (36) and (37) for the velocity laws (a), (b), (c) in (38), we obtain

$$\left.\begin{aligned} 1 + q &= \frac{-k\sigma^2}{(1 + X_{\min})^2}(\exp(-X_{\min}^2/\sigma^2) - \exp(-1/\sigma^2)) + \\ &\quad + \frac{2k\sigma}{(1 + X_{\min})^2} \int_{-X_{\min}/\sigma}^{1/\sigma} \exp(-y^2)\,\mathrm{d}y, && \text{(a)} \\ 1 + q &= \frac{k\sigma}{(1 + X_{\min})} \int_{-X_{\min}/\sigma}^{1/\sigma} \exp(-y^2)\,\mathrm{d}y, && \text{(b)} \\ &\text{and} \\ 1 + q &= \frac{k\sigma^2}{(1 - X_{\min}^2)}(\exp(-X_{\min}^2/\sigma^2) - \exp(-1/\sigma^2)). && \text{(c)} \end{aligned}\right\} \tag{39}$$

For $-X_{\min} \ll 1$ and $-X_{\min} < \sigma < 1$, the corresponding asymptotic expressions are

$$\left.\begin{aligned} 1 + q &= k\sigma(\sqrt{\Pi} - \sigma), && \text{(a)} \\ 1 + q &= k\sigma\sqrt{\Pi}/2, && \text{(b)} \\ 1 + q &= k\sigma^2. && \text{(c)} \end{aligned}\right\} \tag{40}$$

Considering a deep photospheric absorption line with $k = 0.9$ such that $X(\frac{1}{2}) = 0.05, 0.1$, and 0.5, we have summarized in Table I the values of the relative error $1 + q$ resulting between the determinations of the quantity $\dot{M}n$ (level) when making use of relations (22) ($L_{\max} \gg 1$) and (35).

Being aware that the asymptotic expressions (40) consist in upper limits of the true value $1 + q$ and that, for most realistic cases (see Olson and Castor, 1981), $k < 0.9$ and $X(\frac{1}{2}) < 0.1$, we conclude that neglecting the presence of an underlying photospheric absorption line (see relation (22)) leads to an underestimate of the mass-loss rate never exceeding 20%. We also notice from Table I that the relative error $1 + q$ decreases as the steepness of the velocity field near the star increases.

TABLE I

Relative error $1 + q$ between the values of $\dot{M}n$ (level) derived by means of Equations (35) and (22) for $k = 0.9$

$1 + q$ (in %)	Velocity law		
$X(1/2)$	(a)	(b)	(c)
0.05	9.3	4.8	0.3
0.1	17.9	9.6	1.3
0.5	63.3	47.9	32.5

5. Case of a Doublet Transition

Most of P Cygni profiles observed in the ultraviolet spectrum of early-type stars are due to the scattering of line photons by a resonance doublet for which the separation of the two upper atomic levels 2 and 3, expressed in terms of the Doppler velocity Δv_{23}, is usually smaller than the maximal expansion velocity $v_{\max}$ of the envelope where they are formed. For such cases, the radiative transfer in outward-accelerating envelopes is complicated by the fact that photons emitted at a point R in the transition $3 \rightarrow 1$ can interact with atoms at distant points R', via the line transition $1 \rightleftharpoons 2$. In a more general context, the formation of resonance doublet profiles which takes into account these distant interactions between atoms and line photons has been throughout investigated by Surdej (1980, hereafter referred to as Paper II).

Considering as previously a central point-like source, we recall (cf. Paper II) that in terms of the dimensionless frequency X (cf. relation (5)) chosen such that $X = 0$ corresponds to the unshifted position of the red component $1 \rightleftharpoons 2$ of the doublet, the resonance profile is formed in the frequency interval $X \in [-1 - \Delta X_{23}, 1]$, where

$$\Delta X_{23} = \frac{\Delta v_{23}}{v_{\max}} . \tag{41}$$

If we follow a similar reasoning as in Paper I, it is straightforward to derive the expression $E(X)/E_c$ of the doublet profile. Separating the distinct contributions due to the scattering of line photons in the transitions $1 \rightleftharpoons 2$, $1 \rightleftharpoons 3$ and that due to the stellar photons which did not suffer any scattering, we find that

$$\frac{E(X)}{E_c} = \frac{E_{12}(X)}{E_c} + \frac{E_{13}(X)}{E_c} + \frac{E_{abs}(X)}{E_c}, \tag{42}$$

