

# Quasi-viscous accretion flow – I. Equilibrium conditions and asymptotic behaviour

Jayanta K. Bhattacharjee,<sup>1★</sup> Atri Bhattacharya,<sup>2★</sup> Tapas K. Das<sup>2,3★</sup>  
and Arnab K. Ray<sup>4★</sup>

<sup>1</sup>*S. N. Bose National Centre for Basic Sciences, Sector III, Block JD, Salt Lake, Kolkata 700098, India*

<sup>2</sup>*Harish-Chandra Research Institute, Chhatnag Road, Jhansi, Allahabad 211019, India*

<sup>3</sup>*Theoretical Institute for Advanced Research in Astrophysics, 101, Section 2, Kuang Fu Road, Hsinchu, Taiwan*

<sup>4</sup>*Homi Bhabha Centre for Science Education, Tata Institute of Fundamental Research, V. N. Purav Marg, Mankhurd, Mumbai 400088, India*

Accepted 2009 April 27. Received 2009 March 30; in original form 2008 December 28

## ABSTRACT

In a novel approach to studying viscous accretion flows, viscosity has been introduced as a perturbative effect, involving a first-order correction in the  $\alpha$ -viscosity parameter. This method reduces the problem of solving a second-order non-linear differential equation (Navier–Stokes equation) to that of an effective first-order equation. Viscosity breaks down the invariance of the equilibrium conditions for stationary inflow and outflow solutions, and distinguishes accretion from wind. Under a dynamical systems classification, the only feasible critical points of this ‘quasi-viscous’ flow are saddle points and spirals. On large spatial scales of the disc, where a linearized and radially propagating time-dependent perturbation is known to cause a secular instability, the velocity evolution equation of the quasi-viscous flow has been transformed to bear a formal closeness with Schrödinger’s equation with a repulsive potential. Compatible with the transport of angular momentum to the outer regions of the disc, a viscosity-limited length-scale has been defined for the full spatial extent over which the accretion process would be viable.

**Key words:** accretion, accretion discs – black hole physics – hydrodynamics – instabilities.

## 1 INTRODUCTION

The role of viscosity in the formation of accretion discs has, over the years, been recognized to be of paramount importance (Lynden-Bell 1969; Shakura & Sunyaev 1973; Lynden-Bell & Pringle 1974; Pringle 1981; Frank, King & Raine 2002). The standard Keplerian distribution of gaseous matter around a central accretor is determined by viscosity, without which there could be no angular momentum transport on outer length-scales, and, therefore, no infall. Viscosity in a Keplerian disc also has a bearing on the time-scale of the inward radial drift of matter (Frank et al. 2002). So, viscosity leaves its imprint on accretion processes in more ways than one. While these facts are not a matter of doubt anymore, the knowledge of the exact nature of viscosity still proves elusive. No help is also forthcoming from the fact that the observables of an accretion disc have been theoretically shown to be independent of viscosity (Frank et al. 2002). To explain the enhanced outward transport of angular momentum, and the accompanying inflow rate, it has been variously suggested that turbulence, ordinarily hydrodynamic or even magnetohydrodynamic (Balbus & Hawley 1998) holds the key to this as yet unsolved question. As a result, much of the literature in accretion-related studies has been devoted to viscosity from one perspective or the other (Shakura & Sunyaev 1973; Lynden-Bell & Pringle 1974; Piran 1978; Liang & Thomson 1980; Pringle 1981; Matsumoto et al. 1984; Muchotrzeb-Czerny 1986; Abramowicz et al. 1988; Narayan & Yi 1994; Chakrabarti & Titarchuk 1995; Kato, Abramowicz & Chen 1996; Manmoto et al. 1996; Chen, Abramowicz & Lasota 1997; Peitz & Appl 1997; Frank et al. 2002; Afshordi & Paczyński 2003; Chakrabarti & Das 2004; Becker & Subramanian 2005; Umurhan et al. 2006; Das 2007; Lanzafame 2008; Sharma 2008; Subramanian, Becker & Kafatos 2008).

On the other hand, as a subject of general fluid dynamical interest, rotational flows have also been quite thoroughly understood from a conserved and inviscid perspective (Chandrasekhar 1981). Under this general theoretical framework, in contrast to a viscosity-driven accretion process, another model – the sub-Keplerian low angular momentum inviscid flow – has by now also become well-established in accretion studies (Abramowicz & Zurek 1981; Fukue 1987; Chakrabarti 1989; Nakayama & Fukue 1989; Chakrabarti 1990; Kafatos & Yang 1994;

\*E-mail: jkb@bose.res.in (JKB); atri@mri.ernet.in (AB); tapas@mri.ernet.in (TKD); akr@hbcse.tifr.res.in (AKR)

Yang & Kafatos 1995; Molteni, Sponholz & Chakrabarti 1996; Pariev 1996; Lu et al. 1997; Das 2002; Das, Pendharkar & Mitra 2003; Ray 2003a; Barai, Das & Wiita 2004; Das 2004; Abraham, Bilić & Das 2006; Das, Bilić & Dasgupta 2007; Chaudhury, Ray & Das 2006; Goswami et al. 2007; Roy & Ray 2009). This is a particularly expedient and simple physical system to investigate, and is considered especially suitable for describing the rotating flow in the innermost regions of the disc, very close to the event horizon of a black hole. Steady global solutions of inviscid axisymmetric accretion on to a black hole have been meticulously studied over the years, and at present there exists an extensive body of literature devoted to the subject, with special emphasis on the transonic nature of solutions, the multitransonic character of the flow, formation of shocks and the stability of global solutions under time-dependent linearized perturbations.

Having stressed the usefulness of the inviscid model among researchers in accretion astrophysics, it should also be recognized that this model has its own limitations. It is easy to understand that while the presence of angular momentum leads to the formation of an accretion disc in the first place, a physical mechanism should also be found for the outward transport of angular momentum, especially if its distribution is not sub-Keplerian (which, for instance, is the case for strongly coupled black hole binary systems). This should then make possible the inward drift of the accreting matter into the potential well of the accretor. It has already been mentioned that viscosity has been known all along to be just such a physical means to effect infall, although the exact prescription for viscosity in an accretion disc is still a matter of much debate (Papaloizou & Lin 1995; Frank et al. 2002). What is well appreciated, however, is that the viscous prescription should be compatible with an enhanced outward transport of angular momentum. The very well-known  $\alpha$  parametrization of Shakura & Sunyaev (1973) is based on this principle.

So, it transpires that on global scales – especially on the very largest scales of a non-sub-Keplerian disc – the inviscid model will encounter difficulties in the face of the fact that without an effective outward transport of angular momentum, the accretion process cannot be sustained globally. To address this issue, what is being introduced in this paper is the ‘quasi-viscous’ disc model. This model involves prescribing a very small first-order viscous correction in the  $\alpha$ -viscosity parameter of Shakura & Sunyaev (1973), about the zeroth-order inviscid solution. In doing this, a viscous generalization of the inviscid flow can be logically extended to capture the important physical properties of accretion discs on large length-scales, without compromising on the fundamentally simple and elegant features of the inviscid model. This is the single most appealing aspect of the quasi-viscous disc model vis-a-vis many other standard models of axisymmetric flows which involve viscosity. To dwell on this point further, while Keplerian discs explain infall processes satisfactorily, as far as the role of viscosity is concerned, there is the difficulty that the net force driving infall is practically zero (resulting from an exact balance of the centrifugal effects against gravity). On the other hand, while sub-Keplerian inviscid discs are free of this difficulty, they do not account for any direct outward transport of angular momentum<sup>1</sup> – something that is also necessary to bring about infall. The truth probably lies somewhere in between. The quasi-viscous model tries to address that possible area of convergence. While it accounts for an angular momentum transport, it also ensures that there is an effective force in the flow to drive the accretion process from an outer boundary to the event horizon of a black hole accretor.

From a most general fluid dynamical viewpoint, the effect of viscosity is described by a second-order non-linear differential equation – the Navier–Stokes equation (Landau & Lifshitz 1987). The inviscid limit, on the other hand, is mathematically founded on Euler’s equation, which is a first-order non-linear differential equation. The quasi-viscous flow here is based on a perturbative scheme about the inviscid conditions, and so the governing equation for this kind of viscous flow can be suitably approximated to a first-order equation. In fluid dynamics, this is not a particularly unusual mathematical expedient when it comes to accounting for viscosity (Bohr, Dimon & Putkaradze 1993).

