Polynomial State-Space Model Decoupling for the Identification of Hysteretic Systems

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Abstract: Hysteresis is a nonlinear effect that shows up in a wide variety of engineering and scientific fields. The identification of hysteretic systems from input-output data is an important but challenging question, which has been studied by using both tailored parametric white-box identification methods as by using black-box identification methods. The white-box modeling approach is by far the most common in identifying hysteretic systems, and has the advantage of resulting into an interpretable model, but it requires to be adjusted to a specific hysteresis model. A black-box approach can be used more universally, but results in models containing many parameters that cannot easily be interpreted. In the current paper, we propose a two-step identification procedure that combines the best of the two approaches. We employ the Bouc-Wen hysteretic model to generate data that is used for identification. The system is identified using a black-box polynomial nonlinear state-space identification procedure. We reduce the number of parameters in this model by applying a polynomial decoupling method that results in a more parsimonious representation. We compare the full black-box model with the decoupled model and show that the proposed method results in a comparable performance, while significantly reducing the number of parameters.

Keywords: Polynomial Nonlinear State-Space, Hysteretic System, Bouc-Wen, Tensor Decomposition, Canonical Polyadic Decomposition, Decoupling Multivariate Polynomials

1. INTRODUCTION

Hysteresis is encountered in a variety of science and engineering fields, such as electromagnetism, aerodynamics, structural engineering, and even biology, ecology and psychology (Oh et al., 2009; Bernstein, 2007; Noël et al., 2017). The effect of hysteresis is caused by the property of multistability, and is typically expressed in a nonlinear memory effect (Oh et al., 2009; Noël et al., 2017).

System identification of hysteretic systems is challenging because dynamic nonlinearities act upon non-measurable internal state variables (Noël et al., 2017). So far, identification methods for hysteretic systems have mostly been limited to white-box techniques, which assume that the data satisfy a specific model for hysteresis (Noël et al., 2017). The main limitation of white-box identification procedures is that they need to be tailored to a certain (hysteresis) model structure, whereas a real-life identification task may require a prohibitively complicated model description. Recently, a black-box identification approach has been proposed in Noël et al. (2017), which tackles the task by employing a polynomial nonlinear state-space identification procedure. This approach can be applied without requiring a hysteresis model assumption, as it directly aims at minimizing a cost function that involves the (possibly weighted) distance between the model outputs and the measured data. A disadvantage of the black-box approach is that one loses physical intuition, since the parameters cannot be given a meaningful interpretation, and moreover, often a (very) large number of parameters needs to be estimated.

The approach we develop in this paper tries to combine the benefits of black-box and white-box approaches. We start from a polynomial nonlinear state-space (PNLSS) identification approach to identify a hysteretic system (Noël et al., 2017). Since the number of parameters in a PNLSS model increases quickly, it is beneficial to reduce this by including prior knowledge, which can be done for instance by transferring the nonlinearity into the feedback loop (Noël and Kerschen, 2013), or by employing other basis functions, such as splines, radial basis functions, or neural networks (Marconato et al., 2014). In the current paper, we will transform the PNLSS model into a simpler description by applying the approach of Dreesen et al. (2015b) to decouple the nonlinear part of the PNLSS

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model using a tensor decomposition method. This results in a state-space description in which the nonlinearities are contained into univariate (one-to-one) polynomials that operate on a linear combination of the state variables and the input, rather than multivariate polynomials from all state variables and the input to all state updates and the output.

The remainder of this paper is organized as follows. In Section 2 an algorithm for PNLSS estimation is presented. The decoupling method is presented in Section 3. The results for PNLSS and decoupling methods are compared in Section 4. In Section 5 we draw the conclusions and point out open questions for future work.

2. POLYNOMIAL NONLINEAR STATE-SPACE (PNLSS)

Polynomial nonlinear state-space system identification was introduced in Paduart et al. (2010) and further developed by Van Mulders and Vanbeylen (2013); Van Mulders et al. (2012); Paduart et al. (2006); Marconato et al. (2014). The PNLSS algorithm is introduced in detail in Paduart (2007); Paduart et al. (2010). Here the algorithm is reviewed in brief.