with

$$\left.\begin{aligned}
&\frac{E_{12}(X)}{E_c} = \int_{\mathrm{Max}(|X|,\,-X_{\min})}^{1} \frac{P_{12}(X')P_{12}(X',X)}{2X'}\,\mathrm{d}X', \\
&\qquad \text{if} \quad X \in [-1, 1], && \text{(a)} \\
&\frac{E_{13}(X)}{E_c} = \int_{\mathrm{Max}(|X+\Delta X_{23}|,\,-X_{\min})}^{1} \frac{P_{13}(Y')P_{13}(Y', X+\Delta X_{23})}{2Y'} \times \\
&\qquad \times \exp(-\tau_{12}(L(Y',X)))\,\mathrm{d}Y', && \text{(b)} \\
&\qquad \text{if} \quad X \in [-1-\Delta X_{23}, 1-\Delta X_{23}], \\
&\text{and} \\
&\frac{E_{abs}(X)}{E_c} = \left\{\begin{array}{ll}
\exp(-\tau^r_{13}(X+\Delta X_{23})), & \text{if} \quad X \in [-1-\Delta X_{23}, -1], \\
\exp(-\tau^r_{12}(X) - \tau^r_{13}(X+\Delta X_{23})), & \\
 & \text{if} \quad X \in [-1, X_{\min} - \Delta X_{23}], \\
\exp(-\tau^r_{12}(X)), & \text{if} \quad X \in [X_{\min} - \Delta X_{23}, X_{\min}], \\
1, & \text{if} \quad X \in [X_{\min}, 1].
\end{array}\right\} && \text{(c)}
\end{aligned}\right\} \tag{43}$$

The functions $P_{ij}(X')$, $P_{ij}(X', X)$ have their usual meaning (cf. relations (3) and (4)), and we notice in the integral of the contribution (43.b) the presence of the attenuation factor $\exp(-\tau_{12}(L(Y', X)))$ accounting for the distant radiative interactions mentioned above (see also Paper II).

With the definition (7) for the nth order moment of a resonance doublet profile, we obtain

$$W_n = \int_{-1-\Delta X_{23}}^{1} X^n \left(\frac{E(X)}{E_{c_*}} - 1\right) \mathrm{d}X. \tag{44}$$

To perform this last integration is a long but not difficult task. If one takes care of integrating expression (42) along each of the frequency intervals defined in (43.c) and using, when necessary, the variable transformations $Y = X + \Delta X_{23}, X = X'\mu$ or $Y = Y'\mu$ according to a similar procedure as shown in the Appendix, we easily derive the interesting result asserting that

$$W_n = (1 + (-1)^n) \int_{-X_{\min}}^{1} \frac{P_{12}(X')X'^n}{2\beta_{12}^1(X')} \int_0^1 \mu^n \left(\frac{1 - \exp(-\tau_{12}(X', \mu))}{\tau_{12}(X', \mu)}\right) \mathrm{d}\mu\, \mathrm{d}X' +$$

$$+ \int_{-X_{\min}}^{1} \frac{P_{13}(Y')}{2\beta_{13}^1(Y')} \int_{-1}^{1} (Y'\mu - \Delta X_{23})^n \left(\frac{1 - \exp(-\tau_{13}(Y', \mu))}{\tau_{13}(Y', \mu)}\right) \times$$

$$\times \exp(-\tau_{12}(Y', \mu))\, \mathrm{d}\mu\, \mathrm{d}y' -$$

$$- (-1)^n \Bigg[\int_{-X_{\min}}^{-X_{\min} + \Delta X_{23}} X'^n (1 - \exp(-\tau_{12}^r(X')))\, \mathrm{d}X' +$$

$$+ \int_{1}^{1 + \Delta X_{23}} X'^n (1 - \exp(-\tau_{13}^r(X' - \Delta X_{23})))\, \mathrm{d}X' +$$

$$+ \int_{-X_{\min} + \Delta X_{23}}^{1} X'^n (1 - \exp(-\tau_{12}^r(X') - \tau_{13}^r(X' - \Delta X_{23})))\, \mathrm{d}X' \Bigg] . \quad (45)$$

For the case of optically thin lines, i.e., $\tau_{12} < 1$, $\tau_{13} < 1$, the expression of the first order moment W_1 as given by (45) reduces to

$$W_1 = \int_{-X_{\min}}^{1} X'(\tau_{12}^r(X') + \tau_{13}^r(X'))\, \mathrm{d}X' , \quad (46)$$

and we conclude that the moment W_1 for a doublet is the same as that for a single line transition provided that λ_{12} and f_{12} in Equation (22) are replaced by

$$\lambda_D = \frac{f_{12}\lambda_{12} + f_{13}\lambda_{13}}{f_{12} + f_{13}} , \quad (47)$$

and

$$f_D = f_{12} + f_{13} . \quad (48)$$

However, when including for instance the effect of collisions (cf. Section 3) in the treatment of radiative transfer, we do not recover the result (46). Instead, we obtain

$$W_1 = \int_{-X_{\min}}^{1} X'(\tau_{12}^r(X') + \tau_{13}^r(X'))\, \mathrm{d}X' +$$

$$+ \Delta X_{23} \int_{-X_{\min}}^{1} \tau_{13}^r(X')(1 - g(X'))\, \mathrm{d}X' ; \quad (49)$$

and, similarly,

$$W_0 = \int_{-X_{\min}}^{1} (\tau_{12}^r(X') + \tau_{13}^r(X'))(g(X') - 1)\,\mathrm{d}X' , \tag{50}$$

with $g(X')$* being defined in (26).

