International Journal of Solids and Structures 94–95 (2016) 222–237

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

# Homogenization of a cracked saturated porous medium: Theoretical aspects and numerical implementation

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ARTICLE INFO

*Article history:*
Received 22 May 2015
Revised 11 March 2016
Available online 10 May 2016

*Keywords:*
Asymptotic homogenization
Fractured saturated porous medium
Poroelasticity

ABSTRACT

The purpose of this paper is to determine, via a homogenization technique and in the framework of small strains, the macroscopic poroelastic properties of a saturated, deformable, cracked porous medium. The poroelastic matrix is assumed to be homogeneous and the cracks to be connected discontinuities, infilled with a poroelastic material. They are periodically distributed, with the size of the period being small compared to the size of the sample. The considered up-scaling method (based on asymptotic expansions) will provide two uncoupled mechanical and hydraulic problems describing the overall behavior of the material. The degradation of the mechanical properties due to damage is then introduced. Damage depends on cracks' opening, thus making the problem non-linear. A numerical solution of the problem is provided using finite elements. Any stress-strain loading path can be reproduced. The numerical solution of an oedometric test and a biaxial test allows the exploration of the non-linear anisotropic behavior along with the bifurcation phenomenon.



## 1. Introduction

During the past decades, big efforts have been made to better comprehend heterogeneous materials (Auriault, 1991; Biot, 1941; Chambon et al., 2004; Kanouté et al., 2009; Sánchez-Palencia, 1980). Numerical modeling can be successfully applied to continuum problems but difficulties arise when trying to model a heterogeneous microstructure (Kouznetsova et al., 2001). The difference between the scale of the micro- and macro-structures makes it difficult to determine an appropriate mesh size, leading to a computationally expensive problem if one focuses on the micro-scale, or to an approximate description of the microstructural behavior if one focuses on the macroscale problem (Kouznetsova et al., 2001).

Furthermore, macro-scale constitutive laws, calibrated with experimental results, are often adopted. This approach is however less effective when dealing with complex behaviors (Caillerie, 2009). An alternative is provided by homogenization techniques that allow the inclusion of the micro-scale description within the macroscopic problem. In this latter framework, analytic, e.g. mixture theory (Gray and Hassanizadeh, 1979) or semi-analytic, e.g. Eshelby (1957) procedures have been developed. However, these theories cannot describe the micro–macro-behavior for non-linear constitutive laws or non-regular micro-structure configurations in an accurate manner (see, e.g. Kanouté et al., 2009). Numerical homogenization approaches such as direct micro–macro-techniques (Miehe et al., 1999; Nguyen, 2013; Nitka et al., 2011; Smit et al., 1998) overcome these limitations. These techniques use numerical calculations at the (usually periodic) micro-scale level to provide a constitutive law at the macroscale. Although this approach allows more general multi-scale problems to be taken into account, it is highly computationally expensive.

The asymptotic homogenization theory documented in Arbogast et al. (1990), Bensoussan et al. (2011), Papanicolau et al. (1978), Sánchez-Palencia (1980) permits equivalent properties to be obtained and allows an analytic and a numerical approach to be combined. Based on asymptotic expansions (applied to a parameter $e$ that relates the characteristic lengths of the two, well-separated, scales), the homogenized problem can be solved on a generic micro-structural cell (solved using, e.g. finite elements Auriault, 2011) so that the homogenized macroscopic properties are finally obtained.

The proposed approach is developed herein with the purpose of determining the overall poroelastic properties of a saturated cracked deformable porous medium in the framework of small strains. We consider the deformation and the porous flow of the medium to be governed by Biot's equations of poroelasticity. The cracks are thin enough to be considered as curved lines (surfaces in 3D) and interconnected, forming a periodic-network. From a mechanical viewpoint, a crack is here considered (differently from other approaches, e.g. Pensée et al., 2002) as an infilled

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http://dx.doi.org/10.1016/j.ijsolstr.2016.03.017

discontinuity containing a soft poroelastic material that can undergo damage. This corresponds to a weakened elastic zone allowing its two lips to slip and to move apart. The relative motion of the lips induces a change of the porosity of the crack and consequently a change in the fluid flow. Crack propagation is not treated and the opening of cracks is considered to damage the material, thus affecting the transport properties of the medium. This latter point is consistent with the proposed upscaling procedure (based on asymptotic homogenization) that naturally leads to two uncoupled hydraulic and mechanical problems. It is worth noting that the methodology is not adapted to situations where crack propagation matters. In the case of a hydro-mechanical problem, crack propagation induces a sudden change of the stress/strain field that also affects the pressure field (see Pizzocolo et al., 2013; Schrefler et al., 2015). The proposed model is therefore aimed to treat stationary hydraulic cases rather than transitory states.