2.1 PNLSS model

For a single-input single-output system, the PNLSS model is given by the equations

$$\begin{aligned} x(t+1) &= \quad Ax(t)+bu(t) \quad + \quad E\zeta(x(t),u(t)), \\ y(t) &= \underbrace{c^T x(t)+du(t)}_{\text{linear state-space model}} + \underbrace{f^T\eta(x(t),u(t))}_{\text{polynomials in } x \text{ and } u}, \end{aligned} \tag{1}$$

where $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and $x(t) \in \mathbb{R}^n$ denote the input, output and state vector at time instance t, respectively. The linear part of the state-space model is defined by $A \in \mathbb{R}^{n\times n}$ (the state transition matrix), $b \in \mathbb{R}^n$, $c \in \mathbb{R}^n$, and $d \in \mathbb{R}$. The matrix $E \in \mathbb{R}^{n\times n_\zeta}$ and vector $f \in \mathbb{R}^{n_\eta}$ represent the polynomial part of the model and contain the coefficients of state-input monomials in the state and output equation, respectively. The number of monomials in ζ and η are denoted by n_ζ and n_η, respectively. The nonlinear part is represented by polynomials containing all possible monomials of degree d or less, not including constant terms nor linear terms, as they are already captured by the linear state-space part. For example, for a second-order system with one input the monomials of degree two are

$$\zeta(x,u) = \eta(x,u) = \begin{bmatrix} x_1^2 & x_1x_2 & x_1u & x_2^2 & x_2u & u^2 \end{bmatrix}, \tag{2}$$

and the elements of E and f are the corresponding coefficients. It can be verified (Paduart et al., 2010) that the total number of coefficients of nonlinear terms is given by

$$\left(\frac{(n+1+d)!}{d!(n+1)!} - n - 2\right)(n+1), \tag{3}$$

where it is assumed that ζ and η contain the same monomials, and both constants and linear terms are discarded.

2.2 Identification

The identification of the polynomial nonlinear state-space model is as follows (Paduart et al., 2010):

1. Find the best linear approximation (BLA) (Pintelon and Schoukens, 2012; Schoukens et al., 2012) of the system under test as the transfer function $\hat{G}_{BLA}$ and its total covariance $\hat{\sigma}_{G_{BLA}}$.
2. Based on this BLA, calculate the initial estimates of A, b, c, and d by using a linear subspace system identification procedure (Pintelon, 2002).
3. The parameters A, b, c, d, E, and f are tuned by running a Levenberg-Marquardt optimization (Paduart et al., 2010).

3. DECOUPLING POLYNOMIAL FUNCTIONS

Polynomial nonlinear state-space models are flexible and can capture severe nonlinear effects. However, the number of parameters to be estimated is often very high as it grows combinatorially with the polynomial degree and the number of variables (see (3)). Therefore, attributing physical intuition to PNLSS models is not straightforward.

3.1 A decoupled PNLSS model

A simpler PNLSS model may be obtained by decoupling the polynomial functions, that is, by linearly transforming states and input in such a way that the multivariate polynomials in the PNLSS model are decoupled into univariate polynomials. The decoupling method of Dreesen et al. (2015a) computes such a decoupled representation by means of the canonical polyadic decomposition (CPD) of a 3-dimensional matrix or tensor (see Carrol and Chang (1970); Harshman (1970); Kolda and Bader (2009)).

The polynomial decoupling method results in a new polynomial nonlinear state-space as

$$\begin{aligned} x(t+1) &= Ax(t)+bu(t)+W_x g\left(V^T\begin{bmatrix} x(t)\\ u(t)\end{bmatrix}\right), \\ y(t) &= c^T x(t)+du(t)+w_y^T g\left(V^T\begin{bmatrix} x(t)\\ u(t)\end{bmatrix}\right), \end{aligned} \tag{4}$$

where W_x and w_y^T are the linear transformation matrices (more precisely, matrix and vector) for transforming the nonlinear polynomial functions in the PNLSS equation and V is the transformation matrix for transforming states and inputs. For self-containment, the following paragraphs review the algorithm of Dreesen et al. (2015a) using the notation of the PNLSS model.