To remedy this situation, it is in fact necessary to redefine W_1 such that in relation (7) λ_{12} is replaced by λ_D. In doing so, we find the equation

$$W_1^D = W_1 + \frac{f_{13}}{f_D} \Delta X_{23} W_0 , \tag{51}$$

and when replacing W_1 and W_0 in this last relation by their expression given in (49) and (50), we obtain the expected result

$$W_1^D = \int_{-X_{\min}}^{1} X'(\tau_{12}^r(X') + \tau_{13}^r(X'))\,\mathrm{d}X' . \tag{52}$$

We finally conclude that relation (22) between the mass-loss rate and W_1 derived for a single line transition equally applies for the case of a resonance doublet profile provided that λ_{12} in Equations (7) and (22) is replaced by the effective wavelength λ_D (see relation (47)) and f_{12} in Equation (22) is taken to be the doublet oscillator strength f_D (see relation (48)).

6. Finite Dimensions and Limb Darkening of the Stellar Core

In Sections 2–5, we have explicitly assumed that the size of the expanding atmosphere from which arise the observed P Cygni profiles is large with respect to the central stellar core, i.e., $L_{\max} \gg 1$. In the context of deriving mass-loss rates from the analysis of unsaturated P Cygni profiles, we naturally wonder about the validity of all previous results (cf. Equation (22)) when the point-like source approximation is not exactly fulfilled. Therefore, we investigate in the present section how the transfer of line radiation is affected by the effects of occultation and inclination (cf. Paper I) due to the finite size of the central source.

A. Occultation effect

At a distance L from the stellar core (see Figure 1), a certain fraction of emitted line photons, roughly proportional to the dilution factor

$$W = \tfrac{1}{2}(1 - \sqrt{1 - (1/L)^2}) , \tag{53}$$

are back-scattered towards the star and thus remain unobservable. In the frame of an

* Let us note that in the optically thin approximation, the function $g(X')$ has the same form for both line transitions.

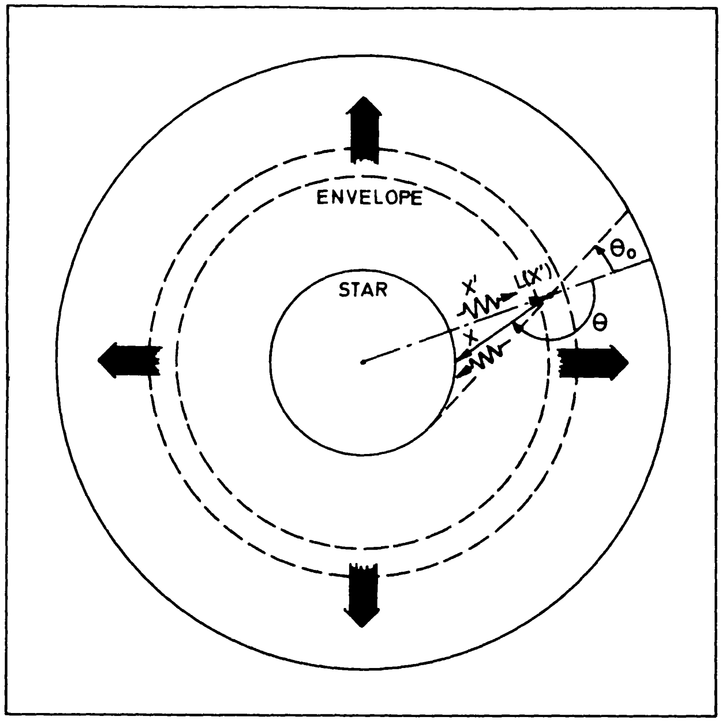

Fig. 1. Occultation by the stellar core of the line photons emitted with a frequency $X = X'\mu$ along a direction $\mu \leq -\mu_0$ (see text). Not to scale.

observer, these lost photons have positive frequencies $X \in [0, 1]$ and there results a gap in the red wing of the observed line profile. This is the well-known occultation effect.

In the following, let us evaluate quantitatively this occultation effect. If a line photon is emitted locally at a distance $L(X')$ along a direction θ with a frequency $X \in]0, 1]$, it will be seen by a fixed observer (see Figure 1) if, and only if

$$\left.\begin{aligned} &\cos(\theta) \geq -\cos(\theta_0)\,, \\ \text{where}\quad &\\ &\cos(\theta_0) = 1 - 2W(L(X'))\,. \end{aligned}\right\} \tag{54}$$

By means of relation (5), Equation (54) may be rewritten as

$$X' \leq -X/(1 - 2W(L(X')))\,. \tag{55}$$

This last condition implies that only those stellar photons having a frequency such that $X' \in [-1, X^N]$, where

$$X^N(1 - 2W(L(X^N))) = -X\,, \tag{56}$$

will contribute to the formation of the line profile $E(X)/E_c$ at the frequency X. Consequently, in the frequency interval $X \in]0, 1]$, the lower limit of integration in Equation (1) must be replaced by $\mathrm{Max}(|X^N|, -X_{\min})$. Let us still remark that in the frequency interval $X \in [X^0, 1]$, where

$$X^0 = 1 - 2W(L_{\max}), \tag{57}$$

there will be no contribution at all by the scattered stellar photons to the line profile.

