An Exact Method for Designing Shewhart $\bar{X}$ and S^2 Control Charts to Guarantee In-Control Performance

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The in-control performance of Shewhart $\bar{X}$ and S^2 control charts with estimated in-control parameters has been evaluated by a number of authors. Results indicate that an unrealistically large amount of Phase I data is needed to have the desired in-control average run length (ARL) value in Phase II. To overcome this problem, it has been recommended that the control limits be adjusted based on a bootstrap method to guarantee that the in-control ARL is at least a specified value with a certain specified probability. In this article we present simple formulas using the assumption of normality to compute the control limits and therefore, users do not have to use the bootstrap method. The advantage of our proposed method is in its simplicity for users; additionally, the control chart constants do not depend on the Phase I sample data.

Keywords: Quality Control; Control Charts; Statistical Process Control (SPC); Adjusted Control Limits; Expected Value of the Run Length; Average Run Length (ARL).

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1. Introduction

Control charts are frequently used for monitoring processes. Often it is assumed that the variable of interest (X) is normally distributed with unknown in-control mean μ_0 and variance σ_0^2, i.e., $X \sim N(\mu_0, \sigma_0^2)$. It is customary to estimate these parameters through m initial samples each of size n. This stage is called Phase I. The control limits are then calculated based on the estimated parameters $(\hat{\mu}_0, \hat{\sigma}_0^2)$ and an out-of-control signal is given when an observation falls beyond the control limits. The goal in Phase II, the monitoring phase, is to detect shifts from the in-control process parameters as quickly as possible. For a recent overview on Phase I issues and methods, readers are referred to Jones-Farmer et al. (2014).

Albers and Kallenberg (2004a) showed that the impact of estimation is considerably greater than what was generally thought. Different Phase I samples give different parameter estimates and hence different control limits are obtained by different users. Therefore, there is variability among different users and hence the effect of estimation on the performance of control charts conditional on the Phase I data should be considered.

Control chart performance is usually evaluated by the average run length (ARL) measure, where the ARL is the expected number of samples until a control chart signals. When the process is in control (out of control), a large (small) ARL value is desired. Consequently, considerable attention has been given to the effects of parameters estimation on the ARL metric. See, for example, Chen (1997), Albers and Kallenberg (2004 a&b), Bischak and Trietsch (2007), Testik (2007) and Castagliola et al. (2009, 2012). For thorough literature reviews on the impact of parameter estimation on the performance of different types of control charts, readers are referred to Psarakis et al. (2014).

Most researchers have determined the required size of the Phase I dataset so that the average of the in-control ARL values among users (AARL) is suitably close to the desired value (ARL_0). To obtain the AARL value, one averages over the distribution of the parameter estimators. Thus, the use of this metric reflects the marginal performance. For example, Quesenberry (1993) concluded that m=400/(n-1) Phase I samples are enough to overcome the effect of estimation errors for the Shewhart $\bar{X}$ chart. Recently, however, some authors have advocated that the standard deviation of the ARL (SDARL) must be accounted for in determining the amount of Phase I data and pointed out that the sole use of the AARL metric can give misleading conclusions. Small SDARL values mean that users would tend to have the in-control ARL values close to the desired value. In most applications though, not enough Phase I data is available to ensure a small enough SDARL to make it possible for the user to obtain the desired in-control ARL. For more information on applications of the SDARL metric, readers are referred to Jones and Steiner (2012), Zhang et al. (2013, 2014), Lee et al. (2013), Aly et al. (2015), Epprecht et al. (2015), Faraz et al. (2015, 2016), Saleh et al. (2015a, 2015b) and Zhao and Driscoll (2016). Also see Gauo and Wang (2017) for closely related work on the two sided S^2 control charts.

Recently, Gandy and Kvaløy (2013) proposed a new method to design control charts by bootstrapping the Phase I data to guarantee the conditional performance of control charts with a pre-specified probability. For example, their approach could be used to adjust the control limits such that 95% of the constructed control charts would have in-control ARL values greater than or equal to 370.40. Moreover, the adjusted control limits increase only slightly the out-of-control ARL (ARL_1) compared to the case when the traditional control limits are used. They concluded that their approach can work well even with a small amount of Phase I data. The approach is effective, accurate and practical and therefore should be encouraged in practice. The difficulty of the bootstrap method, however, is that it requires users to adjust the control limits through a somewhat computationally intensive approach.

In our procedure which involves imposing the normality assumption, we derive the exact distribution of the bootstrap control limits through parametric bootstrap. Here, the parametric bootstrap is simply the substitution approach (See, Remark 1 in Gandy and Kvaløy, 2013, p.6). Therefore, it enables us to calculate the exact formulas for the values that the Gandy and Kvaløy (2013)'s bootstrap method estimates so that the bootstrap computations can be avoided.

In Section 2, we review the use of $\bar{X}$ and S^2 control charts with estimated parameters. In Section 3, following Remark 1 in Gandy and Kvaløy (2013, p.6), we derive the exact solution to the bootstrap method and compare the in-control performance of our method with the bootstrap approach and classical control charts. Finally, concluding remarks and recommendations are given in the last section.

2. $\bar{X}$ and S^2 Control Charts with Estimated Parameters

Consider a process with a quality characteristic which is assumed to be normally distributed with in-control mean μ_0 and variance σ_0^2. Let $X_{ij},\ i = 1,\ 2,\ 3, \ldots$ and $j = 1, 2, \ldots, n$ represent the observations and $\bar{X}_i$ and S_i^2 , $i = 1, 2, \ldots$, be the average and variance of the independent samples (each of size n). For the sake of simplicity, first we consider one sided Shewhart $\bar{X}$ and S^2 charts with upper control limits. When the in-control process parameters are known, the upper control limits for the Shewhart $\bar{X}$ and S^2 charts are calculated as follows (see Montgomery, 2013):

$$UCL_{\bar{X}} = \mu_0 + K\frac{\sigma_0}{\sqrt{n}}$$

$$UCL_{S^2} = \frac{\sigma_0^2}{n-1} L$$

where $K = z_{1-\alpha}$ and $L = \chi^2_{(1-\alpha, n-1)}$, where z_p represents the $100p^{\text{th}}$ percentile of the standard normal distribution and $\chi^2_{(p,v)}$ represents the $100p^{\text{th}}$ percentile of the chi-square distribution with v degrees of freedom.

In practice, the in-control process parameters are usually unknown and therefore must be estimated from historical data $X_{ij}, i = 1, \ldots, m$ and $j = 1, \ldots, n$. Here m is the number of Phase I samples with n the sample size. We let $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_m$ and $S_1^2, S_2^2, \ldots, S_m^2$ be the averages and variances of the m independent samples each of size n. Then the estimators we use for the in-control process mean and variance are as follows (See, Montgomery, 2013):

$$\hat{\mu}_0 = \frac{\sum_{i=1}^{m} \overline{X}_i}{m} \tag{1}$$

$$\hat{\sigma}_0^2 = \frac{\sum_{i=1}^{m} S_i^2}{m} \tag{2}$$

There are several estimators for the process standard deviation. For example, Chakraborti (2006) recommended the use of $\hat{\sigma}_0 = \sqrt{\hat{\sigma}_0^2}$. In our paper, we use the following estimator, which was recommended by Vardeman (1999):

$$\hat{\sigma}_0 = c_4(mn - m + 1)\sqrt{\hat{\sigma}_0^2} \tag{3}$$

where $c_4(y) = \sqrt{\frac{2}{y-1}} \frac{\Gamma(\frac{y}{2})}{\Gamma(\frac{y-1}{2})}$.

When the parameters are estimated, the classical control limits for the Shewhart $\overline{X}$ and S^2 charts are usually estimated as follows (see Montgomery, 2013):

$$\widehat{UCL}_{\overline{X}} = \hat{\mu}_0 + K \frac{\hat{\sigma}_0}{\sqrt{n}} \tag{4}$$

$$\widehat{UCL}_{S^2} = \frac{\hat{\sigma}_0^2}{n-1} L \tag{5}$$

3. The Adjusted Limits for the $\bar{X}$ and S^2 Control Charts

3.1 The $\bar{X}$ Chart Control Limit Constant

Here we wish to derive the control limit constant estimated by the bootstrap algorithm of Gandy and Kvaløy (2013) to adjust the control charts limits such that the conditional in-control ARL meets or exceeds the specified ARL_0 value with a certain probability, say (1-p) 100%. First, we summarize the bootstrap algorithm for the $\bar{X}$ chart as follows:

1- Suppose that the true distribution of Phase I data (P) follows a normal distribution. Using Equations (1) and (2), obtain the estimates of the normal distribution parameters denoted by $\hat{\theta} = (\hat{\mu}_0, \hat{\sigma}_0^2)$. The Phase II observations are then assumed to follow the normal distribution $\hat{P} = N(\hat{\mu}_0, \hat{\sigma}_0^2)$.