Notice that the procedure of Section 2.2 results in an $n+1$ to n multivariate polynomial vector function (represented by E) as well as an $n+1$ to 1 multivariate polynomial vector function (represented by f). From here on, we treat them jointly as a single $n+1$ to $n+1$ multivariate polynomial vector function as

$$p(x(t),u(t)) = \begin{bmatrix} E\zeta(x(t),u(t)) \\ f^T\eta(x(t),u(t)) \end{bmatrix}, \tag{5}$$

where p is a function that maps $n+1$ variables to $n+1$ outputs. Furthermore, let the vector s be defined as

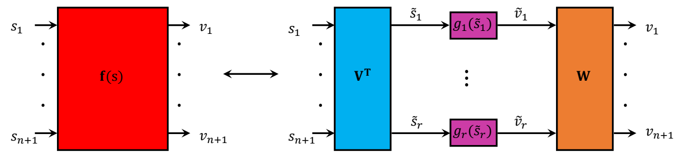


Fig. 1. The outcome of the decoupling procedure. A multivariate polynomial vector function is decomposed into a linear transformation V, followed by a set of parallel univariate polynomials $g_1, \ldots, g_r$, and another linear transformation W.

$$s = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \tag{6}$$

where $s \in \mathbb{R}^{(n+1)\times 1}$. We denote by $p(s)$ the original multivariate polynomial function, which is decoupled into r univariate polynomials as

$$p(s) = Wg(V^T s), \tag{7}$$

where

$$g\left(V^T s\right) = \begin{bmatrix} g_1(\tilde{s}_1) \\ g_2(\tilde{s}_2) \\ \vdots \\ g_r(\tilde{s}_r), \end{bmatrix}, \tag{8}$$

with the univariate functions $g_1(\tilde{s}_1), \ldots, g_r(\tilde{s}_r)$ operate on the transformed variables $\tilde{s}_i = v_i^T s$ and the functions g_i have a fixed user-chosen degree d. Fig. 1 shows the result of the decoupling algorithm. Corresponding to the definition of s, the matrix W is defined as

$$W = \begin{bmatrix} W_x \\ w_y^T \end{bmatrix}, \tag{9}$$

such that

$$\begin{bmatrix} E\zeta(x(t), u(t)) \\ f^T\eta(x(t), u(t)) \end{bmatrix} = \begin{bmatrix} W_x \\ w_y^T \end{bmatrix} g\left(V^T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}\right). \tag{10}$$

The decoupling method of Dreesen et al. (2015b) relies on the fact that the Jacobian of p is given by the expression:

$$J(s) = W \operatorname{diag}(g_1(v_i^T s), \ldots, g_r(v_r^T s))V^T, \tag{11}$$

where v_i denotes the ith column of V. By considering (11) in a set of sampling points $s^{(k)}$, the question of finding the transformation matrices V and W amounts to solving a simultaneous matrix diagonalization problem, which is essentially what the CPD computes.

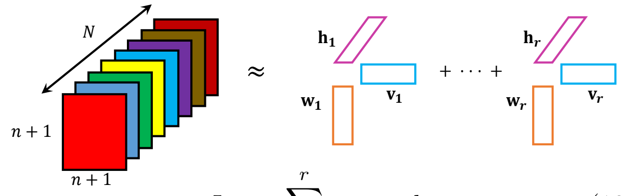
3.2 Summary of decoupling algorithm

The decoupling algorithm is adopted from Dreesen et al. (2015b) and is briefly summarized here.

1. The Jacobian of the polynomial vector function p is evaluated in a set of N sampling points $s^{(k)}$, for $k = 1, 2, \ldots N$, which are drawn from a random normal distribution.
2. The Jacobian matrices are stacked into an $(n+1) \times (n+1) \times N$ tensor (see Fig. 2). We have thus

$$J_{ijk} = \frac{\partial p_i(s_j^{(k)})}{\partial s_j}. \tag{12}$$

3. This tensor is decoupled using the CPD as follows (also see Fig. 2)



$$J_{ijk} \approx \sum_{\ell=1}^{r} w_{i\ell}\, v_{j\ell}\, h_{k\ell}, \tag{13}$$