B. Inclination effect

A second effect caused by the finite size of the stellar core arises from the fact that atoms located at a distance L interact with the stellar photons not only along the radial direction, but also along inclined directions such as $\theta \in [0, \arcsin(1/L)]$ (see Figure 2). We shall call this the 'inclination effect'.

Considering first the contribution of scattered light to the line profile function $E(X)/E_c$, it is clear that in Equation (1) the factor $P(X')$ must be replaced by a new quantity $P^N(X')$ which accounts for the fraction of stellar photons emitted along directions $\theta \in [0, \arcsin(1/L(X'))]$ which undergo at $L(X')$ a first scattering. Denoting by β_{12}^3 (resp. $\beta_{12}^3(\infty)$) the escape probability for a photon emitted at $L(X')$ to strike the stellar source (resp. the point-like source), we have the obvious relation

$$P^N(X')/P(X') = \beta_{12}^3/\beta_{12}^3(\infty). \tag{58}$$

If $\psi(\mu^*)$ is the limb-darkening law of the stellar core such that

$$I_c(\mu^*) = I_c\psi(\mu^*) \tag{59}$$

represents the specific intensity radiated by the central source along a direction

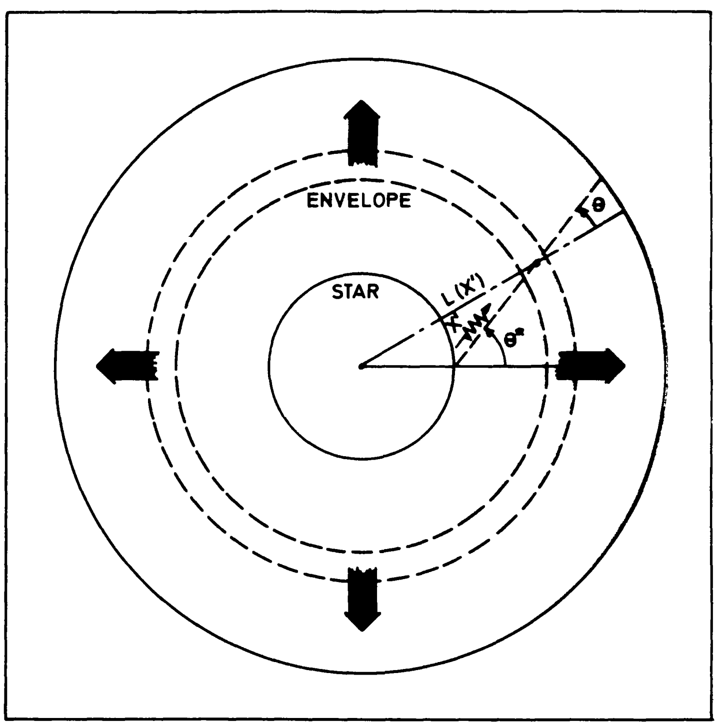

Fig. 2. Inclination of the stellar rays emitted with a limb darkening law $\psi(\mu^*)$ along a direction $\mu^* \in [0, 1]$ (see text). Not to scale.

$\mu^* \in [0, 1]$ (see Figure 2), the expression for β_{12}^3 is easily found to be (see Equation (17) in Surdej, 1981; hereafter referred to as Paper III)

$$\beta_{12}^3 = \tfrac{1}{2} \int\limits_{1-2W(L(X'))}^{1} \psi(\mu^*) \left(\frac{1 - \exp(-\tau_{12}(X', \mu))}{\tau_{12}(X', \mu)} \right) \mathrm{d}\mu\,, \tag{60}$$

with

$$\mu^* = \sqrt{1 - (1 - \mu^2) L^2(X')}\,. \tag{61}$$

If the central source is point-like, i.e. $R^* \to 0$ and $L(X') \to \infty$, the asymptotic expression of β_{12}^3 given by (60) is reduced to

$$\beta_{12}^3(\infty) = \frac{1}{4L^2(X')} \left(\frac{1 - \exp(-\tau_{12}^r(X'))}{\tau_{12}^r(X')} \right) \frac{\overline{I_c}}{I_c}\,, \tag{62}$$

where

$$\overline{I_c} = 2 I_c \int\limits_0^1 \psi(\mu^*) \mu^* \, \mathrm{d}\mu^* \tag{63}$$

represents the mean specific intensity of the stellar continuum $I_c(\mu^*)$ integrated over the disk of the central core. Combining Equations (3) and (58)–(63), we finally obtain

$$P^N(X') = \beta_{12}^3 \tau_{12}^r(X') \frac{I_c}{\overline{I_c}} 4L^2(X')\,. \tag{64}$$