The immediate effect of viscosity on stationary flow solutions has been to break down the invariance of the equilibrium conditions for inflows and outflows, something that is otherwise preserved well in the inviscid limit. The equilibrium conditions of the quasi-viscous flow have been precisely identified, and the nature of the equilibrium points (critical points) has been discerned by devising a first-order autonomous dynamical system from the flow equations. In this manner, it has been shown that the possible critical points in the phase plot can be either saddle points, or spirals, or nodes. In the inviscid limit, an earlier study has shown that only saddle points and centre-type points can exist (Chaudhury et al. 2006). Now centre-type points are a limiting case of spirals (Jordan & Smith 1999). Since the quasi-viscous model represents a generalization of the global inviscid flow, but at the same time also implies that viscosity can be tuned to arbitrarily small values, a likely scenario that emerges is that the centre-type points (associated with inviscid flows) will become spirals on the inclusion of viscosity in the flow, however small. This should have various ramifications, especially about connecting multiple transonic solutions through standing shocks (Chakrabarti 1989; Das 2002).

An earlier work (Bhattacharjee & Ray 2007) on the quasi-viscous flow, driven by the classical Newtonian potential, has revealed an instability – secular instability – when the stationary flow solutions are subjected to small time-dependent perturbations. By analogy, exactly this kind of instability is also seen to develop in Maclaurin spheroids on the introduction of a kinematic viscosity to a first order (Chandrasekhar 1987). Similar features in the quasi-viscous flow have also been argued for in this paper following the earlier study (Bhattacharjee & Ray 2007), but, in this instance, under the pseudo-Schwarzschild generalization.

On large length-scales, the quasi-viscous flow displays some interesting asymptotic behaviour. Under highly subsonic conditions, the pertinent flow equation (Navier–Stokes equation) can be transformed mathematically into an equation that resembles Schrödinger’s equation, albeit with a repulsive potential. This has been physically connected to the accumulation of angular momentum on large length-scales

<sup>1</sup> However, in the absence of any viscous transport of angular momentum, jets launched from accretion discs are supposed to be the only outlet for the intrinsic angular momentum of the infalling matter (Wiita 2001).

(a property of the disc that is very probably also related to the growth of secular instability on the same length-scales), and a limiting scale of length has been derived from this condition.

Finally, it should be worth stressing the fact that the entire treatment presented here has been completely analytic, and to the extent that this work purports to study pseudo-Schwarzschild flows, it has accounted for the use of any kind of generalized pseudo-Newtonian potential to drive the accretion process. This study is the first in a series, in which quasi-viscous accretion around a rotating black hole will also be taken up later. This will reveal the influence of black hole spin angular momentum (Kerr parameter) on various equilibrium and stability criteria for the flow.

## 2 THE EQUATIONS OF THE QUASI-VISCOUS AXISYMMETRIC FLOW AND ITS EQUILIBRIUM CONDITIONS

For the thin disc, under the condition of hydrostatic equilibrium along the vertical direction (Matsumoto et al. 1984; Frank et al. 2002), two of the relevant flow variables are the drift velocity,  $v$ , and the surface density,  $\Sigma$ . In the thin-disc approximation, the latter has been defined by vertically integrating the volume density,  $\rho$ , over the disc thickness,  $H(r)$ . This gives  $\Sigma \cong \rho H$ , and in terms of  $\Sigma$ , the continuity equation is set down as

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma v r) = 0. \quad (1)$$

The axisymmetric accretion flow, driven by the gravitational field of a centrally located black hole, is described in terms of the Newtonian geometry of space and time with the help of what is known as a pseudo-Newtonian potential,  $\phi(r)$ . This paper will make use of such a general expression for the potential, and so the analytical results presented here will hold good under the choice of *any* pseudo-Newtonian potential. Assumption of the hydrostatic equilibrium in the vertical direction will give the condition

$$H = \frac{r}{\sqrt{\gamma}} \frac{c_s}{v_K}, \quad (2)$$

in which the local speed of sound,  $c_s$ , and the local Keplerian velocity,  $v_K$ , are, respectively, defined as  $c_s^2 = \gamma P / \rho$  and  $v_K^2 = r \phi'$ , with the pressure,  $P$ , itself being expressed in terms of a polytropic equation of state,  $P = K \rho^\gamma$  (consequently, the speed of sound may also be given as  $c_s^2 = \partial P / \partial \rho$ ). For a dissipative flow (such as the quasi-viscous flow), the appropriateness of the general polytropic prescription merits a close attention. In considering the thermodynamics of a gaseous system in convective-polytropic equilibrium, it is customary (Chandrasekhar 1939) to connect the heat change,  $dQ$ , to the instantaneous change in temperature,  $dT$ , by a simple proportionality relationship going as  $dQ = b dT$ , in which  $b$  may be treated as a constant (Chandrasekhar 1939). In the isothermal limit,  $b \rightarrow \infty$ , while in the adiabatic limit,  $b \rightarrow 0$ . It can be shown mathematically, by invoking the first law of thermodynamics, that the range of values for the polytropic exponent,  $\gamma$ , varies between unity (the isothermal limit) and  $c_p/c_v$ , which is the ratio of the two coefficients of specific heat capacity of a gas (corresponding simply to the conserved adiabatic limit). Dissipation in the polytropic system can, therefore, be eminently accounted for when  $1 \leq \gamma < c_p/c_v$ . So the polytropic prescription is of a much more general scope than the simple conserved adiabatic case, and is entirely suited for the study of the dissipative quasi-viscous disc here.

Using the relationship between  $c_s$  and  $\rho$ , the disc height can be explicitly set down in terms of the standard fluid flow variables as

$$H = (\gamma K)^{1/2} \frac{\rho^{(\gamma-1)/2} r^{1/2}}{\sqrt{\gamma \phi'}}, \quad (3)$$

and with the use of this result, the continuity equation could then be recast as

$$\frac{\partial}{\partial t} [\rho^{(\gamma+1)/2}] + \frac{\sqrt{\phi'}}{r^{3/2}} \frac{\partial}{\partial r} \left[ \frac{\rho^{(\gamma+1)/2} v r^{3/2}}{\sqrt{\phi'}} \right] = 0. \quad (4)$$

The general condition for the balance of specific angular momentum in the flow is given by (Frank et al. 2002)

$$\frac{\partial}{\partial t} (\Sigma r^2 \Omega) + \frac{1}{r} \frac{\partial}{\partial r} [(\Sigma v r) r^2 \Omega] = \frac{1}{2\pi r} \left( \frac{\partial \mathcal{G}}{\partial r} \right), \quad (5)$$

where  $\Omega$  is the local angular velocity of the flow, while the torque is given as

$$\mathcal{G} = 2\pi r v \Sigma r^2 \left( \frac{\partial \Omega}{\partial r} \right), \quad (6)$$

with  $\nu$  being the kinematic viscosity associated with the flow. With the use of the continuity equation, as equation (1) gives it, and going by the Shakura & Sunyaev (1973) prescription for the kinematic viscosity,  $\nu = \alpha c_s H$ , it would be easy to reduce equation (5) to the form (Narayan & Yi 1994; Frank et al. 2002)

$$\frac{1}{v} \frac{\partial}{\partial t} (r^2 \Omega) + \frac{\partial}{\partial r} (r^2 \Omega) = \frac{1}{\rho v r H} \frac{\partial}{\partial r} \left[ \frac{\alpha \rho H c_s^2 r^3}{\sqrt{\gamma} \Omega_K} \left( \frac{\partial \Omega}{\partial r} \right) \right], \quad (7)$$

with  $\Omega_K$  being defined from  $v_K = r \Omega_K$ .