2- Since the true distribution is assumed to be known, generate B bootstrap estimates from $\hat{P}$ by generating μ_i^* from $N(\hat{\mu}_0, \frac{\hat{\sigma}_0^2}{nm})$ and σ_i^{*2} from $\hat{\sigma}_0^2 \frac{\chi_v^2}{v}$; $v = m(n-1)$ as the given parameter estimates for each of the bootstrap samples. Note that μ_i^* and σ_i^{*2} are independent and one may simply use $\sigma_i^* = \sqrt{\sigma_i^{*2}}$ to estimate the bootstrap standard deviations, however in this paper we use $\sigma_i^* = c_4\sqrt{\sigma_i^{*2}}$. We refer to the bootstrap estimates as $\theta_i^* = (\mu_i^*, \sigma_i^{*2})$, $i = 1, 2, \ldots, B$, where B should be a large number, e.g., $B = 500$.

3- For each bootstrap $\bar{X}$ chart, find the control limit constant K_i such that the desired false alarm probability (α), the reciprocal of the desired in-control ARL value, is achieved. That is,

$$\alpha = 1 - \Pr\left(\bar{X} < \mu_i^* + K_i \frac{\sigma_i^*}{\sqrt{n}}\right) = 1 - \Phi\left(\sqrt{n}\frac{\mu_i^* - \hat{\mu}_0}{\hat{\sigma}_0} + K_i \frac{\sigma_i^*}{\hat{\sigma}_0}\right)$$

$$\rightarrow \Phi\left(\sqrt{n}\frac{\mu_i^* - \hat{\mu}_0}{\hat{\sigma}_0} + K_i \frac{\sigma_i^*}{\hat{\sigma}_0}\right) = 1 - \alpha$$

where $\Phi(z)$ represents the cumulative distribution function of the standard normal distribution at point z. Note that the quantity K_i adjusts the $\bar{X}$ chart limit to give the desired ARL_0 value when the Phase II sample means are generated from $\hat{P}$ and the limits are obtained using θ_i^*, i = 1, 2, 3, ..., B. Therefore, an exact solution can be found by solving the following equation:

$$\sqrt{n}\frac{\mu_i^* - \hat{\mu}_0}{\hat{\sigma}_0} + K_i \frac{\sigma_i^*}{\hat{\sigma}_0} = \Phi^{-1}(1-\alpha) = z_{1-\alpha}$$

In fact, in order to ensure the desired in-control performance, each bootstrap control limit constant should be adjusted to

$$K_i = z_{1-\alpha} \times \frac{\hat{\sigma}_0}{\sigma_i^*} - \sqrt{n}\frac{\mu_i^* - \hat{\mu}_0}{\hat{\sigma}_0} \times \frac{\hat{\sigma}_0}{\sigma_i^*} = \frac{1}{\sqrt{m}} \frac{K\sqrt{m} - z}{\sqrt{w}} \tag{6}$$

where $= z_{1-\alpha}$, $z = \sqrt{nm}\frac{\mu_i^* - \hat{\mu}_0}{\hat{\sigma}_0}$ and $\sqrt{w} = \frac{\sigma_i^*}{\hat{\sigma}_0}$.

4- Find the (1-p) quantile of the bootstrap control limit constant K_i to use as the multiplier in Equation (4) to guarantee the in-control performance with probability (1-p)100%.

Note that repeating the bootstrap method for a given Phase I sample gives different results. Thus, there is within Phase I sample variation. In addition, the bootstrap algorithm gives different result for different Phase I samples, which we refer to as between Phase I sample variation. Therefore, the bootstrap adjusted $\bar{X}$ chart's control limit constants are approximations. To improve the approximation, Faraz et al. (2015) proposed to run the algorithm for a certain number of times, e.g., r = 1000 times, and then use the average as the final control limit constant in the design of the S^2 chart. We show here that there is an exact solution for the control limit constant and Faraz et al. (2015) estimated this value through repeating the bootstrap algorithm for the S^2 chart. The proposed solution is simple and can be considered as a parametric solution to the bootstrap method when $B \to \infty$ and $r \to \infty$. In this case, Equation (6) gives

$$K_i = \frac{1}{\sqrt{m}} \frac{K\sqrt{m} - Z}{\sqrt{W}}$$

where Z and W are independent random variables. Furthermore, $K\sqrt{m} - Z \sim N(K\sqrt{m}, 1)$ and $vW \sim \chi_v^2$, where $v = m(n-1)$. Thus, $K_i = \frac{T'}{\sqrt{m}}$, where T' follows a non-central *t*-student distribution with *v* degrees of freedom and non-centrality parameter $K\sqrt{m}$. Therefore, the (1-*p*) quantile of the non-central *t*-student distribution, say K_p^*, gives the exact solution to the bootstrap method. That is, we use the upper control limit constant

$$K_p^* = t_{(1-p,v,K\sqrt{m})}/\sqrt{m} \tag{7}$$

where $t_{(1-p,v,K\sqrt{m})}$ is the (1-*p*) quantile of the non-central *t*-student distribution with *v* degrees of freedom and non-centrality parameter $K\sqrt{m}$.

In the following we prove that equation (7) is consistent with Remark 1 in Gandy and Kvaløy (2013). Following Gandy and Kvaløy (2013)'s notation, the quantity we are interested in is $c_{ARL}(P,\theta) = \inf\{c > 0: ARL(P,\theta) \geq \alpha\}$, the aim is to find q_p such that $\Pr\{c_{ARL}(\hat{P},\hat{\theta}) - c_{ARL}(P,\hat{\theta}) \geq q_p\}$. Using the distribution result given in their Remark 1, we get $\Pr\left\{\frac{K-Z/\sqrt{m}}{\sqrt{W}} - K \geq q_p\right\} = 1-p$. Thus, $q_p = K - \frac{1}{\sqrt{m}} t_{(1-p,v,K\sqrt{m})}$. Finally, we will have the upper control limit constant by $K_p^* = K - q_p = \frac{1}{\sqrt{m}} t_{(1-p,v,K\sqrt{m})}$. This is exactly the formulation given in (7). The derivations for the *S* or S^2 charts are similar.

We can also extend the bootstrap algorithm to the two-sided $\bar{X}$ chart. Let the two-sided bootstrap charts be asymmetric, that is:

$$\widehat{UCL}_i = \mu_i^* + K_{Ui}\frac{\sigma_i^*}{\sqrt{n}} \quad \text{and} \quad \widehat{LCL}_i = \mu_i^* + K_{Li}\frac{\sigma_i^*}{\sqrt{n}}$$