Let us concentrate now on the fraction $E_{\text{abs}}(X)/E_c$ of unscattered stellar photons contributing to the line profile in the frequency interval $X \in [-1, 0[$. As the amount $E_{\text{abs}}(X)$ of unscattered stellar radiation reaching a fixed observer at a frequency $X \in [-1, 0[$ is expressed by

$$E_{\text{abs}}(X) = \tfrac{1}{2} \int\limits_0^1 I_c(\mu^*) \exp(-\tau_{12}(X', \mu)) \mu^* \, \mathrm{d}\mu^*\,; \tag{65}$$

and since, in the absence of any scattering in the atmosphere, we have

$$E_c = \tfrac{1}{2} \int\limits_0^1 I_c(\mu^*) \mu^* \, \mathrm{d}\mu^*\,, \tag{66}$$

and directly find by means of relations (59), (63), (65), and (66) that the quantity

$\exp(-\tau_{12}^r(X'))$ appearing in Equation (1) must be replaced by

$$\frac{E_{\mathrm{abs}}(X)}{E_c} = \frac{2I_c}{\bar{I}_c} \int_0^1 \psi(\mu^*) \exp(-\tau_{12}(X', \mu))\mu^* \, \mathrm{d}\mu^* , \tag{67}$$

with the frequency X' statisfying the equation

$$X' = X/\sqrt{1 - (1 - \mu^{*2})/L^2(X')} , \tag{68}$$

and where μ is derived from Equation (61).

C. Line profile and moments W_n

On account of both the occultation and inclination effects investigated here above, expression (1) of the line profile function transforms into

$$\frac{E(X)}{E_c} = \begin{cases} \displaystyle\int_{\mathrm{Max}(|X|, -X_{\min})}^{1} \frac{P^N(X')P(X', X)}{2X'} \, \mathrm{d}X' + \frac{E_{\mathrm{abs}}(X)}{E_c} , \\ \quad \text{if} \quad X \in [-1, 0] , \\ \displaystyle\int_{\mathrm{Max}(|X^N|, -X_{\min})}^{1} \frac{P^N(X')P(X', X)}{2X'} \, \mathrm{d}X' + 1 , \\ \quad \text{if} \quad X \in \,]0, X^0] , \\ 1, \quad \text{if} \quad X \in \,]X^0, 1] , \end{cases} \tag{69}$$

with the new quantities X^N, X^0, $P^N(X')$ and $E_{\mathrm{abs}}(X)/E_c$ being defined in (56), (57), (64), and (67), respectively.

Adopting a similar reasoning as in the Appendix and making use of relations (5), (20), and (61) in the variable transformations $\mu^* = \mu^*(\mu, X)$ and $X = X(X', \mu)$, we obtain for the expression of the nth order moment (see Equations (8) and (69))

$$\begin{aligned} W_n = {} & (1 + (-1)^n) \int_{-X_{\min}}^{1} \frac{P^N(X')X'^n}{2} \int_0^1 \mu^n P(X', X'\mu) \, \mathrm{d}\mu \, \mathrm{d}X' - \\ & - \int_{-X_{\min}}^{1} \frac{P^N(X')X'^n}{2} \int_{1-2W(L(X'))}^{1} \mu^n P(X', X'\mu) \, \mathrm{d}\mu \, \mathrm{d}X' - \\ & - (-1)^n \int_{-X_{\min}}^{1} \frac{P^N(X')}{2} \frac{\beta_{12}^1(X')}{\beta_{12}^3(X')} X'^n \times \end{aligned} \tag{70}$$

$$\times \int_{1-2W(L(X'))}^{1} \mu^n \psi(\mu^*) P(X', X'\mu)\, d\mu\, dX' .$$

For $n = 0$, the last equation reduces to

$$W_0 = -\int_{-X_{\min}}^{1} P^N(X') \beta_{12}^{3'}(X') / \beta_{12}^{1}(X')\, dX' , \tag{71}$$

where $\beta_{12}^{3'}(X')$ stands for the escape probability $\beta_{12}^{3}(X')$ calculated without limb darkening of the stellar core: i.e., $\psi(\mu^*) = 1$. As expected, the non-zero value of W_0 directly accounts for the equivalent width of the truncated part of the line profile caused by the partial occultation of the rear part of the expanding atmosphere.