In other terms, the upscaling method is aimed at obtaining a material constitutive law for an REV of a porous medium characterized by infilled discontinuities. The numerical behavior law can be finally embedded in any multiscale approach (e.g. Kouznetsova et al., 2001) so that real-scale geomechanics or engineering problems can be treated.

The first part of the paper presents the equations governing the coupled hydro-mechanical problem in the porous matrix and the cracks. The asymptotic homogenization is detailed and the final equations describing the macro-scale problem are presented. In the second part of the paper, the homogenized problem is numerically solved first for the linear case. As a further step, damage is introduced, which makes the problem non-linear. The proposed numerical implementation allows to reproduce any stress/strain loading history: two cases are considered, the first using a strain controlled path (i.e. oedometric test) and the second using a mixed stress/strain condition (i.e. biaxial test). A constitutive non-linear material law can then be obtained for any loading history.

*Notations.*

- The "usual" vectors: positions, normal, tangent, forces, flows, ... are denoted: $\vec{x}$, $\vec{y}$, $\vec{n}$, $\vec{\tau}$, $\vec{u}$, $\vec{v}$, $\vec{T}$, $\vec{q}$, ... $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is an orthonormal basis.
- The dot symbol $\cdot$ denotes the simple contraction between two tensors of any order: $\vec{T} \cdot \vec{v}$, $\vec{T} = \sigma \cdot \vec{n}$, ...
- The colon symbol : denotes the double contraction of two second order tensors: $\sigma : \nabla\vec{v}$, $c : \epsilon(\vec{u})$, ...
- The tensor product $\vec{a} \otimes \vec{b}$ denotes the linear application defined by: $\forall \vec{c}$, $\left(\vec{a} \otimes \vec{b}\right) \cdot \vec{c} = \left(\vec{b} \cdot \vec{c}\right)\vec{a}$.
- $\overrightarrow{\text{grad}} f$ denotes the gradient of the scalar function $f$, $\nabla\vec{u}$ is the gradient of the vector field $\vec{u}$ and $\epsilon(\vec{u})$ denotes the strain tensor associated to the displacement field $\vec{u}$, i.e. the symmetrical part $\nabla\vec{u}^S$ of $\nabla\vec{u}$. The gradients of a field of two space variables $\vec{x}$ and $\vec{y}$ are distinguished by an exponent: $\overrightarrow{\text{grad}}^x f$, $\nabla^y \vec{v}$.
- Whenever the index notation of tensors is used, the Einstein notation for the contraction of tensors is adopted.

## 2. Description of a saturated cracked deformable porous medium

### 2.1. Description of the medium and strong form of the equations

Let us consider a cracked deformable and saturated porous medium occupying, in the small strain framework, a domain $\Omega$. For the sake of simplicity the study is carried out in two dimensions; an extension to 3D is straightforward but is not presented in the following for sake of clarity of the notations. However, some hints about the 3D modeling are given.

The porous parts of the medium are separated by cracks which are curves that joint at points (see Fig. 1); $\Gamma$ denotes the set of all cracks of the medium. To make the writing of the equations of the poroelasticity of the cracks precise, the cracks are (arbitrarily) oriented, let $s$ denote the curvilinear abscissa along a crack and $\vec{\tau}$ its unit vector, assuming the crack is smooth. The unit normal $\vec{n}$ to a crack is the vector obtained by the rotation of angle $+\frac{\pi}{2}$ of the tangent vector $\vec{\tau}$.

The considered porous medium is assumed to be finely periodic. That means, on one hand, that the space distribution of cracks is periodic (see Fig. 1) and, on the other hand, that the size of the period is small with respect to that of the medium. In the asymptotic expansion method of homogenization used in this paper, the ratio of the size of the period to that of the medium is a small parameter intended to go to 0. That means that the periodic cells of the medium are increasingly smaller. The usual way to handle this is to define the cells of the medium as the image of a given cell $Y$ by a homothety of ratio $e$, $e$ being the small parameter of the asymptotic procedure (see Bensoussan et al., 2011 and Sánchez-Palencia, 1980).

A function defined on $Y$ is said to be $Y$-periodic if it takes equal values on opposite sides of the cell $Y$.