where W and V follow immediately from the CPD and

$$h_{k\ell} = g_\ell{}'(\tilde{s}_\ell^{(k)}), \tag{14}$$

is the derivative of the univariate function g_ℓ evaluated in sampling points $\tilde{s}_\ell^{(k)}$, for $k = 1, \ldots, N$.
4. For the ℓ^{th} branch $g'_\ell(\tilde{s}_\ell)$, we solve the following polynomial fitting

$$\begin{bmatrix} (\tilde{s}_i^{(1)})^1 & (\tilde{s}_i^{(1)})^2 & \cdots & (\tilde{s}_i^{(1)})^{d-1} \\ (\tilde{s}_i^{(2)})^1 & (\tilde{s}_i^{(2)})^2 & \cdots & (\tilde{s}_i^{(2)})^{d-1} \\ \vdots & & \vdots & \\ (\tilde{s}_i^{(N)})^1 & (\tilde{s}_i^{(N)})^2 & \cdots & (\tilde{s}_i^{(N)})^{d-1} \end{bmatrix} \begin{bmatrix} c'_{i,2} \\ \vdots \\ c'_{i,d-1} \end{bmatrix} = \begin{bmatrix} h_{1i} \\ h_{2i} \\ \vdots \\ h_{Ni} \end{bmatrix}, \tag{15}$$

leading to the coefficients of g'_ℓ. Notice that, again, the constant and linear terms are not considered.
5. Solving the symbolic integration

$$g_\ell(\tilde{s}_\ell) = \int g'_\ell(\tilde{s}_\ell) d\tilde{s}_\ell, \tag{16}$$

determines the functions g_ℓ up to the correct value of the integration constants.

The result of this algorithm is a decoupled nonlinear state space model with $(2n+d+1)r$ nonlinear parameters. Notice that the number of parameters now increases linearly with the degree as opposed to the combinatorial increase for the multivariate polynomials in (3).

This decoupled model can be used as an initialization for further tuning V, W and the coefficients of the g_ℓ in order to minimize the output error. Furthermore, the degree d of the nonlinearity can now be increased without introducing a prohibitively large number of parameters.

It should be noted that the decoupling algorithm does not guarantee that the decoupled PNLSS model is stable. If the decoupled model would be unstable for the estimation data set, the optimization of the model parameters cannot be done as the Jacobians needed for the Levenberg-Marquardt optimization routine are obtained by simulating nonlinear state-space models with the same dynamics as the PNLSS model (Paduart et al., 2010). A practical workaround for this problem is to either re-initialize the decoupling algorithm by regenerating a collection of N sampling points, or to initialize the elements of W_x or the coefficients of the polynomials g_ℓ as zeros. In the latter case, the dynamics in the state equation are linear and can be guaranteed to be stable in bounded-input, bounded-output sense.

4. RESULTS

This section illustrates the decoupling approach presented in the previous section on a Bouc-Wen hysteretic system.

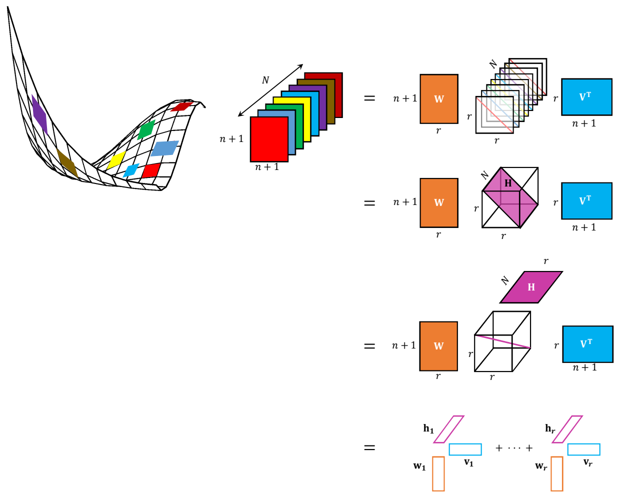


Fig. 2. Decoupling multivariate polynomials is done by collecting the Jacobian matrices in a set of sampling points and stacking them into a 3D tensor. The simultaneous matrix diagonalization of the Jacobian matrices is equivalent to the CPD of this 3D tensor, and returns the transformation matrices V, W and information about the internal functions g_ℓ.