Going back to equation (4), a new variable is defined as  $f = \rho^{(\gamma+1)/2} v r^{3/2} / \sqrt{\phi'}$ , whose steady value, as it is very easy to see from equation (4), can be closely identified with the constant matter flux rate. In terms of this new variable, equation (4) can be modified as

$$\frac{\partial}{\partial t} [\rho^{(\gamma+1)/2}] + \frac{\sqrt{\phi'}}{r^{3/2}} \left( \frac{\partial f}{\partial r} \right) = 0, \quad (8)$$

while equation (7) can be rendered as

$$\frac{1}{v} \frac{\partial}{\partial t} (r^2 \Omega) + \frac{\partial}{\partial r} (r^2 \Omega) = \frac{\alpha \gamma K}{f} \frac{\partial}{\partial r} \left[ \frac{f}{\sqrt{\gamma}} \left( \frac{f^2 \Omega_K}{\rho^2 v^3} \right) \frac{\partial \Omega}{\partial r} \right]. \quad (9)$$

The inviscid disc model is given by the requirement that  $r^2 \Omega = \lambda$ , in which  $\lambda$  is the constant specific angular momentum. The quasi-viscous disc that is being proposed here will introduce a first-order correction in terms involving  $\alpha$ , the Shakura & Sunyaev (1973) viscosity parameter, about the constant angular momentum solution. Mathematically, this will be represented by the prescription of an effective specific angular momentum,

$$\lambda_{\text{eff}}(r) = r^2 \Omega = \lambda + \alpha r^2 \tilde{\Omega}, \quad (10)$$

with the form of  $\tilde{\Omega}$  having to be determined from equation (9), under the stipulation that the dimensionless  $\alpha$ -viscosity parameter is much smaller than unity. This smallness of the quasi-viscous correction induces only very small changes on the constant angular momentum background, and, therefore, neglecting all orders of  $\alpha$  higher than the first, and ignoring any explicit time-variation of the viscous correction term, the latter being a standard method adopted also for Keplerian flows (Lightman & Eardley 1974; Shakura & Sunyaev 1976; Pringle 1981; Frank et al. 2002), the dependence of  $\tilde{\Omega}$  on  $v$  and  $\rho$  is obtained as

$$\tilde{\Omega} = - \frac{2\lambda\gamma K}{\sqrt{\gamma} r^2} \left[ \frac{f^2 \Omega_K}{\rho^2 v^3 r^3} + \int \frac{f^2 \Omega_K}{\rho^2 v^3 r^3} \left( \frac{1}{f} \frac{\partial f}{\partial r} \right) dr \right]. \quad (11)$$

With  $\tilde{\Omega}$  thus defined, it becomes possible under stationary conditions to set down equation (10) in a modified form as

$$\lambda_{\text{eff}}(r) = \lambda - 2\alpha \frac{\lambda}{\sqrt{\gamma}} \left( \frac{c_s^2}{v v_K} \right). \quad (12)$$

Lastly, the equation for radial momentum balance in the flow will also have to be modified under the condition of quasi-viscous dissipation. This has to be done according to the scheme outlined in equation (10) by which, the centrifugal term,  $\lambda_{\text{eff}}^2(r)/r^3$ , of the radial momentum balance equation, will have to be corrected up to a first order in  $\alpha$ . This will finally lead to the result

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial P}{\partial r} + \phi'(r) - \frac{\lambda^2}{r^3} - 2\alpha \frac{\lambda}{r^3} (r^2 \tilde{\Omega}) = 0, \quad (13)$$

with  $\tilde{\Omega}$  being given by equation (11), and  $P$  being expressed as a function of  $\rho$  with the help of a polytropic equation of state, as it has been mentioned earlier. One aspect of equation (13) deserves special mention. Accretion disc models based on the  $\alpha$  parametrization do not usually account for the explicit presence of any viscosity-dependent term along the radial component of the momentum equation (viscosity is brought in through the azimuthal component only), although an attempt towards this end has been made by Kato et al. (1996). A similar principle has been followed in equation (13) above, whose steady solution is given as

$$v \frac{dv}{dr} + \frac{1}{\rho} \frac{dP}{dr} + \phi'(r) - \frac{\lambda^2}{r^3} + 4\alpha \frac{\lambda^2}{\sqrt{\gamma} r^3} \left( \frac{c_s^2}{v v_K} \right) = 0, \quad (14)$$

from which the first integral cannot be obtained analytically because of the dissipative  $\alpha$ -dependent term. In the inviscid limit, though, the integral is easily obtained. This case will be governed by conserved conditions, and its solutions have been well-known in accretion literature (Chakrabarti 1989; Das 2002; Das et al. 2003). They will either be open solutions passing through saddle points or closed paths about centre-type points. The slightest presence of viscous dissipation, however, will radically alter the nature of solutions seen in the inviscid limit, and it may be easily understood that solutions forming closed paths about centre-type points will, under conditions of small-viscous correction, be changed to solutions of the spiralling kind (Liang & Thomson 1980; Matsumoto et al. 1984; Afshordi & Paczyński 2003). This state of affairs is appreciated very easily by the analogy of the simple harmonic oscillator. In the undamped state, the phase trajectories of the oscillator will, very much like the solutions of the inviscid flow, be either closed paths about centre-type points or open paths through saddle points (Jordan & Smith 1999). With the presence of even very weak damping, the closed paths change into spiralling solutions. A more detailed analysis in this regard will be carried out in Section 3.

## 2.1 The fixed points for polytropic flows

The pressure,  $P$ , is prescribed by an equation of state for the flow (Chandrasekhar 1939). As a general polytropic, it is given as  $P = K \rho^\gamma$ , where  $K$  is a measure of the entropy in the flow and  $\gamma$  is the polytropic exponent. In terms of  $\gamma$ , the polytropic index,  $n$ , is defined as  $n = (\gamma - 1)^{-1}$  (Chandrasekhar 1939). These definitions are necessary to recast the first integral of equation (4), which is easily obtained for stationary conditions. Using the relation between  $c_s$  and  $\rho$ , afforded by the polytropic condition, the final expression for the integral could be presented as

$$c_s^{2(2n+1)} \frac{v^2 r^3}{\phi'} = \frac{\gamma}{4\pi^2} \dot{\mathcal{M}}^2, \quad (15)$$

where  $\dot{\mathcal{M}} = (\gamma K)^n \dot{m}$  (Chakrabarti 1990) with  $\dot{m}$ , an integration constant itself, being physically the matter flow rate.

To obtain the critical points (the equilibrium points) of the flow, it should be necessary to combine both equations (14) and (15), along with the polytropic definition of the equation of state, to arrive ultimately at

$$(v^2 - \beta^2 c_s^2) \frac{d}{dr}(v^2) = \frac{2v^2}{r} \left[ \frac{\lambda^2}{r^2} \left( 1 - \frac{4\alpha c_s^2}{\sqrt{\gamma v^2 r \phi'}} \right) - r\phi' + \frac{1}{2}\beta^2 c_s^2 \left( 3 - r \frac{\phi''}{\phi'} \right) \right], \quad (16)$$

with  $\beta^2 = 2(\gamma + 1)^{-1}$ . The critical points of the flow will be given by the condition that the entire right-hand side of equation (16) will vanish along with the coefficient of  $d(v^2)/dr$ . Explicitly written down, following some rearrangement of terms, this will give the two critical point conditions as

$$v_c^2 = \beta^2 c_{sc}^2 = 2 \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} - \frac{8\alpha\lambda^2\beta^{-2}}{\sqrt{\gamma v_c^2 r_c^5 \phi'(r_c)}} \right]^{-1}, \quad (17)$$

with the subscript 'c' labelling the critical point values.

The roots of  $r_c$  could be fixed in terms of  $\gamma$ ,  $\alpha$ ,  $\lambda$  and  $\mathcal{M}$  in the  $r-v^2$  phase portrait of the stationary flow. In order to do so, the latter condition in equations (17) can be further modified with the help of the former condition to eliminate  $v_c$ . This will lead to

$$2 \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right]^{-1} = \beta^2 c_{sc}^2 - \frac{8\alpha\lambda^2 c_{sc}^2}{\sqrt{\gamma \beta^2 c_{sc}^2 r_c^5 \phi'(r_c)}} \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right]^{-1}, \quad (18)$$

which is a relation that gives  $r_c$  as a function of  $\gamma$ ,  $\alpha$ ,  $\lambda$  and  $c_{sc}$ . To eliminate the dependence on  $c_{sc}$ , it will be necessary to substitute  $v$  in equation (15) by using the critical point conditions. This will give

$$c_{sc}^2 = \left[ \frac{\gamma \mathcal{M}^2 \phi'(r_c)}{4\pi^2 \beta^2 r_c^3} \right]^{1/2(n+1)}, \quad (19)$$

a result with the obvious implication that the dependence of  $r_c$  will finally be given as  $r_c = f_1(\gamma, \alpha, \lambda, \mathcal{M})$ . One very interesting consequence of the presence of viscosity in equation (18) is that the sign of the square root on the right-hand side has to be chosen according to whether one is studying inflow solutions or outflow solutions. For inflows, a negative sign would have to be extracted from the square root, while for outflows the chosen sign would have to be positive. This suggests that the invariance of the coordinates of the critical points in the  $r-v^2$  plane would be lost because of dissipation, as opposed to the fully conserved inviscid case (Chaudhury et al. 2006). Viscosity, therefore, will distinguish accretion solutions from wind solutions.