For $n = 1$, we derive from Equation (70)

$$W_1 = \int_{-X_{\min}}^{1} \frac{P^N(X')X'}{\beta_{12}^{3}(X')} \left[\gamma_{12}(X') - \frac{\beta_{12}^{3}(X')}{\beta_{12}^{1}(X')} \gamma_{12}'(X') \right] dX' , \tag{72}$$

where the escape probability

$$\gamma_{12}(X') = \tfrac{1}{2} \int_{1-2W(L(X'))}^{1} \psi(\mu^*) \left(\frac{1 - \exp(-\tau_{12}(X', \mu))}{\tau_{12}(X', \mu)} \right) \mu\, d\mu \tag{73}$$

is directly proportional to the stellar radiative force acting on an atom at a distance $L(X')$ (see Surdej, 1978). As before, the quantity $\gamma_{12}'(X')$ stands for $\gamma_{12}(X')$ calculated with $\psi(\mu^*) = 1$.

Within the optically thin approximation, Equation (72) reduces to

$$W_1 = \int_{-X_{\min}}^{1} X' \tau_{12}^{r}(X') \left(1 - \beta_{12}^{3}(X') \frac{I_c}{\bar{I}_c} \right) dX' ; \tag{74}$$

and since it is well established that the quantity $\beta_{12}^{3} I_c / \bar{I}_c$ is almost independent of the limb-darkening of the stellar core (see Section 5.1 in Paper III), we conclude that the moment W_1 is itself also negligibly affected by any limb darkening effects. We shall subsequently illustrate this result with a numerical application for the case of a strong limb-darkening law.

Assuming that there is mass conservation of the species under study and that

$$\beta_{12}^{3}(X') \frac{I_c}{\bar{I}_c} \sim W(L(X')) , \tag{75}$$

we obtain by means of Equations (19) and (74) a new relation between W_1 and $\dot{M}n$ (level)

$$W_1 = \frac{\Pi e^2}{mc} f_{12}\lambda_{12} \frac{A\,(\text{element})\,\dot{M}n\,(\text{level})}{4\Pi\bar{\mu}M_{\text{amu}}v_{\max}^2 R^*} q^c\,, \tag{76}$$

where the quantity

$$q^c = -\left(\frac{1}{2} + \frac{\Pi}{8}\right) + \frac{1}{2L_{\max}} + \frac{1}{4L_{\max}} \sqrt{1 - \left(\frac{1}{L_{\max}}\right)^2} + \frac{1}{4} \arcsin\left(\frac{1}{L_{\max}}\right) \tag{77}$$

is a correcting factor essentially due to the finite size of the stellar core (see Equation (22) for comparison). Let us point out that under the same conditions, CLS have derived in another formalism a very similar relation between W_1 and $\dot{M}n$ (level). For their particular choice of the velocity field $X' = (1 - 1/L)$ and opacity distribution $\tau_{12}^r(X') \sim (1 - X')$, a correcting factor $q^c = -0.60$ characterizes their results rather than we have $qc(L_{\max} \to \infty) = -0.89$ under the only assumption that there is mass-conservation in the flow.

If we include the strong limb-darkening law $\psi(\mu^*) = \mu^{*2}$ in Equation (74), we would easily find that result (76) is recovered with, however, the factor q^c replaced by

$$q^\psi = -\left(\frac{\Pi}{2} - \frac{2}{3}\right) - \frac{2}{3}L_{\max} + \frac{2}{3}\left(L_{\max} + \frac{1}{2L_{\max}}\right)\sqrt{1 - \left(\frac{1}{L_{\max}}\right)^2} + {}$$
$$+ \arcsin\left(\frac{1}{L_{\max}}\right). \tag{78}$$

With all previous relations, we are now able to calculate the relative error affecting the determinations of the mass-loss rate based on the expressions of the moment W_1 when including or neglecting the finite size and/or a strong limb darkening law for the stellar core. By means of Equations (22), (76)–(78), we can directly state that the relative error, expressed in %, which results between the two determinations of $\dot{M}n$ (level) when

TABLE II

Relative error $1 + q^c$ (resp. $1 + q^\psi$) between the values of $\dot{M}n$ (level) derived by means of Equations (76), (77), (resp. (76), (78)), and (22) as a function of $L_{\max}$

$L_{\max}$	$1 + q^c$ (in %)	$1 + q^\psi$ (in %)
2	59.6	58.5
3	43.7	42.6
4	35.6	34.5
5	30.7	29.5
10	20.7	19.6
50	12.7	11.6
100	11.7	10.6
∞	10.7	9.6

taking or not taking into account the finite size (resp. the limb-darkening law $\psi(\mu^*) = \mu^{*2}$) of the central source is equal to $1 + q^c$ (resp. $1 + q^\psi$). We have reported in Table II the values of these relative errors as a function of $L_{\max}$.

From the results in the first column of Table II, we conclude that neglecting the finite size of the stellar core leads to an underestimate, usually less than 50%, in the determination of the mass-loss rate. Furthermore, by comparing the values for $1 + q^c$ and $1 + q^\psi$ in that same table, we confirm the very negligible effect ($\sim 1\%$) of a strong limb-darkening on the evaluation of the quantity $\dot{M}n$ (level).