In Noël et al. (2017) the system is introduced and the reason why the PNLSS model is needed is introduced in detail. The system is introduced briefly here and the results of the PNLSS model on the Bouc-Wen hysteretic benchmark (Noël and Schoukens, 2016) with and without applying the decoupling approach are compared.

4.1 The system under test

The hysteretic system under study is simulated through Bouc-Wen model which is described by the following differential equations

$$m_L\ddot{y} + c_L\dot{y} + k_L y + z(y,\dot{y}) = u(t),$$
$$\dot{z}(y,\dot{y}) = \alpha\dot{y} - \beta(\gamma\,|\dot{y}|\,|z|^{\nu-1}\,z + \delta\dot{y}\,|z|^{\nu}), \qquad (17)$$

where c_L, and k_L are the parameters of the linear restoring force contribution, and α, β, γ, δ, and ν are the parameters of the Bouc-Wen hysteresis loop (Bouc, 1967; Wen, 1976), and (Roberts and Spanos, 1990).

4.2 Comparison

The benchmark provides two test data sets, one with a random phase multisine input and the other one with a swept sine input. The Bouc-Wen model is excited with random phase multisine input force ($u(t)$) with standard deviation levels of 50 N and 55 N. The latter is used for generating the Full PNLSS model to avoid extrapolation problem which results usually to an unstable model (either the Full PNLSS or the decoupled model). The final goal is to achieve a good model performance for a 50 N excitation.

The excitation signal has twenty phase realizations and seven periods and flat power spectrum in the frequency band [5 150] Hz. The first two periods are discarded for transient removal, and just the last five periods are taken into account for estimation. The BLA analysis of the system proposes to have third order dynamics (Noël et al., 2017).

For the Full PNLSS model the nonlinearities are of second and third degree. The nonlinearity in the output equation is zero ($F = 0$). The decoupled model has three branches ($r = 3$) and the degree of the polynomials is increased to eleven ($d = 11$) in each branch. The extracted models (Full PNLSS and decoupled PNLSS) are tested on the test data set from the benchmark data (Noël and Schoukens, 2016).

The estimated models are compared based on their RMS error on the test data:

$$e_{RMSt} = \sqrt{\frac{1}{N_t}\sum_{t=1}^{N_t}(y_{mod}(t) - y_t(t))^2}, \qquad (18)$$

(Noël and Schoukens, 2016), where $y_{mod}(t)$ is the output of the model, $y_t(t)$ is the output of the test data and N_t is the number of samples in the last period of the test data ($N_t = 8192$ for the multisine and $N_t = 15300$ for the swept sine). Both models are tested on the test data and the RMS errors (e_{RMSt}) are 1.8703×10^{-5} (multisine) and 1.2024×10^{-5} (swept sine) for the Full PNLSS model and 1.8996×10^{-5} (multisine) and 1.4019×10^{-5} (swept sine) for the decoupled model. The decoupled model has a slightly higher RMS error than that of the PNLSS model on the test data, but the decoupled model only has 67 parameters, while the Full PNLSS model has 106.

The results on the test data are plotted in Fig. 3 and Fig. 4. Fig. 3 shows the output spectrum (blue), error of linear model (cyan), the error of the PNLSS model (green) and the error of the decoupled model (red) in the frequency domain. The decoupled model shows a slightly higher error than Full PNLSS near the resonance frequency, while in other frequencies it shows more or less the same error as Full PNLSS. A comparison of RMS error of the multisine excitation for both models also shows the decoupled model error is slightly worse, but negligible (1.57 %).

In Fig. 4 both models are tested on the swept sine data. The swept sine frequency band is [20 50] Hz, with sweep rate of 10 Hz/min. The figure shows errors of linear (blue), PNLSS (green) and decoupled (red) models in the time domain. The decoupled model error time series is almost the same as Full PNLSS error time series, even in the transient region.