The slope of the continuous solutions which could possibly pass through the critical points is to be obtained by applying the L'Hospital rule on equation (16) at the critical points. This will give a quadratic equation for the slope of stationary solutions at the critical points themselves in the  $r-v^2$  phase portrait. The resulting expression will read as

$$\left[ \frac{d}{dr}(v^2) \Big|_c \right]^2 + (\mathcal{Z}_1 + \alpha \mathcal{Z}_2) \left[ \frac{d}{dr}(v^2) \Big|_c \right] + (\mathcal{Z}_3 + \alpha \mathcal{Z}_4) = 0, \quad (20)$$

for which the constant coefficients  $\mathcal{Z}_1$ ,  $\mathcal{Z}_2$ ,  $\mathcal{Z}_3$  and  $\mathcal{Z}_4$  are given by

$$\begin{aligned} \mathcal{Z}_1 &= \frac{2}{\gamma} \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{c_{sc}^2}{r_c} \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right], & \mathcal{Z}_2 &= - \left( \frac{3\gamma - 1}{\gamma} \right) \frac{2\lambda^2 c_{sc}^2}{\sqrt{\gamma v_c^2 r_c^7 \phi'(r_c)}}, \\ \mathcal{Z}_3 &= \frac{c_{sc}^2}{\gamma} \left[ \frac{6\lambda^2}{r_c^4} + 2\phi''(r_c) + \frac{2}{\gamma + 1} c_{sc}^2 \left\{ \frac{3}{r_c^2} + \frac{d}{dr} \left( \frac{\phi''}{\phi'} \right) \Big|_c \right\} + \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{v_c^2}{r_c^2} \left\{ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right\}^2 \right], \\ \mathcal{Z}_4 &= \frac{8\lambda^2 c_{sc}^2}{\sqrt{\gamma v_c^2 r_c^7 \phi'(r_c)}} \frac{c_{sc}^2}{\gamma} \left[ \sqrt{r_c^7 \phi'(r_c)} \frac{d}{dr} \left\{ (r^7 \phi')^{-1/2} \right\} \Big|_c - \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{1}{r_c} \left\{ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right\} \right]. \end{aligned}$$

## 2.2 The fixed points for isothermal flows

For an isothermal flow, the necessary equation of state is given by  $P = \rho \kappa T / \mu m_H$ , in which  $\kappa$  is Boltzmann's constant,  $T$  is the constant temperature,  $m_H$  is the mass of a hydrogen atom and  $\mu$  is the reduced mass, respectively. The definition for  $H$  in equation (2) will have to be modified slightly by setting  $\gamma = 1$  (Afshordi & Paczyński 2003; Chaudhury et al. 2006). The local speed of sound will also be modified to become a global constant of the flow, going by the definition  $c_s^2 = \partial P / \partial \rho$ .

Going back to equation (14) and using the linear dependence between  $P$  and  $\rho$  as the proper equation of state will lead to

$$v \frac{dv}{dr} + c_s^2 \frac{d}{dr} (\ln \rho) + \phi'(r) - \frac{\lambda^2}{r^3} + 4\alpha \frac{\lambda^2}{r^3} \left( \frac{c_s^2}{v v_K} \right) = 0, \quad (21)$$

while the first integral of the stationary continuity condition from equation (4) will give

$$\frac{\rho^2 v^2 r^3}{\phi'} = \frac{\dot{m}^2}{4\pi^2 c_s^2}. \quad (22)$$

As it has been done for polytropic flows, the two foregoing equations can be combined to obtain

$$(v^2 - c_s^2) \frac{d}{dr}(v^2) = \frac{2v^2}{r} \left[ \frac{\lambda^2}{r^2} \left( 1 - \frac{4\alpha c_s^2}{\sqrt{v^2 r \phi'}} \right) - r\phi' + \frac{1}{2} c_s^2 \left( 3 - r \frac{\phi''}{\phi'} \right) \right], \quad (23)$$

from which the critical point conditions are easily identified as

$$v_c^2 = c_s^2 = 2 \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} - \frac{8\alpha \lambda^2}{\sqrt{v_c^2 r_c^5 \phi'(r_c)}} \right]^{-1}. \quad (24)$$

Fixing the critical point is a much simpler task for isothermal flows. As it has been done for the polytropic case,  $v_c$  has to be eliminated first from the second critical point condition in equations (24) to obtain

$$2 \left[ r_c \phi'(r_c) - \frac{\lambda^2}{r_c^2} \right] \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right]^{-1} = c_s^2 - \frac{8\alpha \lambda^2 c_s^2}{\sqrt{c_s^2 r_c^5 \phi'(r_c)}} \left[ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right]^{-1}. \quad (25)$$

In this expression, the speed of sound,  $c_s$ , is globally constant, and so having arrived at the critical point conditions, it should be easy to see that  $r_c$  and  $v_c$  have already been fixed in terms of a global constant of the system. The speed of sound can further be written in terms of the temperature of the system as  $c_s = \Theta T^{1/2}$ , where  $\Theta = (\kappa/\mu m_H)^{1/2}$ , and, therefore, it should be entirely possible to give a functional dependence for  $r_c$ , as  $r_c = f_2(\alpha, \lambda, T)$ . The slope of the solutions passing through the critical points in the  $r-v^2$  phase portrait is obtained simply by setting  $\gamma = 1$  in equation (20).

### 3 THE CHARACTER OF THE FIXED POINTS AND RELATED ASPECTS OF FLOW STABILITY

The stationary fluid equations describing a viscous flow are in general second-order non-linear differential equations (Landau & Lifshitz 1987). There is as yet no standard prescription for deriving analytical solutions from these equations. Therefore, for any understanding of the behaviour of the flow solutions, a numerical integration is in most cases the only recourse. On the other hand, an alternative approach could be made to this question, if the governing equations are set up to form a standard first-order dynamical system (Jordan & Smith 1999). The mathematical formalism of the stationary quasi-viscous flow is premised on two first-order differential equations, given by equations (14) and (15). Of these, the former is the result of an approximation to obtain an appropriate first-order differential equation to describe a viscous flow. This kind of approximation is quite usual in general fluid dynamical studies (Bohr et al. 1993), and short of carrying out any numerical integration, this approach allows for gaining physical insight into the behaviour of flows to a surprising extent. Towards this end, for the stationary polytropic flow, described by equation (16), it should be necessary to parametrize this equation and set up a coupled autonomous first-order dynamical system as (Jordan & Smith 1999)

$$\begin{aligned} \frac{d}{d\tau}(v^2) &= 2v^2 \left[ \frac{\lambda^2}{r^2} \left( 1 - \frac{4\alpha c_s^2}{\sqrt{\gamma v^2 r \phi'}} \right) - r\phi' + \frac{1}{2} \beta^2 c_s^2 \left( 3 - r \frac{\phi''}{\phi'} \right) \right] \\ \frac{dr}{d\tau} &= r (v^2 - \beta^2 c_s^2), \end{aligned} \quad (26)$$

in which  $\tau$  is an arbitrary mathematical parameter. With respect to accretion studies in particular, this kind of parametrization has been reported before (Muchotrzeb-Czerny 1986; Ray & Bhattacharjee 2002; Afshordi & Paczyński 2003; Chaudhury et al. 2006; Goswami et al. 2007; Mandal, Ray & Das 2007; Roy & Ray 2009). This opens the way to explore the mathematical nature of the critical points much more thoroughly.