Conditionally on Phase I, the lower and upper control limits constants should be adjusted such that the desired Type I error rate is ensured with probability (1-*p*), that is we should have:

$$\Pr\left(\Pr\left(\mu_i^* + K_{Li}\frac{\sigma_i^*}{\sqrt{n}} \le \bar{X} \le \mu_i^* + K_{Ui}\frac{\sigma_i^*}{\sqrt{n}} | PHASE\ I\right) \ge 1-\alpha\right) \ge 1-p$$

$$\rightarrow \Pr\left(\Pr\left(\frac{\sqrt{n}(\mu_i^* - \hat{\mu}_0)}{\hat{\sigma}_0} + K_{Li}\frac{\sigma_i^*}{\hat{\sigma}_0} \le \frac{\sqrt{n}(\bar{X} - \hat{\mu}_0)}{\hat{\sigma}_0} \le \frac{\sqrt{n}(\mu_i^* - \hat{\mu}_0)}{\hat{\sigma}_0} + K_{Ui}\frac{\sigma_i^*}{\hat{\sigma}_0}\right) \ge 1-\alpha\right) \ge 1-p$$

$$\Pr\left(\Pr\left(a \le \frac{\sqrt{n}(\bar{X} - \hat{\mu}_0)}{\hat{\sigma}_0} \le b\right) \ge 1-\alpha\right) \ge 1-p$$

where $a = \sqrt{n}\frac{\mu_i^* - \hat{\mu}_0}{\hat{\sigma}_0} + K_{Li}\frac{\sigma_i^*}{\hat{\sigma}_0}$ and $b = \sqrt{n}\frac{\mu_i^* - \hat{\mu}_0}{\hat{\sigma}_0} + K_{Ui}\frac{\sigma_i^*}{\hat{\sigma}_0}$. Since $\sqrt{n}\frac{\bar{X} - \hat{\mu}_0}{\hat{\sigma}_0}$ follows the standard normal distribution, the probability $\Pr\left(a \le \sqrt{n}\frac{\bar{X} - \hat{\mu}_0}{\hat{\sigma}_0} \le b\right) \ge 1-\alpha$ is ensured if $a \le z_{\frac{\alpha}{2}}$ and $b \ge z_{1-\frac{\alpha}{2}}$. Therefore, we impose $\Pr\left(a \le z_{\frac{\alpha}{2}} \,\&\, b \ge z_{1-\frac{\alpha}{2}}\right) \ge 1-p$. That is

$$\Pr\left(\sqrt{n}\frac{\mu_i^* - \widehat{\mu}_0}{\widehat{\sigma}_0} + K_{Li}\frac{\sigma_i^*}{\widehat{\sigma}_0} \le z_{\frac{\alpha}{2}} \ \&\ \sqrt{n}\frac{\mu_i^* - \widehat{\mu}_0}{\widehat{\sigma}_0} + K_{Ui}\frac{\sigma_i^*}{\widehat{\sigma}_0} \ge z_{1-\frac{\alpha}{2}}\right) \ge 1-p$$

$$\rightarrow \Pr\left(K_{Li} \le z_{\frac{\alpha}{2}}\frac{\widehat{\sigma}_0}{\sigma_i^*} - \sqrt{n}\frac{\mu_i^* - \widehat{\mu}_0}{\sigma_i^*} \,\&\, K_{Ui} \ge z_{1-\frac{\alpha}{2}}\frac{\widehat{\sigma}_0}{\sigma_i^*} - \sqrt{n}\frac{\mu_i^* - \widehat{\mu}_0}{\sigma_i^*}\right) \ge 1-p$$

Using the Bonferroni inequality for intersection of two events, we have

$$\Pr\left(K_{Li} \ge z_{\frac{\alpha}{2}}\frac{\widehat{\sigma}_0}{\sigma_i^*} - \sqrt{n}\frac{\mu_i^* - \widehat{\mu}_0}{\sigma_i^*}\right) + \Pr\left(K_{Ui} \le z_{1-\frac{\alpha}{2}}\frac{\widehat{\sigma}_0}{\sigma_i^*} - \sqrt{n}\frac{\mu_i^* - \widehat{\mu}_0}{\sigma_i^*}\right) \le p$$

Since the random variables $z_{\frac{\alpha}{2}}\frac{\widehat{\sigma}_0}{\sigma_i^*} - \sqrt{n}\frac{\mu_i^* - \widehat{\mu}_0}{\sigma_i^*}$ and $z_{1-\frac{\alpha}{2}}\frac{\widehat{\sigma}_0}{\sigma_i^*} - \sqrt{n}\frac{\mu_i^* - \widehat{\mu}_0}{\sigma_i^*}$ follow the non-central *t*-student distribution, the left side of the last expression holds if, for example, the lower and upper control limits constants are set to the $\frac{p}{2}$ and $1-\frac{p}{2}$ quantiles of the non central t with v=m(n-1) degrees of freedom and non-centrality parameters $z_{\alpha/2}\sqrt{m}$ and $z_{1-\alpha/2}\sqrt{m}$, respectively. That is, the lower and upper adjusted control limits can be calculated as follows:

$$\widehat{UCL}_{\bar{X}} = \hat{\mu}_0 + t_{\left(1-\frac{p}{2},v,z_{1-\alpha/2}\sqrt{m}\right)}/\sqrt{m}\frac{\hat{\sigma}_0}{\sqrt{n}}$$

$$\widehat{LCL}_{\bar{X}} = \hat{\mu}_0 + t_{\left(\frac{p}{2},v,z_{\alpha/2}\sqrt{m}\right)}/\sqrt{m}\frac{\hat{\sigma}_0}{\sqrt{n}}.$$

Since $t_{\left(1-\frac{p}{2},v,z_{1-\alpha/2}\sqrt{m}\right)}/\sqrt{m} = -\,t_{\left(\frac{p}{2},v,z_{\alpha/2}\sqrt{m}\right)}/\sqrt{m}$, we have

$$\widehat{UCL}_{\bar{X}} = \hat{\mu}_0 + K_p^*\frac{\hat{\sigma}_0}{\sqrt{n}}$$
$$\widehat{LCL}_{\bar{X}} = \hat{\mu}_0 - K_p^*\frac{\hat{\sigma}_0}{\sqrt{n}}. \tag{8}$$

where $K_p^* = t_{\left(1-\frac{p}{2},v,z_{1-\alpha/2}\sqrt{m}\right)}/\sqrt{m}$.

3.2 Performance Comparisons

Albers and Kallenberg (2005) provided some corrections to adjust the classical $\bar{X}$ control chart limits such that the resulting in-control ARL value is less than a fraction (1-ε) of the desired ARL value (ARL_0) with a pre-determined probability p, i.e.,

$$\Pr\big(ARL < ARL_0(1-\varepsilon)\big) = p$$

Note that the performance requirement presented in our paper is equivalent to that of Albers and Kallenberg (2005) when $\varepsilon = 0$. Their adjusted control limits constant K_p depends on the estimator used for the standard deviation. For the standard deviation estimator given in Equation (3), their control limits constant when $\varepsilon = 0$ is

$$K_p = z_{1-\frac{\alpha}{2}}\left(1+\frac{z_{1-\frac{p}{2}}}{\sqrt{2v}}\right) \quad (9)$$

where $v = m(n-1)$. Tables 1 and 2 compare the two control limits constants K_p and K_p^*, given in (8) and (9), respectively, for 10,000,000 simulated $\bar{X}$ charts. The tables give the changes in in-control performance measures such as the average of the in-control ARL (AARL), the median of the in-control ARL (MRL) and the standard deviation of the in-control ARL (SDARL) as well as the proportion of population that falls within control limits (%Pop). Results indicate that the adjusted limits given in (8) provide users with more conservative designs and hence have better in-control performance.

[Insert Tables 1 and 2 here]

Some comparisons between control limits constants K_p^* , K_p and the ones based on the bootstrap method are provided in Figure 1 for the case where *K*=3, *m* = 50, *n* = 5, *B* = 500 and ARL_0 = 370. Using Equations (7) and (9) we obtain K_p^* = 3.641 and K_p = 3.247, respectively. For bootstrap results we consider two cases: *a*) we simulate a Phase I dataset from the standard normal distribution and then we repeat the bootstrap algorithm *r* = 10,000 times, *b*) we simulate 10,000 Phase I datasets and then we run the bootstrap algorithm once for each Phase I dataset. Figure 1 gives the histogram of adjusted control limits constants for the $\bar{X}$ chart for both cases. Figure 1(a) clearly depicts the within Phase I sample variation of the bootstrap method and Figure 1(b) the between Phase I sample variation. The results clearly indicate that bootstrap results are centered at K_p^* and that the variation about this value is small. Furthermore, it illustrates that K_p is too small. These results are consistent with the Table 1 results.

[Insert Figure 1]

Finally, we provide users with the required percentiles of the non-central *t*- distributions for different combinations of m, n, p and α in Table 3. These values were calculated using MATLAB R2010a software. Other software, such as MINITAB and JMP, can be used to find the required percentiles.