5. CONCLUSION

The Bouc-Wen system's behavior is captured by the Full PNLSS model. A decoupled PNLSS model reaches a similar accuracy (1.57 % higher than the RMS error of PNLSS model) for the random phase multisine excitation with lower amplitude data, while the number of parameters in the decoupled model is less than two third of the number of parameters in the Full PNLSS model (67 instead of 106). The order of the polynomials in the decoupled model can also be increased without having the problem of blowing up the number of parameters, as it is the case for the Full PNLSS model. The connection between the hysteresis loop's characteristics and the properties of univariate polynomial (g_i) functions can be a part of future study.

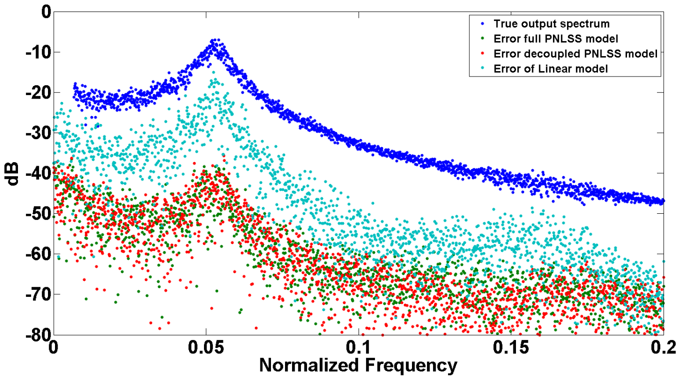
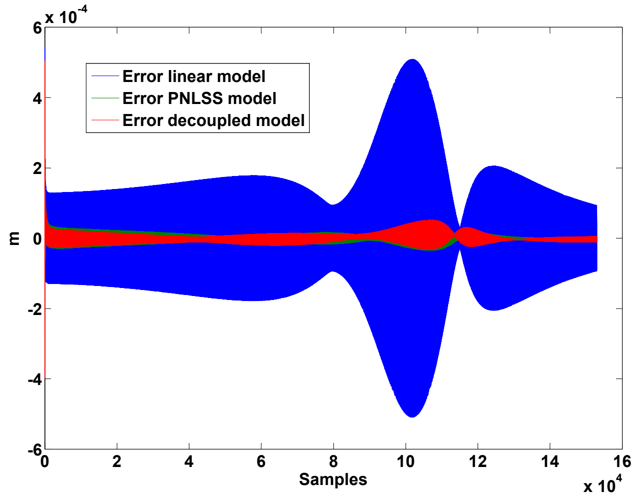


Fig. 3. The output spectrum (blue), the error of linear model (cyan) and the error of PNLSS (green) for second and third degree monomials of states and inputs in state updates ($F = 0$), and the error for the decoupled model with three branches with eleventh degree polynomials (red).



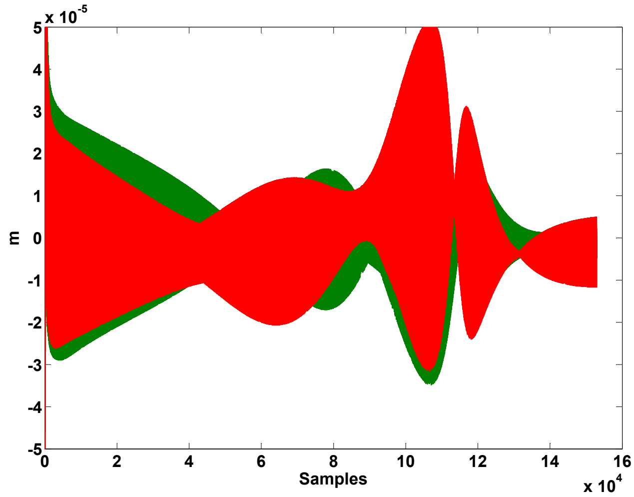


Fig. 4. Top: The error in the time domain for the linear model (blue), for the PNLSS (green) and the error of decoupled PNLSS (red) for second and third degree monomials of states and inputs in state updates ($F = 0$), and the error for the decoupled model with three branches with eleventh degree polynomials (red). Bottom: Zoom of the top figure to differentiate the error time series of decoupled and Full PNLSS models.

REFERENCES

Bernstein, D.S. (2007). Ivory ghost [Ask The Experts]. *IEEE Control Systems Magazine*, 27, No. 5, 16–17.