The critical points (which give the equilibrium conditions in the flow) have themselves been fixed in terms of the flow constants. About these fixed point values, upon using a perturbation prescription of the kind  $v^2 = v_c^2 + \delta v^2$ ,  $c_s^2 = c_{sc}^2 + \delta c_s^2$  and  $r = r_c + \delta r$ , it becomes possible to derive a set of two autonomous first-order linear differential equations in the  $\delta r - \delta v^2$  plane, with  $\delta c_s^2$  itself being expressed in terms of  $\delta r$  and  $\delta v^2$  from equation (15) as

$$\frac{\delta c_s^2}{c_{sc}^2} = -\frac{\gamma - 1}{\gamma + 1} \left[ \frac{\delta v^2}{v_c^2} + \left\{ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right\} \frac{\delta r}{r_c} \right]. \quad (27)$$

The resulting coupled set of linear equations in  $\delta r$  and  $\delta v^2$  will have the form

$$\begin{aligned} \frac{d}{d\tau}(\delta v^2) &= A\delta v^2 + B\delta r \\ \frac{d}{d\tau}(\delta r) &= C\delta v^2 + D\delta r, \end{aligned} \quad (28)$$

in which the constant coefficients  $A$ ,  $B$ ,  $C$  and  $D$  are to be read as

$$A = \left( \frac{\gamma - 1}{\gamma + 1} \right) \mathcal{X} v_c^2 + \left( \frac{3\gamma - 1}{\gamma + 1} \right) \frac{4\alpha\lambda^2 c_{sc}^2}{\sqrt{\gamma v_c^2 r_c^5 \phi'(r_c)}},$$

$$B = -2v_c^2 \left[ \frac{2\lambda^2}{r_c^3} + \phi'(r_c) + r_c \phi''(r_c) + \frac{\beta^2 \phi''(r_c)}{2 \phi'(r_c)} c_{sc}^2 \mathcal{Y} + \frac{\beta^2}{2} \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{c_{sc}^2}{r_c} \mathcal{X}^2 \right]$$

$$+ \frac{8\alpha\lambda^2 v_c^2 c_{sc}^2}{\sqrt{\gamma v_c^2 r_c^7 \phi'(r_c)}} \left[ \left( \frac{\gamma - 1}{\gamma + 1} \right) \left\{ 3 - r_c \frac{\phi''(r_c)}{\phi'(r_c)} \right\} + \frac{5}{2} + \frac{r_c \phi''(r_c)}{2\phi'(r_c)} \right],$$

$$C = \left( \frac{2\gamma}{\gamma + 1} \right) r_c, \quad D = - \left( \frac{\gamma - 1}{\gamma + 1} \right) \mathcal{X} v_c^2,$$

under the further definition that

$$\mathcal{X} = r_c \frac{\phi''(r_c)}{\phi'(r_c)} - 3, \quad \mathcal{Y} = 1 + r_c \frac{\phi'''(r_c)}{\phi''(r_c)} - r_c \frac{\phi''(r_c)}{\phi'(r_c)}.$$

Solutions of the type  $\delta v^2 \sim \exp(\Omega\tau)$  and  $\delta r \sim \exp(\Omega\tau)$  in equations (28) will deliver the eigenvalues,  $\Omega$ , which are the growth rates of  $\delta v^2$  and  $\delta r$ , as

$$\Omega^2 - (A + D)\Omega + (AD - BC) = 0. \quad (29)$$

Under a further definition that  $\mathcal{P} = A + D$ ,  $\mathcal{Q} = AD - BC$  and  $\Delta = \mathcal{P}^2 - 4\mathcal{Q}$ , the solution of the foregoing quadratic equation can be written as

$$\Omega = \frac{\mathcal{P} \pm \sqrt{\Delta}}{2}. \quad (30)$$

Once the position of a critical point,  $r_c$ , has been ascertained, it is then a straightforward task to find the nature of that critical point by using  $r_c$  in equation (30) and all its associated quantities. Since it has been discussed in Section 2 that  $r_c$  is a function of  $\alpha$ ,  $\lambda$  and  $T$  for isothermal flows, and a function of  $\gamma$ ,  $\alpha$ ,  $\lambda$  and  $\mathcal{M}$  for polytropic flows, it effectively implies that  $\Omega$  can, in principle, be rendered as a function of the flow parameters for either kind of flow. For isothermal flows, starting from equation (23), a similar expression for the related eigenvalues may likewise be derived. The algebra in this case is much simpler, and it is easy to show that for isothermal flows the relevant results could be derived by simply setting  $\gamma = 1$  in equations (28).

The nature of the possible critical points can also be predicted from the form of  $\Omega$  in equation (30). If  $\Delta > 0$ , then a critical point can be either a saddle or a node (Jordan & Smith 1999). The precise nature of the critical point will then be dependent on the sign of  $\mathcal{Q}$ . If  $\mathcal{Q} < 0$ , then the critical point will be a saddle point. Such points are always notoriously unstable in terms of their sensitivity to generating a solution through them, after starting from a boundary value far away from the critical point (Ray & Bhattacharjee 2002, 2007a; Roy & Ray 2007). On the other hand, if  $\mathcal{Q} > 0$ , then the critical point will be a node. Such a point may or may not be stable, depending on the sign of  $\mathcal{P}$ . If  $\mathcal{P} < 0$ , then the node will be stable.

A completely different class of critical points will result when  $\Delta < 0$ . These points will be like a spiral (a focus). Once again, the stability of the spiral will depend on the sign of  $\mathcal{P}$ . If  $\mathcal{P} < 0$ , then the spiral will be stable. For inflow solutions in the quasi-viscous disc, the form of  $\mathcal{P}$  (deriving from the sum of  $A$  and  $D$ ) shows that it will *always* be negative. This is because for inflows the negative root of  $v_c^2$  (i.e.  $v_c < 0$ ) has to be extracted from the square root in the definition of  $A$ . The obvious implication that follows will be that if the critical point is either a spiral or a node, then it will be stable, with flow solutions in the neighbourhood of the critical point converging towards it.

The quasi-viscous prescription is based on the requirement that viscosity will only have a small perturbative effect about the conserved inviscid flow. In other words, one could tune the viscosity parameter,  $\alpha$ , to arbitrarily small (but non-zero) values. In this kind of a situation, it is much more likely than not that  $\Delta < 0$ , and that the stable critical point will be a spiral [nodal points, however, cannot be ruled out completely, as Afshordi & Paczyński (2003) have shown]. Therefore, the most likely picture that emerges as far as the phase portrait of the flow is concerned is that there will be adjacent unstable saddle points and stable spiral points (adjacent points cannot be both stable or unstable simultaneously). This argument is in keeping with an earlier study (Chaudhury et al. 2006) on the inviscid disc, where a generic conclusion that was drawn about the critical points was that for a conserved pseudo-Schwarzschild axisymmetric flow driven by any potential, the only admissible critical points would be either saddle points or centre-type points. For a saddle point,  $\Omega^2 > 0$ , while for a centre-type point,  $\Omega^2 < 0$ , with  $\Omega^2$  being real on both occasions. Noting that a centre-type point is merely a special case ( $\mathcal{P} = 0$ ) of a spiral, the introduction of viscosity as a small perturbative effect certainly represents a physical generalization. But with this, what is lost from the phase portrait of the flow are homoclinic trajectories connecting a saddle point to itself, or even heteroclinic trajectories connecting two saddle points, although one might still argue that heteroclinic paths will exist to connect saddle points with spirals.

Once the behaviour of all the physically relevant critical points has been understood in this way, a complete qualitative picture of the flow solutions passing through these points (if they are saddle points), or in the neighbourhood of these points (if they are spiral points), can be constructed, along with an impression of the direction that these solutions can have in the phase portrait of the flow (Jordan & Smith 1999). So what does that imply for multitransonicity, especially about flow solutions which can be generated very far away from the black hole accretor to reach its event horizon eventually? Many earlier studies (Chakrabarti 1989, 1990; Das 2002; Das et al. 2003; Chaudhury et al. 2006) have taken up this question in great detail, and it has been shown that for certain parameter-space values pertaining to the inviscid disc, three critical points can result. These are located in such a manner that a centre-type point is flanked by two saddle points (Chakrabarti

1990) through which transonic solutions can pass. For very small values of viscosity, it is now conceivable that the centre-type point in the middle will become a stable spiral. This is in fact very much in keeping with the conclusion of Liang & Thomson (1980) that the number of independent transonic solutions cannot exceed one plus the number of spiral singularities, and the critical transonic solution whenever it exists is unique in relevant situations. This line of reasoning arguably has also a strong bearing on another feature that is closely associated with multitransonicity in accretion flows – shocks, with or without dissipation (Chakrabarti 1989, 1990; Das 2002; Das et al. 2003; Chakrabarti & Das 2004; Das 2007; Fukumura & Kazanas 2007; Lanzafame 2008).