[Insert Table 3 here]

3.3 The S^2 Chart's Control Limit Constant

For the S^2 control chart, steps 3 and 4 in the bootstrap algorithm should be revised as follows:

3- For each bootstrap S^2 chart find the control limit constant L_i such that the desired false alarm rate (α) is achieved. Given the bootstrap estimate σ_i^{*2}, we have

$$1-\alpha = \Pr\left(S_i^2 < \frac{\sigma_i^{*2}}{n-1} L_i\right) = \Pr\left(\frac{(n-1)S_i^2}{\hat{\sigma}_0^2} < \frac{\sigma_i^{*2}}{\hat{\sigma}_0^2} L_i\right)$$

$$= \chi^2\left(\frac{\sigma_i^{*2}}{\hat{\sigma}_0^2} L_i, n-1\right)$$

where $\chi^2(x, df)$ represents the cumulative distribution function of a chi-square random variable at point x and with df degrees of freedom. The quantity L_i, i=1, 2, 3, ..., B, adjusts the S^2 chart limits to give the desired ARL_0 value when the Phase II sample data are generated from $\hat{P}$ and the limits are obtained using θ_i^*, i=1, 2, 3, ..., B. Therefore, the exact solution can be found by solving the following equation:

$$\frac{\sigma_i^{*2}}{\hat{\sigma}_0^2} L_i = \chi^2_{(1-\alpha, n-1)}$$

where $\chi^2_{(p, df)}$ represents the $p(100)^{\text{th}}$ percentile of the chi-square distribution with df degrees of freedom. In fact, in order to ensure the desired in-control performance, each bootstrap S^2 chart's control limit constant should be adjusted using

$$L_i = \frac{\chi^2_{(1-\alpha,n-1)}}{w} \tag{10}$$

where $w = \frac{\sigma_i^{*2}}{\hat{\sigma}_0^2}$.

4- Find the (1-p) quantile of the bootstrap control limit constants L_i to obtain the control limit constant that guarantees the in-control performance with probability (1-p)100%.

Since the Phase II observations in the bootstrap method are assumed to come from $\hat{P}$ and that $\frac{v\sigma_i^{*2}}{\hat{\sigma}_0^2} \sim \chi^2_v$, where $v = m(n-1)$, Equation (10) can be rewritten as

$$L_i = \frac{v\chi^2_{(1-\alpha,n-1)}}{\chi^2_v}$$

Hence, the control limit constant can be obtained as

$$L_p^* = v\chi^2_{(1-\alpha,n-1)}/\chi^2_{(p,v)} \tag{11}$$

For the S chart, the results are straightforward. That is,

$$\widehat{UCL}_S = \sqrt{\frac{v\chi^2_{(1-\alpha,n-1)}/\chi^2_{(p,v)}}{n-1}}\,\hat{\sigma}_0 \tag{12}$$

where $\hat{\sigma}_0 = c_4\sqrt{\hat{\sigma}_0^2}$. For the classic R chart, we apply the transformation $\hat{\sigma}_0 = \bar{R}/d_2$ where $\bar{R}$ is the estimated range in Phase I and d_2 is a constant (for more information, readers are referred to Montgomery, 2013). Then we have

$$\widehat{UCL}_R = \sqrt{\frac{v\chi^2_{(1-\alpha,n-1)}/\chi^2_{(p,v)}}{n-1}}\,\frac{\bar{R}}{d_2} \tag{13}$$

Figures 2 and 3 give the histogram of bootstrap control limit constants for the S^2 chart with an upper control limit for 10,000 different Phase I samples for B=500 and B=1000, respectively. Using Equation (11), we have L_p^*=18.59, which is approximately the mean of the bootstrap control limit constants. Note that L = 16.2489. Figures 2 and 3 indicate that the bootstrap method's results are distributed around L_p^* and they are less variable about the L_p^* value as B and r increase. Hence, we recommend the use of the proposed control limit constant L_p^*.

[Insert Figures 2-3 here]

Finally, we compared the in-control performance of the classical and adjusted charts for m = 25 and 50, n = 5, B = 500, ARL_0 = 370 and p = 0.20. We simulated 10,000 different Phase I datasets and for each dataset estimated the in-control parameters. We then constructed three control charts using the classical method, our adjustment method using Equation (7) and the bootstrap method for each set of Phase I data. Then we calculated the in-control ARL value for each of the three resulting charts. Figures 5 and 6 show the boxplots of the in-control ARL values for 10,000 $\bar{X}$ charts with and without adjustment using Equation (7) and for the bootstrap method for m =25 and m = 50. The results indicate that only 20% of the adjusted charts using Equation (7) have an in-control ARL value less than 370 while, as expected, almost 50% of the classical charts have an in-control ARL below 370. The adjusted charts based on the bootstrap method show close in-control performance to our adjusted chart. The results again suggest the simplicity and accuracy of the proposed method.

[Insert Figures 4-5]

Figure 6 shows the boxplot of the in-control ARL for 10,000 S^2 charts with and without adjustment using Equation (11) and with the bootstrap method. The results again show the simplicity and accuracy of the adjusted S^2 chart using Equation (11).

[Insert Figure 6]

Figures 7-9 show illustrates the out-of-control ARL distributions of the $\bar{X}$ and S^2 charts for different values of shift and the guarantee probability p. As we expected, the classical charts show better out-of-control performance due to having tighter limits, however, the increased out-of-control ARLs with use of the adjusted limits are the cost of avoiding low in-control ARL values and can be compensated by using the variable sampling schemes such as variable sample sizes (VSS), variable sampling intervals (VSI), variable sample sizes and sampling intervals (VSSI) and variable parameters (VP). These topics are left for further research.

[Insert Figures 7-9]

3.4 An Illustrative Example

We illustrate the simplicity of our method through a case study in Golestan Company, Iran. Golestan is packing and distributing its products all over Iran. Its most important products are saffron, pistachio, black/herbal tea and rice. Since 2005, Golestan and the British TWININGS company have established a commercial partnership.

Golestan Saffron pistachio is a type of hazelnut pistachio which is the most favorite type of pistachio in Iran. Here, the quality characteristic of interest is the net weight of 100-g packs of pistachios. When the process was assumed to be in-control, we sampled *m*=20 samples each of size *n*=5 as the available Phase I dataset. We then estimated the process parameters as $\hat{\mu}_0 = 102.66$ and $\hat{\sigma}_0^2 = S_p^2 = 10.03$. Using Equation (3), the process standard deviation is estimated as $\hat{\sigma}_0 = c_4(mn - m + 1)\sqrt{S_p^2} = 3.16$ where $c_4(mn - m + 1) = 0.9969$. We wish to establish the Shewhart $\bar{X}$ and S^2 control charts to monitor the packing process to satisfy the desired ARL_0=370 with probability 0.85. According to the Table 3, in order to be 85% sure that the in-control ARL exceeds the desired value 370, the $\bar{X}$ chart's control limits constant should be adjusted from 3 to 3.522. Therefore, the control limits are easily calculated by using Equation (8) as follows:

$$UCL_{\bar{X}} = 102.66 + 3.522\frac{3.16}{\sqrt{5}} = 107.64$$

$$LCL_{\bar{X}} = 102.66 - 3.522\frac{3.16}{\sqrt{5}} = 97.68$$

The adjusted control limit constant and the upper control limit for the S^2 chart are calculated using Equations (11) and (5), respectively, as follows:

$$L_p^* = \frac{80\chi^2_{(1-0.0027\,,4)}}{\chi^2_{(0.15\,,80)}} = \frac{80 \times 16.251}{66.994} = 19.41$$

$$UCL_{S^2} = \frac{\hat{\sigma}_0^2}{n-1}L_p^* = \frac{10.03}{4} \times 19.41 = 48.67$$

Figure 10 shows the Shewhart control charts with adjusted control limits (straight lines) and the classical control limits (dash lines) for a Phase II dataset of N=100 simulated samples each of size n=5. The first and the last 25 samples are normally distributed with the estimated parameters. That is, the process was assumed to be in control. The samples 26 – 35 are simulated from $a\ N(\hat{\mu}_0 - \hat{\sigma}_0, \hat{\sigma}_0)$ distribution; i.e. , when the process mean shifts one standard deviation to the left. The samples 36 – 45 are simulated from a $N(\hat{\mu}_0 + \hat{\sigma}_0, \hat{\sigma}_0)$ distribution or when the process mean shifts one standard deviation to the right. The samples 46 – 55 simulated from $N(\hat{\mu}_0, 1.5\hat{\sigma}_0)$ distribution that is when the process standard deviation increases by 50%. The samples 56 – 65 are simulated from $N(\hat{\mu}_0 - \hat{\sigma}_0, 1.5\hat{\sigma}_0)$ and samples 66 – 75 from $N(\hat{\mu}_0 + \hat{\sigma}_0, 1.5\hat{\sigma}_0)$ or when the process mean and the standard deviation shift simultaneously. The results indicate that the effect of the adjustments on the power of the charts are minor. The advantage of the adjusted charts is having very good in-control performance. In order to improve their power to detect shifts one may apply other alternatives such as designing adjusted charts with variable sampling policies or other methods more sensitive to small shifts such as CUSUM or EWMA charts. This topic is left for further research.