In practice, the relative size $L_{\max}$ of the expanding atmosphere where is formed an observed line profile is unknown. As it is reasonable to assume that $L_{\max} \gg 1$, the final relation which should be adopted between $\dot{M}n$ (level) and the observed value W_1 of an unsaturated P Cygni profile is

$$\dot{M}n\,(\text{level}) = \left(\frac{\Pi e^2}{mc}\,\frac{f_{12}\lambda_{12}A\,(\text{element})}{4\Pi\bar{\mu}M_{\text{amu}}v_{\max}^2R^*}\right)^{-1} W_1/q^c(\infty)\,, \tag{79}$$

where $q^c(\infty) = -0.8927$.

By using this formula, we can ascertain that due to the unknown value of $L_{\max}$, the relative error $(1 - q^c/q(\infty))$ affecting the determination of $\dot{M}n$ (level) on the basis of the observed moment W_1 for an unsaturated P Cygni profile should be less than about 30% (see Table III).

TABLE III

Relative error $1 - q^c/q^c(\infty)$ between the values of $\dot{M}n$ (level) derived by means of Equations (76) and (79) as a function of $L_{\max}$

$L_{\max}$	$1 - q^c/q^c(\infty)$ (in %)
2	54.8
3	37.0
4	27.9
5	22.3
10	11.2
50	2.2
100	1.1
∞	0.0

7. Conclusions

In the framework of the Sobolev approximation for the transfer of line photons (two-level atom model), we have studied the relation between the first order moment $W_1 \propto \int (E(\lambda)/E_c - 1)(\lambda - \lambda_0)\,d\lambda$ and the quantity $\dot{M}n$ (level) under various physical and geometrical conditions.

Considering first an outward-accelerating envelope around a central point-like source, we have derived the general expression of the nth order moment W_n of a line profile (conservative scattering) in terms of the usual parameters inherent to Sobolev-type theories. For the case of optically thin lines and under the reasonable assumption that there is mass-conservation of the species in the flow (see Section 2), we have stated that the relation between W_1 and $\dot{M}n$ (level), initially established by Castor *et al.*(1981), is in fact independent of any adopted velocity field $v(r)$, and a fortiori of any ad hoc distribution for the fictive radial opacity $\tau_{12}^r(X')$.

Investigating the effects due to collisions ($\varepsilon \neq 0$) and/or the superposition of a rotational velocity field $v^{\perp}(r)$ in the atmosphere, we have concluded that none of these effects alter the previous results. We have then shown that neglecting the presence of an underlying photospheric absorption line has the net result of underestimating the mass-loss rate by a factor of 20% at maximum.

Most of P Cygni profiles observed in the ultraviolet spectrum of early-type stars are associated with resonance doublets for which the separation Δv_{23} of the two upper atomic levels 2 and 3 is usually smaller than the maximum expansion velocity $v_{\max}$ of the envelope where they are formed. For the case of such unresolved doublet profiles, we have derived identical results to those for a single line transition. Care should however be taken in adopting the wavelength λ_D and the total oscillator strength f_D of the resonance doublet when calculating W_1 and $\dot{M}n$ (level).

Since we do not really know what is the relative size $L_{\max}$ of the expanding envelope with respect to the stellar core, we have investigated in the last section the core effects (occultation, inclination and limb darkening) onto the determination of the mass-loss rate. This study has led us to refine the value of the multiplicative constant fixing the ratio of W_1 to $\dot{M}n$ (level) (see formula (79)). From detailed calculations we can ascertain that the relative error affecting the determination of the mass-loss rate by means of this last formula should be less than 30%, whatever the unknown value of $L_{\max}$ is. As to the limb-darkening, we have shown that its effects were always negligible.

Recalling that the error affecting the measurement of W_1 due to uncertainties in the setting of the wavelength scale λ and/or of the continuum level E_c should not exceed 10% (see Castor *et al.*, 1981) and in view of all previous results concerning the various physical and geometrical effects investigated in the present paper, we can safely conclude that Equation (79) offers a very powerful method of deriving mass-loss rates from the first order moment of unsaturated P Cygni profiles. The total uncertainty possibly affecting this determination should be smaller than 60% (20% + 30% + 10%), or still more conservatively than a factor 2. The only problem which still defies us is how 'to guess' a precise estimate for the fractional abundance n (level) of the elements observed in the stellar winds.

Acknowledgment

My thanks are due to Jean-Pierre Swings for reading the manuscript.