Liang & Thomson (1980) have further suggested that in realistic physical situations, models with spiral singularities are unstable. Many earlier works have taken up the question of the stability of viscous thin-disc accretion (Lightman & Eardley 1974; Shakura & Sunyaev 1976; Livio & Shaviv 1977; Piran 1978; Kato 1978; Kato, Honma & Matsumoto 1988; Chen & Taam 1993; Manmoto et al. 1996; Kato et al. 1996; Umurhan & Shaviv 2005). Regarding quasi-viscous accretion discs in particular, Bhattacharjee & Ray (2007) have already shown that stationary flow solutions driven by the simple Newtonian potential are unstable under time-dependent perturbations. For a flow that has been modelled to be driven by a general gravitational potential,  $\phi$  (as opposed to the choice of any particular kind of mathematical form for  $\phi$ ), the same feature can be shown to hold true. To this end, a time-dependent perturbation is introduced about the stationary solutions of the flow variables,  $v$  and  $\rho$ , according to the scheme  $v(r, t) = v_0(r) + v'(r, t)$  and  $\rho(r, t) = \rho_0(r) + \rho'(r, t)$ , in which the subscript ‘0’ implies stationary values. Going by equations (4) and (8), a further linearized dependence involving  $f$  can be derived as

$$\frac{f'}{f_0} = \left( \frac{\gamma + 1}{2} \right) \frac{\rho'}{\rho_0} + \frac{v'}{v_0}, \quad (31)$$

which gives a relation for the perturbation,  $f'(r, t)$ , on the constant background accretion flow rate,  $f_0(r)$ . For spherically symmetric flows, this Eulerian perturbation scheme has been applied by Petterson, Silk & Ostriker (1980) and Theuns & David (1992), while for inviscid axisymmetric flows, the same method has been used equally effectively by Ray (2003a) and Chaudhury et al. (2006). In terms of  $f'$ , a linearized equation of motion for the perturbation can be derived as

$$\frac{\partial^2 f'}{\partial t^2} + 2 \frac{\partial}{\partial r} \left( v_0 \frac{\partial f'}{\partial t} \right) + \frac{1}{v_0} \frac{\partial}{\partial r} \left[ v_0 (v_0^2 - \beta^2 c_{s0}^2) \frac{\partial f'}{\partial r} \right] = 4\alpha \frac{\lambda^2}{\sqrt{\gamma}} \frac{\sigma}{v_0 r^3} \left[ \frac{\partial f'}{\partial t} + \left( \frac{3\gamma - 1}{\gamma + 1} \right) v_0 \frac{\partial f'}{\partial r} - \frac{1}{\sigma} \int \sigma \frac{\partial}{\partial r} \left( \frac{\partial f'}{\partial t} \right) dr \right], \quad (32)$$

in which  $\sigma = c_{s0}^2/(v_0 v_K)$  and  $c_{s0}$  is the local speed of sound in the steady state. It shall be important to realize here that the choice of a driving potential, Newtonian or pseudo-Newtonian has no explicit bearing on the form of the equation, and the case of  $\alpha = 0$  corresponds to a stable configuration (Ray 2003a; Chaudhury et al. 2006). The form of equation (32) is identical to the one derived by Bhattacharjee & Ray (2007) for flows driven by the Newtonian potential (except for the fact that here all  $\alpha$ -dependent terms are scaled by a factor of  $1/\sqrt{\gamma}$ ), and so none of the conclusions regarding secular instability, drawn for the case of the Newtonian potential, will be qualified upon using any of the pseudo-Newtonian potentials (Paczyński & Wiita 1980; Nowak & Wagoner 1991; Artemova, Björnsson & Novikov 1996) which are regularly invoked in accretion-related literature to describe rotational flows on to a Schwarzschild black hole. This shall be especially true for flows on large length-scales, where all pseudo-Newtonian potentials converge to the Newtonian limit. On these very length-scales, the perturbation (treated as high-frequency travelling waves) has been known to exhibit an exponential growth behaviour under a WKB analysis (Bhattacharjee & Ray 2007). This has disturbing implications, since all physically meaningful inflow solutions will have to pass through the large length-scales, and connect the outer boundary of the accretion disc to the event horizon.

The flow will also be unstable when equation (32) is used to treat the perturbation as a standing wave between two chosen boundaries (Bhattacharjee & Ray 2007). It need not always be true that standing and travelling waves will simultaneously exhibit the same qualitative properties as far as stability is concerned. Many instances in fluid dynamics bear this out. In the case of binary fluids, standing waves exhibit instability, as opposed to travelling waves (Cross & Hohenberg 1993; Bhattacharya & Bhattacharjee 2005), while the whole physical picture is quite opposite for the fluid dynamical problem of the hydraulic jump (Bohr et al. 1993; Singha, Bhattacharjee & Ray 2005; Ray & Bhattacharjee 2007b). Contrary to all this, the axisymmetric stationary quasi-viscous flow is greatly disturbed both by a standing wave and by a travelling wave (Bhattacharjee & Ray 2007). This provides convincing evidence of its unstable character, and it is very much in consonance with similar conclusions drawn from some earlier studies. For high-frequency radial perturbations, Chen & Taam (1993) have found that inertial-acoustic modes are locally unstable throughout the disc, with the outward travelling modes growing faster than the inward travelling modes in most regions of the disc. On the other hand, Kato et al. (1988) have revealed a growth in the amplitude of a non-propagating perturbation at the critical point, which, however, becomes stable in the inviscid regime.

This kind of instability – one that manifests itself only if some dissipative mechanism (viscous dissipation in the case of the quasi-viscous rotational flow) is operative – is called *secular instability* (Chandrasekhar 1987). It should be very much instructive here to furnish a parallel instance of the destabilizing influence of viscous dissipation in a system undergoing rotation: that of the effect of viscous dissipation in a Maclaurin spheroid (Chandrasekhar 1987). In studying ellipsoidal figures of equilibrium, Chandrasekhar (1987) has discussed that a secular instability develops in a Maclaurin spheroid, when the stresses derive from an ordinary viscosity which is defined in terms of a coefficient of kinematic viscosity (as the  $\alpha$  parametrization is for an accretion disc), and when the effects arising from viscous dissipation are considered as small perturbations on the inviscid flow, to be taken into account in the first order only. It is exactly in this spirit that the ‘quasi-viscous’ approximation has been prescribed for the thin accretion disc, although, unlike a Maclaurin spheroid, an astrophysical accretion disc is an open system. In the context of this system, it will also be interesting to note that for sufficiently small viscosity (once again afforded by the  $\alpha$  parametrization), Kato et al. (1996) have demonstrated the possibility of a thermal instability under short-wavelength perturbations.



The linear analysis of Kato et al. (1996) was extended further to the non-linear regime by Manmoto et al. (1996), and it was found that the disturbance created in the flow persisted on the infall time-scale.

Curiously enough, the geometry of the fluid flow also seems to be having a bearing on the issue of stability. The same kind of mathematical study on the stability of a quasi-viscous rotational flow was also carried out earlier for a viscous spherically symmetric accreting system. In that treatment (Ray 2003b), viscosity was found to have a stabilizing influence on the system, causing a viscosity-dependent decay in the amplitude of a linearized standing-wave perturbation. This is quite in keeping with the understanding that the respective roles of viscosity are at variance with each other in the two distinctly separate cases of spherically symmetric flows and disc flows. While viscosity contributes to the resistance against infall in the former case, in the latter it aids the infall process.

#### 4 ASYMPTOTIC BEHAVIOUR OF THE QUASI-VISCOUS DISC

A previous study (Ray & Bhattacharjee 2007a) has dwelt on how it should be possible to select the stationary transonic solution of the inviscid axisymmetric flow through a non-perturbative time-dependent criterion (such as the minimization of the total specific energy of the flow). However, in the case of a quasi-viscous disc, a straightforward application of the methods employed for the inviscid disc would not be possible on two counts. First, the quasi-viscous disc being a dissipative system (i.e. with energy being allowed to be drained away from this system), there should be no occasion to look for the selection of a particular solution, and a selection criterion thereof, on the basis of energy minimization. Secondly, the fact that the quasi-viscous disc is unstable on large length-scales is reason enough to believe that no solution – transonic or otherwise – might be free of time-dependence. Therefore, a long-time evolution of the quasi-viscous disc towards a stationary end is not something that might be hoped for. Nevertheless, this kind of a disc system does exhibit some interesting asymptotic features on large length-scales.