4. Concluding Remarks and further research

In this article, we have provided simple expressions that can be used to adjust $\bar{X}$ and S^2 control chart limits to guarantee the in-control performance of the charts such that the in-control ARL for each user takes a value greater than the desired value with a certain probability. We strongly encourage use of this approach when the in-control run length performance is to be controlled. We also encourage the implementation of this design method for other types of control charts.

Finally, we suggest some related topics for the further research. The proposed adjusted control limits rely on the normality and independence assumptions. Of course, even if the single observations are not normal the sample means approach a normal distribution as the sample size gets larger because of the Central Limit Theorem. But if the mean of Phase I dataset samples are neither normally distributed nor independent (see for example, Schoonhoven et al, 2011; Abbasi et al. 2015) then the adjusted limits cannot be obtained based on the percentiles of the student-t distribution. The present methodology can be extended to control charts for a short-run production process (for example, ANOM and Q charts). Moreover, outliers would definitely have negative effects on the control charts performance. The study of the effects of non-normality, the existence of outliers and/or auto-correlation on the performance of our method remain as topics for further research.

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Table 1. A comparison between control limit constants K_p when $\varepsilon = 0$ and K_p^* for different values of *n, m* and α at level of p = 0.1.

ARL₀	***n***	***m***	K_p^*				K_p			
			AARL	***MARL***	***SDARL***	***%Pop***	***AARL***	***MARL***	***SDARL***	***%pop***
200	4	25	1828.921	1004.528	3173.933	96.2%	855.2724	528.1343	1186.844	87.5%
		50	778.4369	615.4476	601.8786	97.0%	508.5758	414.4225	358.2169	89.9%
		75	574.0267	498.1403	320.5464	97.3%	418.0139	368.0283	218.5888	91.0%
		100	486.4517	439.9925	221.5595	97.5%	374.7911	341.9536	161.8987	91.6%
	5	25	1298.165	854.2947	1567.099	96.7%	641.1421	457.1865	649.4659	87.2%
		50	656.6077	554.7347	409.0123	97.6%	431.4652	372.6555	246.1982	89.8%
		75	509.2917	459.3968	235.9316	97.9%	369.6827	337.1015	160.5047	90.9%
		100	444.9703	413.3737	170.3431	98.2%	340.6098	318.5328	123.5919	91.8%
	10	25	757.7241	640.8877	476.4714	98.2%	381.1392	333.7881	208.641	85.1%
		50	496.1632	462.6456	188.8873	99.0%	315.9443	297.7228	110.0874	88.6%
		75	415.3988	397.9115	121.5051	99.3%	290.5319	279.7887	79.21298	90.0%
		100	374.9959	363.6006	91.88492	99.4%	276.5996	269.0871	63.82121	90.8%
370.4	4	25	4673.904	2188.306	10565.63	95.8%	2069.095	1115.529	3674.288	87.6%
		50	1712.607	1274.013	1561.144	96.6%	1096.438	845.2324	911.0545	90.0%
		75	1207.573	1010.051	779.9594	97.0%	868.5771	738.9093	526.6864	91.0%
		100	999.3998	880.3283	522.5669	97.2%	763.2114	679.147	379.5723	91.6%
	5	25	3072.238	1811.742	4587.633	96.4%	1458.081	946.0492	1811.902	87.3%
		50	1396.686	1129.072	1010.535	97.2%	902.9849	749.1136	600.2057	89.8%
		75	1046.374	919.3477	555.636	97.6%	751.7595	668.8769	375.4674	90.9%
		100	898.6995	819.1936	392.8374	97.9%	682.691	627.1113	283.8403	91.8%
	10	25	1598.604	1296.289	1155.63	97.9%	784.4002	662.3168	496.2416	85.3%
		50	997.3776	913.3544	430.7966	98.8%	626.7668	581.0267	249.2169	88.7%
		75	820.322	776.9879	271.6059	99.1%	568.2596	541.5818	176.29	90.0%
		100	733.0328	705.0232	203.0892	99.2%	536.5238	518.0175	140.6298	90.8%

%Pop indicates the proportion of the users having control charts with in-control $ARL \geq ARL_0$

Table 2. A comparison between control limit constants K_p when $\varepsilon = 0$ and K_p^* for different values of *n, m* and α at level of p = 0.2.

ARL₀	*n*	*m*	K_p^*				K_p			
			AARL	*MARL*	*SDARL*	*%Pop*	*AARL*	*MARL*	*SDARL*	*%Pop*
200	4	25	1068.958	638.9319	1569.099	91.0%	622.2575	402.2221	783.375	80.4%
		50	565.9749	457.822	408.1379	92.4%	414.1697	342.2055	278.4659	83.3%
		75	449.7037	394.6577	238.8803	93.1%	355.3751	315.1173	179.5634	84.6%
		100	396.4556	360.9709	173.3731	93.4%	326.2673	298.9422	137.0097	85.4%
	5	25	823.3821	571.2844	887.8246	92.0%	493.3954	361.7416	467.89	80.0%
		50	495.7641	425.0102	291.3912	93.4%	362.8809	316.0402	199.5129	83.0%
		75	409.264	372.0456	181.3489	94.1%	321.8562	294.8926	135.6818	84.4%
		100	370.161	345.5342	136.6338	94.7%	302.6017	283.8465	107.176	85.6%
	10	25	543.534	467.7955	319.8874	94.8%	324.5856	286.4358	171.6544	77.3%
		50	398.5374	373.5912	145.374	96.5%	282.8868	267.2415	96.31368	81.5%
		75	349.1369	335.2934	98.76639	97.2%	265.7569	256.236	71.16785	83.1%
		100	323.3163	314.0102	76.96247	97.5%	256.1277	249.3747	58.17622	84.1%
370.4	4	25	2550.619	1327.04	4803.455	90.5%	1431.864	818.4763	2283.433	80.5%
		50	1200.976	919.1512	1014.619	91.9%	866.8242	680.1636	683.8124	83.3%
		75	921.0738	780.9521	564.732	92.5%	721.7195	619.3738	421.8841	84.6%
		100	797.0845	708.3474	399.7588	92.9%	651.9609	583.6448	314.7237	85.4%
	5	25	1849.821	1165.413	2439.485	91.4%	1079.036	725.7577	1245.136	80.1%
		50	1024.748	843.8853	698.5648	92.8%	741.6455	622.1346	474.109	83.1%
		75	822.7357	729.4398	417.9874	93.5%	642.3586	574.7464	311.3889	84.4%
		100	733.9174	673.0097	308.8345	94.1%	596.7511	550.3167	241.6319	85.6%
	10	25	1114.052	922.4332	754.2172	94.1%	653.4435	557.0393	399.2465	77.5%
		50	786.6946	724.8775	326.1201	95.9%	552.9883	514.1548	214.9566	81.5%
		75	679.3557	645.5668	217.6181	96.6%	513.4526	490.1664	156.341	83.2%
		100	624.4078	601.7383	168.2392	97.0%	491.798	475.3138	126.9054	84.1%

%Pop indicates the proportion of the users having control charts with in-control $ARL \geq ARL_0$

Table 3. Values of our control limit constants K_p^* for different combinations of m, n, p and α = 0.0027.