Bouc, R. (1967). Forced vibrations of a mechanical system with hysteresis. In *Proceedings of the 4th Conference on Nonlinear Oscillations, Prague, Czechoslovakia.*

Carrol, J.D. and Chang, J.J. (1970). Analysis of individual differences in multidimensional scaling via an n-way generalization of "Eckart-Young" decomposition. *Psychrometrika*, 35, 283–319.

Dreesen, P., Ishteva, M., and Schoukens, J. (2015a). Decoupling multivariate polynomials using first-order information and tensor decompositions. *SIAM Journal on Matrix Analysis and Applications*, 36, Issue 2, 864–879.

Dreesen, P., Schoukens, M., Tiels, K., and Schoukens, J. (2015b). Decoupling static nonlinearities in a parallel Wiener-Hammerstein system: A first-order approach. In *IEEE International Instrumentation and Measurement Technology Conference (I^2MTC) Proceedings, Pisa, Italy*, 987–992.

Harshman, R.A. (1970). Foundations of the PARAFAC procedure: Models and conditions for an "explanatory" multi-modal factor analysis. *UCLA Working Papers in Phonetics*, 16, 1–84.

Kolda, T.G. and Bader, B.W. (2009). Tensor decompositions and applications. *SIAM Review*, 51, No. 3, 455–500.

Marconato, A., Sjöberg, J., Suykens, J.A.K., and Schoukens, J. (2014). Improved initialization for nonlinear state-space modeling. *IEEE Transactions on Instrumentation and Measurement*, 63, 972–980.

Noël, J.P., Fakhrizadeh Esfahani, A., Kerschen, G., and Schoukens, J. (2017). A nonlinear state-space approach to hysteresis identification. *Mechanical Systems and Signal Processing*, 84, Part B, 171–184.

Noël, J.P. and Kerschen, G. (2013). Frequency-domain subspace identification for nonlinear mechanical systems. *Mechanical Systems and Signal Processing*, 40, 2–35.

Noël, J.P. and Schoukens, M. (2016). Hysteretic benchmark with a dynamic nonlinearity. In *Workshop on nonlinear system identification benchmarks, pp. 7-14, Brussels, Belgium.*

Oh, J., Drincic, B., and Bernstein, D.S. (2009). Nonlinear feedback models of hysteresis. *IEEE Control Systems Magazine*, 100–119.

Paduart, J. (2007). *Identification of Nonlinear Systems using Polynomial Nonlinear State Space Models*. Ph.D. thesis, Vrije Universiteit Brussel (VUB).

Paduart, J., Lauwers, L., Swevers, J., Smolders, K., Schoukens, J., and Pintelon, R. (2010). Identification of nonlinear systems using polynomial nonlinear state space models. *Automatica*, 46, 647–656.

Paduart, J., Schoukens, J., Pintelon, R., and Coen, T. (2006). Nonlinear state space modelling of multivariable systems. In *14th IFAC Symposium on System Identification, Newcastle, Australia.*

Pintelon, R. (2002). Frequency-domain subspace system identification using non-parametric noise models. *Automatica*, 38, 1295–1311.

Pintelon, R. and Schoukens, J. (2012). *System Identification: a Frequency Domain Approach*. John Wiley & Sons, Inc.

Roberts, J.B. and Spanos, P.D. (1990). *Random vibration and statistical linearization*. John Wiley & Sons Ltd.

Schoukens, J., Pintelon, R., and Rolain, Y. (2012). *Mastering System Identification in 100 Exercises*. John Wiley & Sons, Inc.

Van Mulders, A. and Vanbeylen, L. (2013). Comparison of some initialisation methods for the identification of nonlinear state-space models. In *IEEE International Instrumentation and Measurement Technology Conference (I^2MTC), Minneapolis, MN*, 807–811.

Van Mulders, A., Vanbeylen, L., and Schoukens, J. (2012). Robust optimization method for the identification of nonlinear state-space models. In *IEEE International Instrumentation and Measurement Technology Conference (I^2MTC), Graz, Austria.*, 1423–1428.

Wen, Y.K. (1976). Method for random vibration of hysteretic systems. *Journal of the engineering mechanics division*, 102.2, 249–263.