On such length-scales of an accretion disc, all pseudo-Schwarzschild flows converge to the Newtonian limit, i.e.  $\phi(r) \simeq -GM/r$ . As a result, the stationary solution of equation (1), through equation (4), can be expressed on the same length-scales simply as

$$\rho_0^{(\gamma+1)/2} v_0 r^{5/2} \simeq -\frac{\dot{m}}{2\pi} \sqrt{\frac{GM}{K}}, \quad (33)$$

with  $\dot{m}$  being the conserved matter inflow rate. The negative sign arises because for inflows,  $v_0$  goes with a negative sign. From equation (33), with  $\rho_0$  approaching a constant ambient value on large length-scales, the drift velocity,  $v_0$ , can consequently be seen to go asymptotically as  $r^{-5/2}$ . Bearing in mind that for inflows,  $v_0 < 0$ , the asymptotic dependence of the effective angular momentum can be shown from equation (12) to be

$$\lambda_{\text{eff}}(r) \simeq \lambda \left[ 1 + 2 \frac{\alpha}{\sqrt{\gamma}} \left( \frac{r}{r_1} \right)^3 \right], \quad (34)$$

where  $r_1$  is a scale of length, which, to an order-of-magnitude, is given by  $r_1^3 \simeq \sqrt{\gamma} GM \dot{m} [2\pi c_s^3(\infty) \rho(\infty)]^{-1}$ . This asymptotic behaviour is entirely to be expected, because the physical role of viscosity is to transport angular momentum to large length-scales of the accretion disc.

The distribution of matter in a viscous disc takes place on a time-scale determined by viscosity, and so a study of the time-dependent properties in a viscous disc is one of the few means of acquiring some impression about the role of viscosity, especially since the observables in a steady disc are largely independent of viscosity (Frank et al. 2002). With that objective in mind, it will be worthwhile first to try to understand the structure of the governing time-dependent differential equation for the flow on large length-scales. To do so, it shall be necessary to invoke the approximation that very far from the accretor in the outer regions of the flow, on highly subsonic scales of velocity, the density variations are negligibly small compared to the time evolution of the velocity field. The evolution will consequently follow the general Navier–Stokes equation in the limit of  $\partial_j v_j = 0$ , which can be set down as

$$\partial_t v_i + v_j \partial_j v_i = \nu \partial_j \partial_j v_i - \partial_i V, \quad (35)$$

where the potential function  $V \equiv V(r, t) = nc_s^2 + \phi(r) + \lambda^2/2 r^2$ , with  $n$  being the usual polytropic index. Quite evidently, equation (35) is a non-linear differential equation, but with the help of the Hopf–Cole transformation (Regev 2006),

$$v_i = -\frac{2\nu}{\xi} \partial_i \xi, \quad (36)$$

it can be reduced to a linear form in the variable,  $\xi$ , going as

$$2\nu \partial_i \xi = 2\nu^2 \partial_i \partial_i \xi + V \xi. \quad (37)$$

The potential function,  $V$ , will in general be modified by the addition of an integration constant, which might physically be identified from the ambient conditions of the fluid. This, of course, will require the knowledge of the boundary conditions for the scalar function  $\xi$ , something whose inherent difficulties would be appreciated soon. In the outer regions of the disc, the speed of sound, which is a scalar function of the density, would asymptotically assume a constant value.

An interesting aspect of equation (37) is that for the one-to-one correspondence of  $2\nu$  with  $i\hbar$ , and of  $m$  with 1, there is an exact equivalence between this equation and Schrödinger's equation,

$$i\hbar \partial_i \psi = -\frac{\hbar^2}{2m} \partial_i \partial_i \psi + V \psi. \quad (38)$$

For the steady limit of the potential function,  $V$  (a requirement that is satisfied on large length-scales), it is easy to carry out a separation of variables in equation (37), and the resulting stationary eigenvalue equation in  $\xi$  may then be expressed in a Hamiltonian form as

$$-2v^2\nabla^2\xi + \tilde{V}\xi = E\xi, \quad (39)$$

in which  $\tilde{V} = -V$ . A comparison between equation (39) and the stationary Schrödinger equation ought to be instructive and insightful here. To have any notion of how  $\xi$  evolves in time, one would have to determine the eigenvalues given by  $E$ , and it is here that a great stumbling block is encountered. To solve the second-order differential equation given by equation (39), two boundary conditions are imperative, and they in turn would characterize the eigenvalues. The outer boundary condition on  $\xi$  is relatively easy to prescribe. For  $r \rightarrow \infty$  and  $v \rightarrow 0$ , the scalar function,  $\xi$ , will asymptotically approach a constant value, and there is much similarity in this with the asymptotic behaviour of another scalar function, the density,  $\rho$ . Knowing the precise inner boundary condition on  $\xi$ , however, is a most difficult problem. First of all, it depends on the nature of the accretor itself. While a black hole will have an event horizon, a neutron star or an ordinary star will have a physical surface. This fact alone has much influence over the inner boundary condition. Apart from this, realistically speaking, various astrophysical processes near the surface of the accretor will affect the flow (Petterson et al. 1980). In any case, equation (39) is valid for large-scales only. Yet, short of actually having to solve equation (39), it should still be possible to derive some information on how viscosity affects the flow in the outer regions of the disc. It is important to see that  $\tilde{V}$  in equation (39) assumes the properties of a repulsive potential. More than the pressure effects, this repulsive nature is reflective of the cumulative transfer of angular momentum to large length-scales of an accretion disc. Referring to equation (34), one can see that in the outer regions of the disc, the effective specific force,  $\varphi$ , is given to a first order in  $\alpha$  by

$$\varphi(r) \simeq -\frac{GM}{r^2} + \frac{\lambda^2}{r^3} + 4\frac{\alpha}{\sqrt{\gamma}} \left( \frac{\lambda^2}{r_1^3} \right), \quad (40)$$

from which it is evident that on scales of  $r \simeq r_1$ , the transport of angular momentum will give rise to an asymptotic constant non-zero force opposed to gravity. For the viscous disc, this gives rise to a repulsive effect on large length-scales, in opposition to gravitational attraction. From this argument, one may go a step beyond and conjecture that  $r_1(\sqrt{\gamma}/\alpha)^{1/3}$  defines a limiting length-scale for accretion that the outward transport of angular momentum imposes. This is also the length-scale on which secular instability is likely to be most conspicuous. In units of the Schwarzschild radius of the black hole,  $2GM/c^2$ , this limiting length-scale is seen to be

$$r_{\text{SI}} \simeq \frac{c^2}{2c_s(\infty)} \left[ \frac{\gamma\dot{m}}{2\pi\alpha(GM)^2\rho(\infty)} \right]^{1/3}. \quad (41)$$

All of this is quite compatible with how viscosity redistributes an annulus of matter in a Keplerian flow around an accretor; the inner region drifting in because of dissipation, and consequently, through the conservation of angular momentum and its outward transport, making it necessary for the outer regions of the matter distribution to spread even further outwards (Pringle 1981; Frank et al. 2002). This state of affairs is qualitatively not altered in anyway for the quasi-viscous flow, except for the fact that with viscosity being very weak here, the outward transport of angular momentum can perceptibly cause an outward drift of matter only on very large scales. The time-scale for this drift is expected to be on the viscous time-scale (Frank et al. 2002), and such a time-scale could be defined as  $t_{\text{visc}} \sim r_{\text{SI}}^2/v$ . Knowing that  $v = \alpha c_s H$ , a scaling behaviour for the viscous time-scale, for the drift of matter on length-scales of  $r_{\text{SI}}$ , could be found to be  $t_{\text{visc}} \sim \alpha^{-7/6}$ . It is obvious that once  $\alpha = 0$ , i.e. for the inviscid limit, the viscosity-defined scales in length and time would be shifted to infinity.

## 5 CONCLUDING REMARKS

One very important physical role of viscosity in an accretion disc is that it determines the distribution of matter in the disc. The manner in which viscosity redistributes an annulus of matter in a Keplerian flow around an accretor is very well known, with the inner region of this disc system drifting in because of dissipation, and consequently making it necessary for the outer regions of the matter distribution to spread out even farther, because of the conservation of angular momentum and its outward transport (Pringle 1981; Frank et al. 2002).

Viscosity also gives rise to secular instability if the disc is quasi-viscous (Bhattacharjee & Ray 2007). This casts much doubt on the long-term viability of the accretion flow, and its temporal evolution towards a stationary state. It may rightly be argued that the instability that develops on the large subsonic scales of a quasi-viscous disc is intimately related to the cumulative transfer of angular momentum on these very length-scales. The accumulation of angular momentum in this region may create an abrupt centrifugal barrier against any further smooth inflow of matter. However, this adverse effect could disappear if there should be some other means of transporting angular momentum from the inner regions of the disc. Astrophysical jets could readily afford such a means (Wiita 2001), insofar as jets actually cause a physical drift of angular momentum vertically away from the plane of the disc, instead of along its radial length. This will be all the more true if this off-the-plane angular momentum drift happens on length-scales that are much smaller than the scale indicated by equation (41).