n	***m***	α = 0.0027				α = 0.005			
		***p*=0.10**	***p*=0.15**	***p*=0.20**	***p*=0.25**	***p*=0.10**	***p*=0.15**	***p*=0.20**	***p*=0.25**
2	10	4.907	4.608	4.395	4.230	4.607	4.325	4.125	3.968
	15	4.414	4.203	4.051	3.930	4.143	3.944	3.800	3.686
	20	4.160	3.992	3.869	3.772	3.904	3.745	3.629	3.536
	25	4.002	3.859	3.754	3.671	3.754	3.619	3.520	3.441
	30	3.891	3.766	3.674	3.600	3.650	3.532	3.444	3.374
	50	3.652	3.563	3.497	3.443	3.424	3.340	3.277	3.226
	100	3.436	3.378	3.335	3.299	3.220	3.165	3.124	3.090
3	10	4.240	4.061	3.930	3.825	3.987	3.817	3.692	3.593
	15	3.956	3.822	3.723	3.643	3.717	3.590	3.496	3.420
	20	3.801	3.691	3.609	3.542	3.570	3.465	3.387	3.324
	25	3.701	3.605	3.534	3.477	3.475	3.384	3.316	3.261
	30	3.630	3.545	3.481	3.429	3.407	3.327	3.266	3.217
	50	3.470	3.408	3.361	3.323	3.256	3.196	3.152	3.115
	100	3.321	3.279	3.248	3.222	3.114	3.074	3.044	3.019
4	10	4.015	3.873	3.768	3.683	3.779	3.643	3.543	3.462
	15	3.794	3.685	3.605	3.539	3.568	3.464	3.387	3.324
	20	3.671	3.581	3.513	3.458	3.450	3.364	3.299	3.246
	25	3.590	3.512	3.452	3.404	3.373	3.298	3.241	3.195
	30	3.532	3.462	3.409	3.365	3.318	3.250	3.199	3.158
	50	3.401	3.349	3.309	3.277	3.192	3.142	3.104	3.073
	100	3.276	3.241	3.214	3.191	3.072	3.038	3.012	2.991
5	10	3.899	3.776	3.684	3.610	3.672	3.554	3.466	3.394
	15	3.709	3.614	3.542	3.484	3.490	3.398	3.330	3.274
	20	3.602	3.522	3.462	3.413	3.387	3.310	3.252	3.205
	25	3.532	3.461	3.408	3.365	3.319	3.252	3.201	3.159
	30	3.481	3.418	3.37	3.331	3.27	3.21	3.164	3.126
	50	3.364	3.317	3.281	3.252	3.158	3.113	3.078	3.05
	100	3.252	3.22	3.195	3.175	3.05	3.019	2.995	2.976

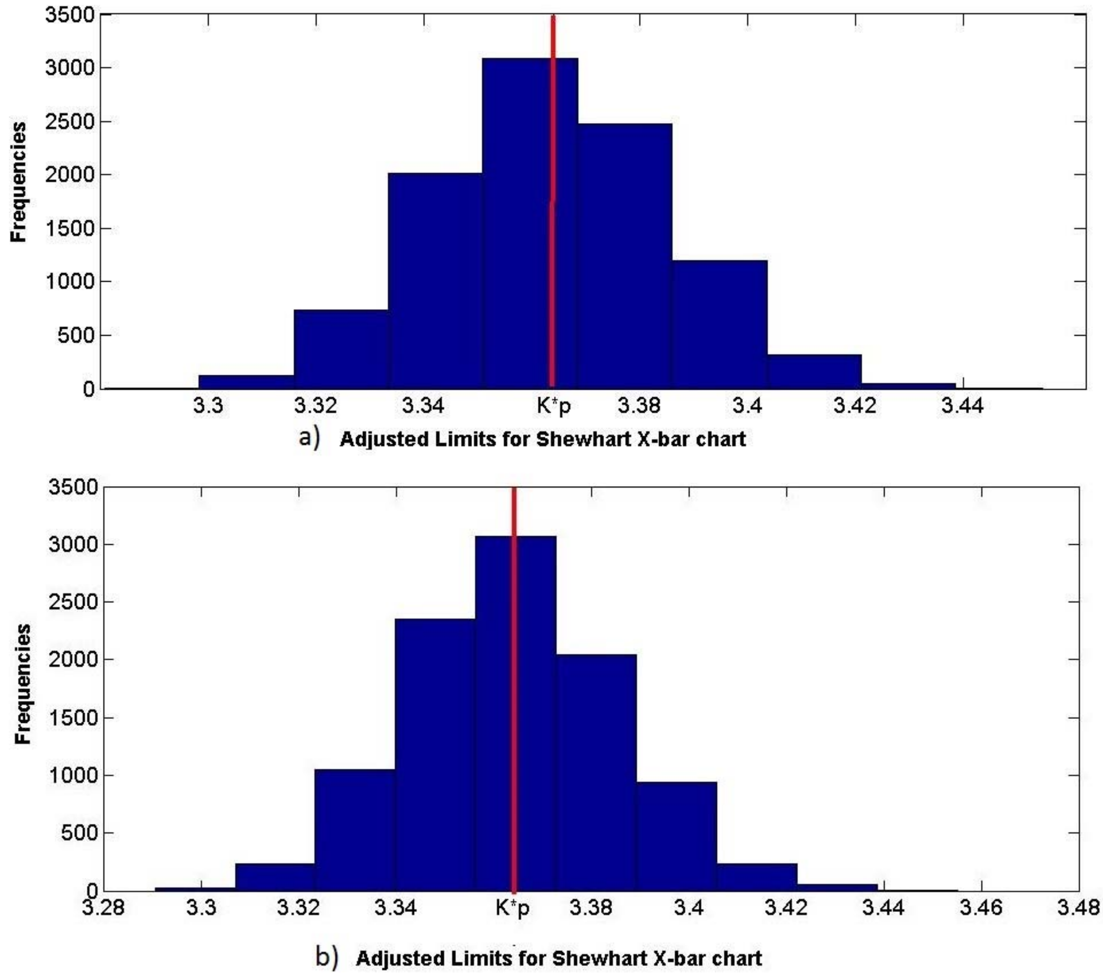

Figure 1. The histogram of the bootstrap $\bar{X}$ chart control limit constants with *B*=500, *m*=50, *n*=5, *p*=0.1 and ARL_0=370 for *a*) repeating 10,000 times the bootstrap method for a given Phase I sample *b*) 10,000 different Phase I samples.

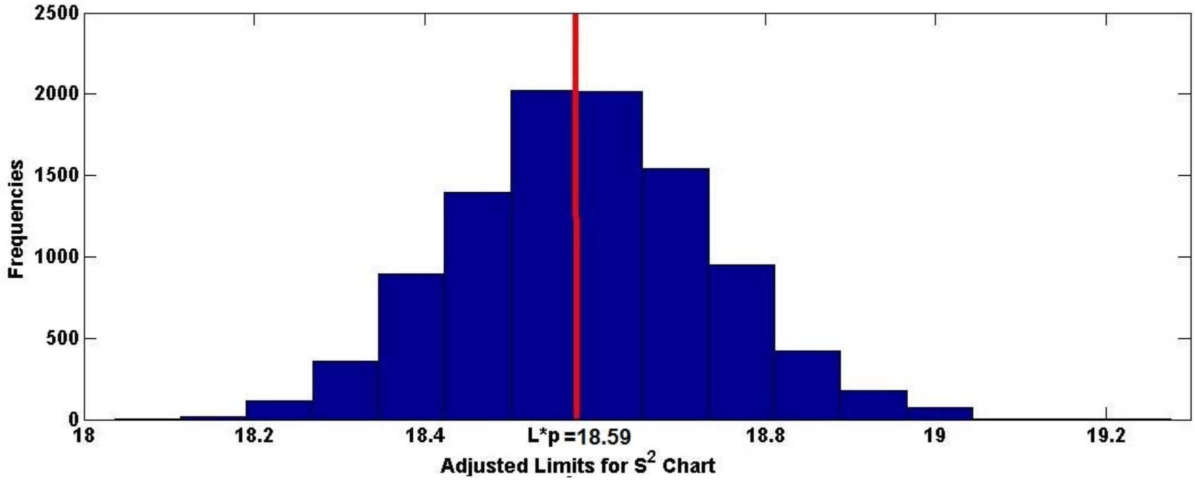

Figure 2. The histogram of the bootstrap S^2 chart control limit constants with B = 500, m = 50, n = 5, p = 0.1 and ARL_0 = 370 for 10,000 different Phase I samples.