Appendix

Inserting expression (1) of the line profile function $E(X)/E_c$ into Equation (8) for the moment W_n leads to the result

$$W_n = \left[\int_{-1}^{X_{\min}} \mathrm{d}X \int_{-X}^{1} \mathrm{d}X' + \int_{X_{\min}}^{-X_{\min}} \mathrm{d}X \int_{-X_{\min}}^{1} \mathrm{d}X' + \right.$$

$$\left. + \int_{-X_{\min}}^{1} \mathrm{d}X \int_{X}^{1} \mathrm{d}X' \left(\frac{X^n P(X') P(X', X)}{2X'} \right) \right] -$$

$$- \int_{-1}^{X_{\min}} X'^n (1 - \exp(-\tau_{12}^r(X'))) \, \mathrm{d}X' \,. \tag{A1}$$

We have sketched in Figure A1 the domain of integration defined by the first set of

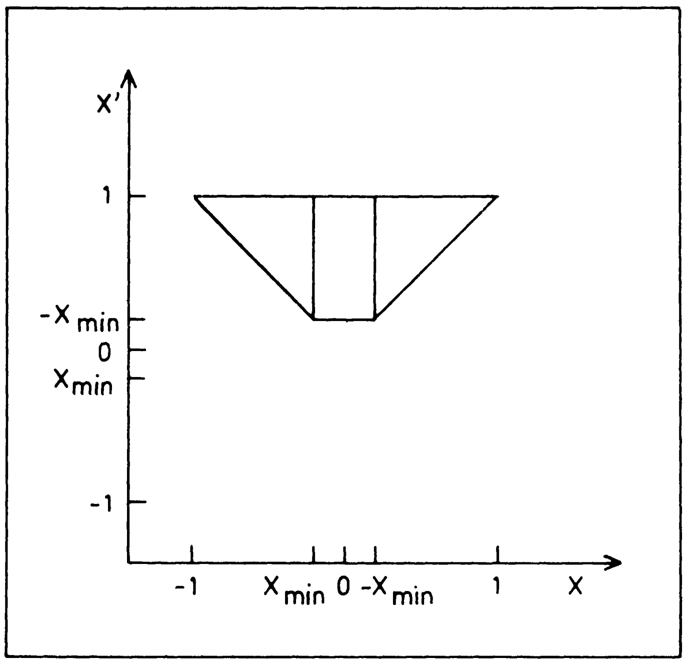

Fig. A1. Domain of integration in the plane (X, X') of the function $X^n P(X') P(X', X)/2X'$ in Equation (A1).

integrals in the plane (X, X'). Interchanging the order of integration between X and X' in the last equation, and since

$$\left. \begin{aligned} &(-X)^n = (-1)^n X^n \,, \\ \text{and that} \quad & \\ &P(X', -X) = P(X', X) \,, \end{aligned} \right\} \tag{A2}$$

we easily find that Equation (A1) reduces to

$$W_n = (1 + (-1)^n) \int_{-X_{\min}}^{1} \mathrm{d}X' \int_{0}^{X'} \mathrm{d}X \left(\frac{X^n P(X') P(X', X)}{2X'} \right) - \\ - \int_{-1}^{X_{\min}} X'^n (1 - \exp(-\tau_{12}^r(X'))) \, \mathrm{d}X' . \tag{A3}$$

Applying the variable transformations

$$\left. \begin{aligned} X &= X'\mu \quad , \\ \mathrm{d}X &= X' \, \mathrm{d}\mu \, , \end{aligned} \right\} \tag{A4}$$

and

$$\left. \begin{aligned} X' &= -X \quad , \\ \mathrm{d}X' &= -\mathrm{d}X \, , \end{aligned} \right\} \tag{A5}$$

to the first and second members of Equation (A3), respectively, and adopting the implicit convention that for $X' \in [-X_{\min}, 1]$ $\tau_{12}^r(X')$ is evaluated at $-X'$, we finally obtain the desired expression for W_n as

$$W_n = (1 + (-1)^n) \int_{-X_{\min}}^{1} \frac{P(X') X'^n}{2\beta_{12}^1(X')} \int_{0}^{1} \mu^n \left(\frac{1 - \exp(-\tau_{12}(X', \mu))}{\tau_{12}(X', \mu)} \right) \mathrm{d}\mu \, \mathrm{d}X' - \\ - (-1)^n \int_{-X_{\min}}^{1} X'^n (1 - \exp(-\tau_{12}^r(X'))) \, \mathrm{d}X' . \tag{A6}$$

Note added in proof. A recent work by Surdej (1982, submitted to *Astron. Astrophys.*) confirms the validity of the present results (cf. Equation (79)) irrespective of the Sobolev approximation used for the transfer of line rotation in outward-accelerating envelopes as well as of the assumption of mass-conservation for the species in the flow.

For practical use, Equation (79) may be rewritten as

$$\dot{M}(-M_\odot/\text{year})\bar{n}(\text{level}) = -1.19 \times 10^{21} \, v_{\max}^2 \, (\mathrm{km\ s^{-1}}) \times \\ \times R^*(-R_\odot) W_1 / f_{12} \lambda_{12} (10^3 \text{ Å}) \, A(\text{element})) ,$$

where

$$\bar{n}(\text{level}) = \int_{1}^{L_{\max}} (W(L) - 1) \frac{n(\text{level})}{L^2} \, \mathrm{d}L \bigg/ \int_{1}^{L_{\max}} (W(L) - 1) \frac{\mathrm{d}L}{L^2}$$

is a representative value for the fractional abundance of the relevant ion in the expanding atmosphere.

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