Stability could be restored through many other means. A recent work by Mach & Malec (2008) has shown numerically how it should be possible to have stable steady accretion solutions for transonic flows of a self-gravitating gas. This stability argument holds true even for perturbations in the non-linear regime. Another work by Nagakura & Yamada (2008) has established the stability of an accretion shock (connecting transonic solutions passing through two distinct saddle points) under axisymmetric perturbations.

## ACKNOWLEDGMENTS

This research has made use of NASA's Astrophysics Data System. The authors express their gratitude to Rajaram Nityananda and Paul J. Wiita for some helpful comments. AKR would like to acknowledge the kind hospitality provided by HRI, Allahabad, India. The work of TKD was partially supported by the Theoretical Institute for Advanced Research in Astrophysics (TIARA) operated under Academia Sinica and the National Science Council Excellence Projects programme in Taiwan, administered through grant NSC 96-2752-M-007-007-PAE. An anonymous referee is to be thanked for useful suggestions in improving the draft.

## REFERENCES

- Abraham H., Bilić N., Das T. K., 2006, *Class. Quantum Gravity*, 23, 2371
- Abramowicz M. A., Zurek W. H., 1981, *ApJ*, 246, 314
- Abramowicz M. A., Czerny B., Lasota J. P., Szuszkiewicz E., 1988, *ApJ*, 332, 646
- Afshordi N., Paczyński B., 2003, *ApJ*, 592, 354
- Artemova I. V., Björnsson G., Novikov I. D., 1996, *ApJ*, 461, 565
- Balbus S. A., Hawley J. F., 1998, *Rev. Mod. Phys.*, 70, 1
- Barai P., Das T. K., Wiita P. J., 2004, *ApJ*, 613, L49
- Becker P. A., Subramanian P., 2005, *ApJ*, 622, 520
- Bhattacharjee J. K., Ray A. K., 2007, *ApJ*, 668, 409
- Bhattacharya S., Bhattacharjee J. K., 2005, *Proc. Indian Natl. Sci. Acad.*, 71A, 1
- Bohr T., Dimon P., Putkaradze V., 1993, *J. Fluid Mech.*, 254, 635
- Chakrabarti S. K., 1989, *ApJ*, 347, 365
- Chakrabarti S. K., 1990, *Theory of Transonic Astrophysical Flows*, World Scientific, Singapore
- Chakrabarti S. K., Das S., 2004, *MNRAS*, 349, 649
- Chakrabarti S. K., Titarchuk L. G., 1995, *ApJ*, 455, 623
- Chandrasekhar S., 1939, *An Introduction to the Study of Stellar Structure*, The University of Chicago Press, Chicago
- Chandrasekhar S., 1981, *Hydrodynamic and Hydromagnetic Stability*. Dover Publications, New York
- Chandrasekhar S., 1987, *Ellipsoidal Figures of Equilibrium*. Dover Publications, New York
- Chaudhury S., Ray A. K., Das T. K., 2006, *MNRAS*, 373, 146
- Chen X., Taam R., 1993, *ApJ*, 412, 254
- Chen X., Abramowicz M. A., Lasota J. P., 1993, *ApJ*, 476, 61
- Cross M. C., Hohenberg P. C., 1993, *Rev. Mod. Phys.*, 65, 851
- Das T. K., 2002, *ApJ*, 577, 880
- Das T. K., 2004, *MNRAS*, 349, 375
- Das S., 2007, *MNRAS*, 376, 1659
- Das T. K., Pendharkar J. K., Mitra S., 2003, *ApJ*, 592, 1078
- Das T. K., Bilić N., Dasgupta S., 2007, *J. Cosmol. Astropart. Phys.*, 06, 009
- Frank J., King A., Raine D., 2002, *Accretion Power in Astrophysics*. Cambridge Univ. Press, Cambridge
- Fukue J., 1987, *PASJ*, 39, 309
- Fukumura K., Kazanas D., 2007, *ApJ*, 669, 85
- Goswami S., Khan S. N., Ray A. K., Das T. K., 2007, *MNRAS*, 378, 1407
- Jordan D. W., Smith P., 1999, *Non-linear Ordinary Differential Equations*. Oxford Univ. Press, Oxford
- Kafatos M., Yang R. X., 1994, *MNRAS*, 268, 925
- Kato S., 1978, *MNRAS*, 185, 629
- Kato S., Honma F., Matsumoto R., 1988, *MNRAS*, 231, 37
- Kato S., Abramowicz M. A., Chen X., 1996, *PASJ*, 48, 67
- Landau L. D., Lifshitz E. M., 1987, *Fluid Mechanics*. Butterworth-Heinemann, Oxford
- Lanzafame G., 2008, in Pogorelov N. V., Audit E., Zank G. P., eds, *ASP Conf. Ser. Vol. 385, Numerical Modeling of Space Plasma Flows: Astronom 2007*. Astron. Soc. Pac., San Francisco, p. 115
- Liang E. P. T., Thomson K. A., 1980, *ApJ*, 240, 271
- Lightman A. P., Eardley D. M., 1974, *ApJ*, 187, L1
- Livio M., Shaviv G., 1977, *A&A*, 55, 95
- Lu J. F., Yu K. N., Yuan F., Young E. C. M., 1997, *A&A*, 321, 665
- Lynden-Bell D., 1969, *Nat*, 223, 690
- Lynden-Bell D., Pringle J. E., 1974, *MNRAS*, 168, 603
- Mach P., Malec E., 2008, *Phys. Rev. D*, 78, 124016
- Mandal I., Ray A. K., Das T. K., 2007, *MNRAS*, 378, 1400
- Manmoto T., Takeuchi M., Mineshige S., Matsumoto R., Negoro H., 1996, *ApJ*, 464, L135
- Matsumoto R., Kato S., Fukue J., Okazaki A. T., 1984, *PASJ*, 36, 71
- Molteni D., Sponholz H., Chakrabarti S. K., 1996, *ApJ*, 457, 805
- Muchotrzeb-Czerny B., 1986, *Acta Astron.*, 36, 1
- Nagakura H., Yamada S., 2008, *ApJ*, 689, 391
- Nakayama K., Fukue J., 1989, *PASJ*, 41, 271
- Narayan R., Yi I., 1994, *ApJ*, 428, L13
- Nowak A. M., Wagoner R. V., 1991, *ApJ*, 378, 656
- Paczyński B., Wiita P. J., 1980, *A&A*, 88, 23

- Papaloizou J. C. B., Lin D. N. C., 1995, *ARA&A*, 33, 505  
Pariev V. I., 1996, *MNRAS*, 283, 1264  
Peitz J., Appl S., 1997, *MNRAS*, 286, 681  
Peterson J. A., Silk J., Ostriker J. P., 1980, *MNRAS*, 191, 571  
Piran T., 1978, *ApJ*, 221, 652  
Pringle J. E., 1981, *ARA&A*, 19, 137  
Ray A. K., 2003a, *MNRAS*, 344, 83  
Ray A. K., 2003b, *MNRAS*, 344, 1085  
Ray A. K., Bhattacharjee J. K., 2002, *Phys. Rev. E*, 66, 066303  
Ray A. K., Bhattacharjee J. K., 2007a, *Class. Quantum Gravity*, 24, 1479  
Ray A. K., Bhattacharjee J. K., 2007b, *Phys. Lett. A*, 371, 241  
Regev O., 2006, *Chaos and Complexity in Astrophysics*. Cambridge Univ. Press, Cambridge  
Roy N., Ray A. K., 2007, *MNRAS*, 380, 733  
Roy N., Ray A. K., 2009, *MNRAS*, in press (arXiv:0903.0082)  
Shakura N. I., Sunyaev R. A., 1973, *A&A*, 24, 337  
Shakura N. I., Sunyaev R. A., 1976, *MNRAS*, 175, 613  
Sharma M., 2008, *MNRAS*, 391, 1369  
Singha S. B., Bhattacharjee J. K., Ray A. K., 2005, *Eur. Phys. J. B*, 48, 417  
Subramanian P., Becker P. A., Kafatos M., 2008, in Hasan S. S., Gangadhara R. T., Krishan V., eds, *Turbulence, Dynamos, Accretion Disks, Pulsars and Collective Plasma Processes*. Springer, Berlin, p. 249  
Theuns T., David M., 1992, *ApJ*, 384, 587  
Umurhan O. M., Shaviv G., 2005, *A&A*, 432, L31  
Umurhan O. M., Nemirovsky A., Regev O., Shaviv G., 2006, *A&A*, 446, 1  
Wiita P. J., 2001, preprint (astro-ph/0103020)  
Yang R. X., Kafatos M., 1995, *A&A*, 295, 238

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