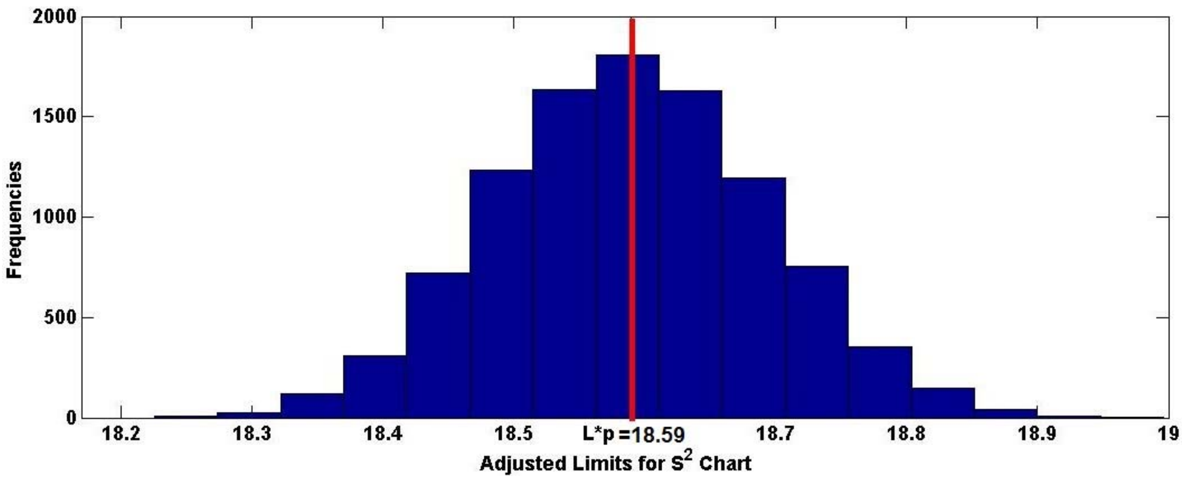

Figure 3. The histogram of the bootstrap S^2 chart's control limit constants with B=500, m=50, n=5, p=0.1 and ARL_0=370 for 10,000 different Phase I samples.

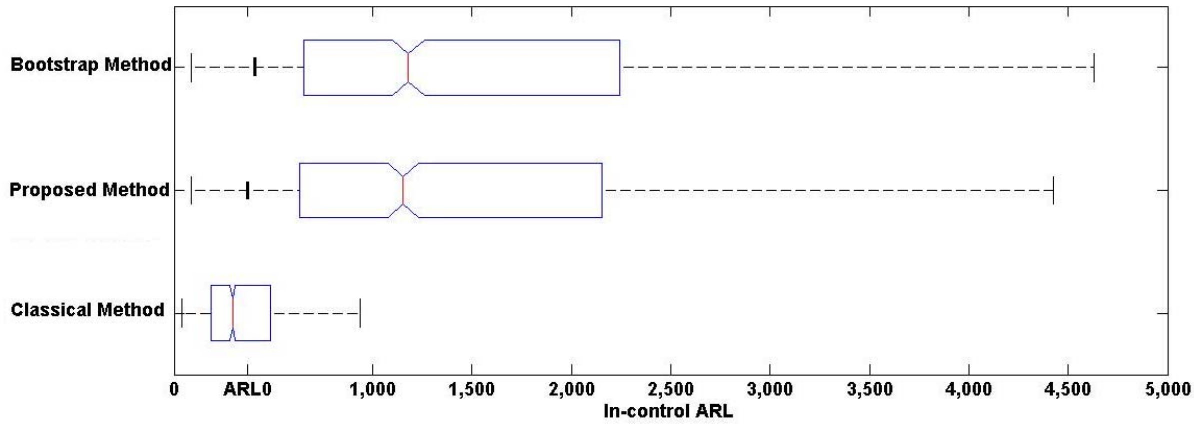

Figure 4. The box-plot of the in-control ARL for 10,000 classical, our adjusted, and bootstrap adjusted $\bar{X}$ chart for m = 25, n = 5, p = 0.2 and ARL_0 = 370.

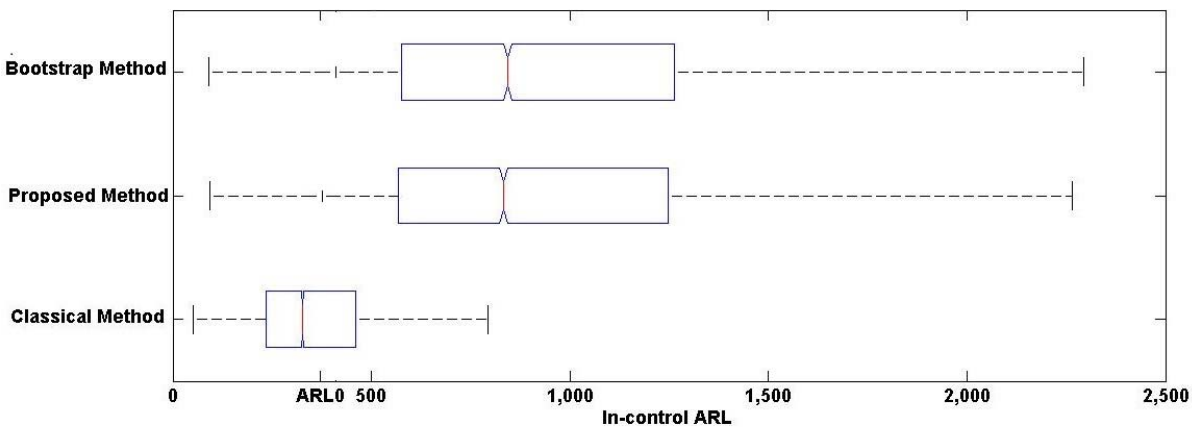

Figure 5. The box-plot of the in-control ARL for 10,000 classical, our adjusted, and bootstrap adjusted $\bar{X}$ chart for m = 50, n = 5, p = 0.2 and ARL_0 = 370.

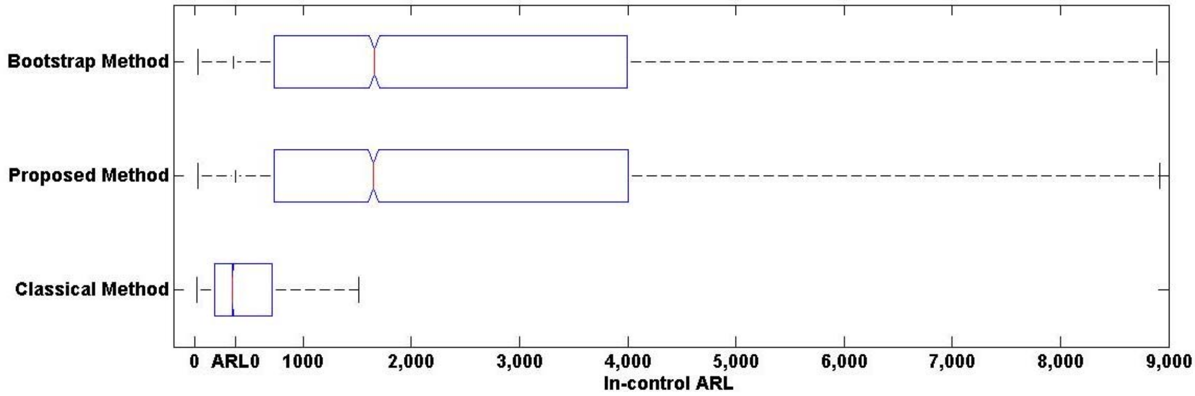

Figure 6. The box-plot of the in-control ARL for 10,000 classical, our adjusted, and bootstrap adjusted S^2 chart for m = 25, n = 5, p = 0.1 and ARL_0 = 370.

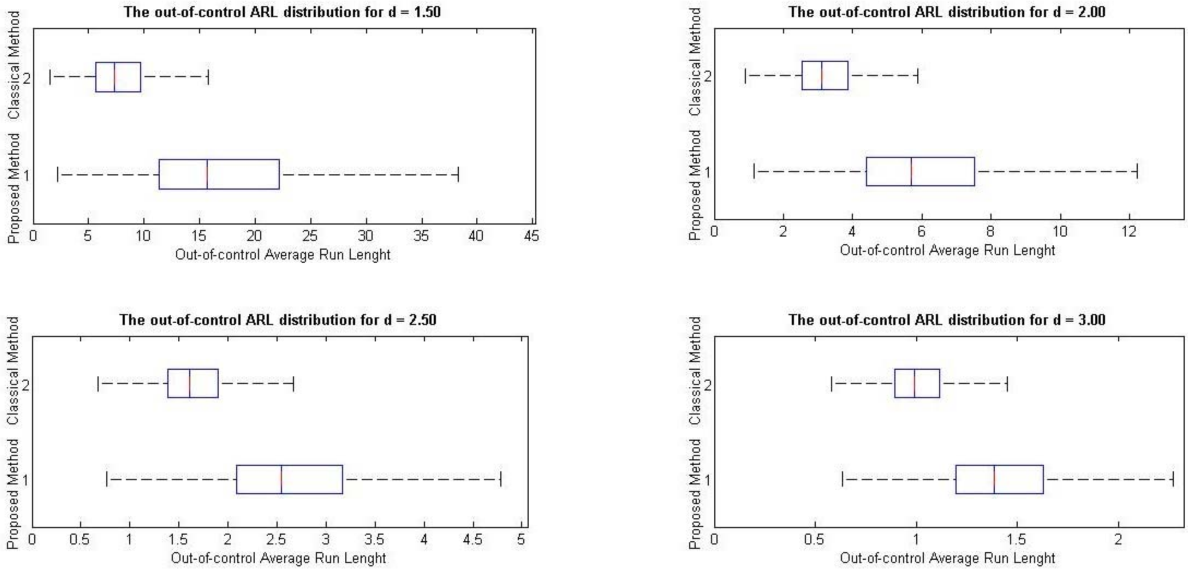

Figure 7. The box plot of the out-of-control ARL with and without limit adjustment for 10,000,000 simulated $\bar{X}$ charts with m = 50, n = 5, p=0.1 and ARL_0 = 370.4.

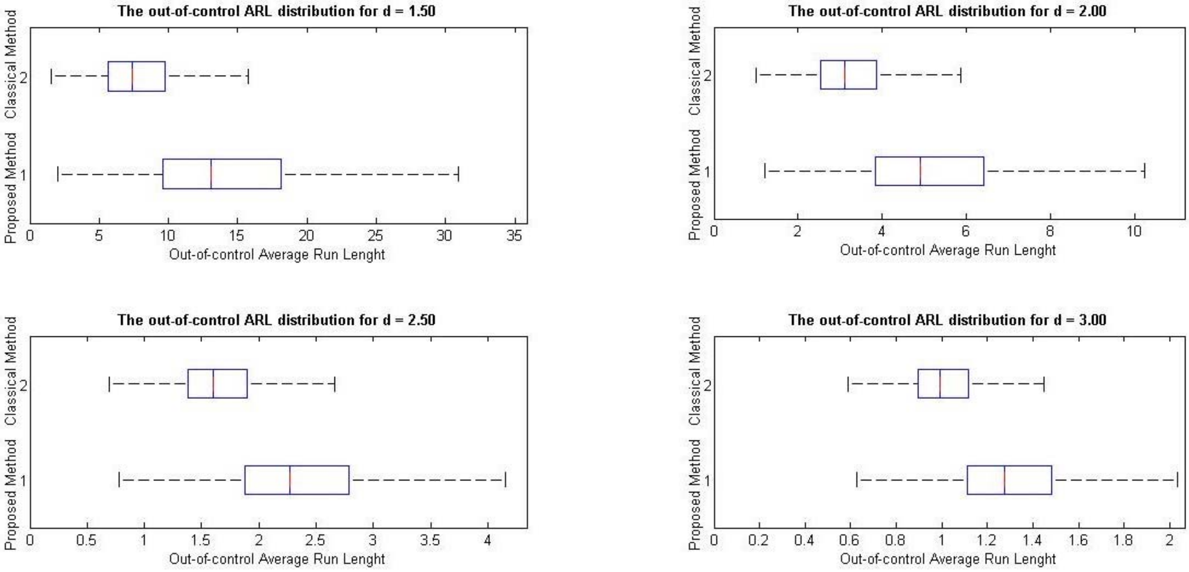

Figure 8. The box plot of the out-of-control ARL with and without limit adjustment for 10,000,000 simulated $\bar{X}$ charts with m = 50, n = 5, p=0.2 and ARL_0 = 370.4.

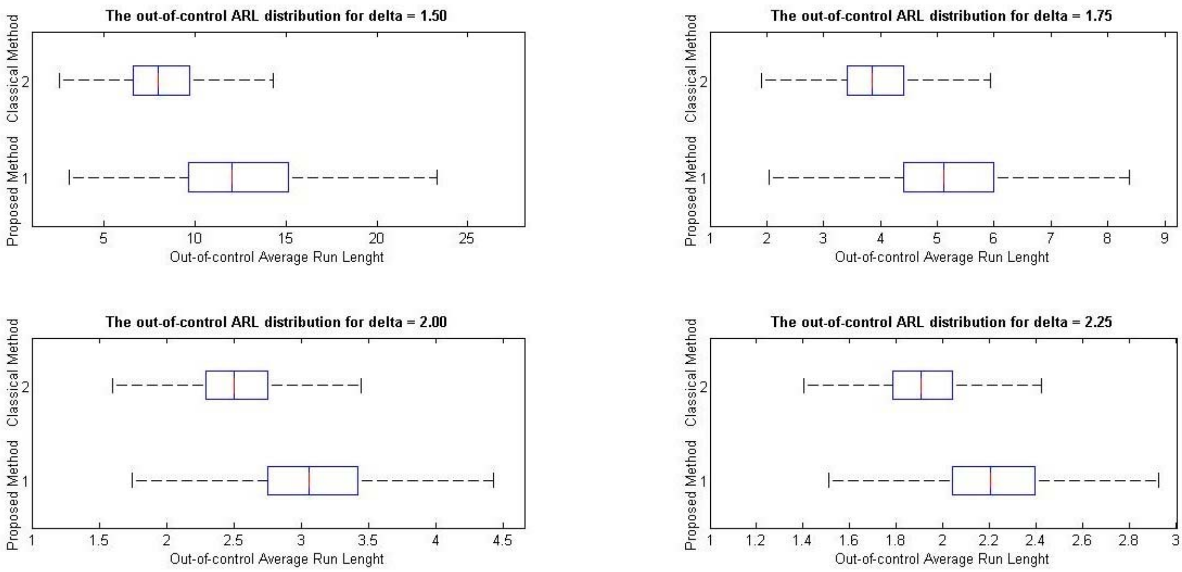

Figure 9. The box plot of the out-of-control ARL with and without limit adjustment for 10,000,000 simulated S^2 charts with m = 50, n = 5, p=0.1 and ARL_0 = 370.4.

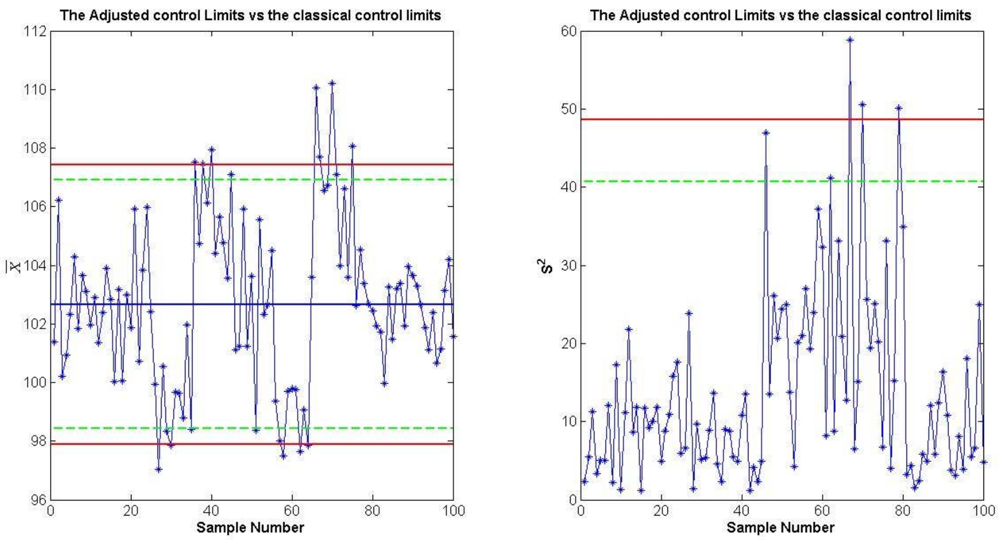

Figure 10. The Shewhart control charts with the adjusted control limits (straight lines) and the classical control limits (dash lines) for a Phase II dataset of N=100 simulated in-control and out-of-control samples each of size n=5.