*Biot's equations of the porous parts.* In the porous parts of the medium $\Omega$, the deformation of the medium and the flow of fluid are governed by Biot's equations that read, see (see Biot, 1941; 1955; Auriault, 2005 or Coussy, 2004):

$$\text{div}\ \sigma = 0 \tag{1a}$$

$$\sigma = c : \epsilon(\vec{u}) - p\alpha \tag{1b}$$

$$\kappa = \alpha : \epsilon(\vec{u}) + \beta p \tag{1c}$$

$$\text{div}\ \vec{q} + \dot{\kappa} = 0 \tag{1d}$$

$$\vec{q} = -k\overrightarrow{\text{grad}}\, p \tag{1e}$$

where $\eta$ denotes the porosity of the porous matrix. $\vec{u}$ is the displacement field and $\dot{\vec{u}}$ its time derivative, $\sigma$ is the total Cauchy stress tensor and $p$ is the pore pressure. $\vec{q} = \eta\left(\vec{v} - \dot{\vec{u}}\right)$ is the relative fluid flow, $\vec{v}$ being the velocity of the fluid. $c$ is the fourth order tensor of elastic stiffness, $\alpha$ is the second order tensor of Biot coefficients, $\beta$ is the Biot modulus and $k$ is the permeability of the medium. $\kappa$ denotes the variation - due to the displacement $\vec{u}$ - of the porosity, see Coussy (2004), which reads in terms of the porosity of the porous matrix $\eta$ and of its variation $\delta\eta$ due to the deformation of the medium:

$$\kappa = \delta\eta + \eta\ \text{div}\,\vec{u}$$

*Equations on the cracks.* The cracks separating the porous parts of the medium are very soft and highly permeable. That means that the lips of the cracks can slide and open and, in order to maintain coherence, that the stress vector $\vec{T} = \sigma \cdot \vec{n}$ is continuous on the cracks. The displacement field $\vec{u}$ is then discontinuous on the cracks and its jump $\vec{u}^+ - \vec{u}^-$ through a crack where $\vec{u}^+$ is the value of $\vec{u}$ on the side toward which $\vec{n}$ points and $\vec{u}^-$ is the value of $\vec{u}$ on the opposite side, is denoted by $[[\vec{u}]]$. The assumption of high permeability means that fluid pressure $p$ is continuous on the cracks but the fluid flow is discontinuous, the jump of the normal flow is $[[\vec{q}]] \cdot \vec{n}$ where $[[\vec{q}]]$ denotes the jump of $\vec{q}$ across the cracks.

According to these assumptions (see Appendix A), the poroelastic behavior of the cracks is modeled by the following equations:

$$\vec{T} = C \cdot [[\vec{u}]] - p\vec{A} \tag{2a}$$

$$\kappa^c = \vec{A} \cdot [[\vec{u}]] + Bp \tag{2b}$$

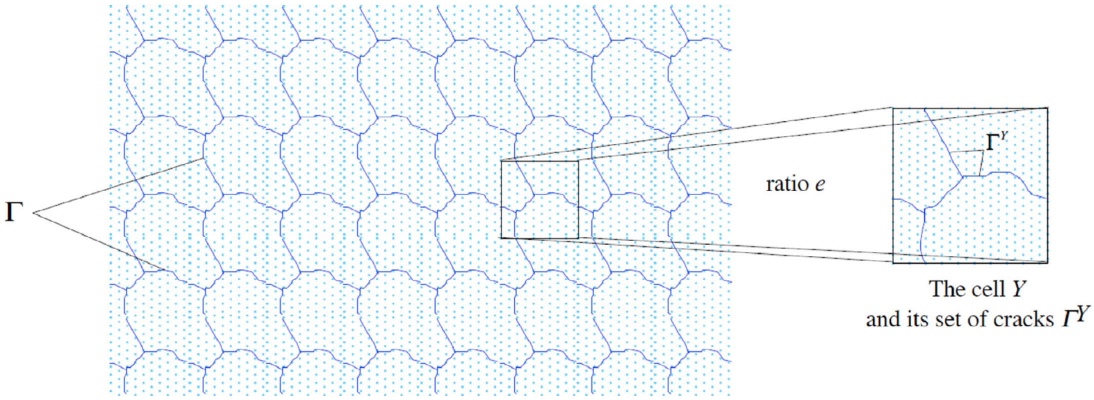



**Fig. 1.** Periodic cracked porous medium.

$$\frac{dQ}{dl} + [[\vec{q}]] \cdot \vec{n} + \dot{\kappa}^c = 0 \tag{2c}$$

$$Q = -K\frac{dp}{dl} \tag{2d}$$

where $l$ is the curvilinear abscissa along the cracks and $\vec{T} = \sigma.\vec{n}$ is the stress vector on the crack and $Q$ denotes the fluid flow of water along the crack.

The set of Eq. (2) presents a structure similar to that of Eq. (1). Eq. (2a) is the elastic constitutive equation analogous to (1b), $C$ and $\vec{A}$ being the elastic stiffness of the crack and its vector of Biot's coefficients. In (2b) $\kappa^c$ is the variation of the porosity of the crack due to its sliding/opening and $B$ is its Biot's modulus. $K$ is the permeability of the crack. More details are given in Appendix A. It can be noted that in the presented formulation the relative motion of the crack lips is not restricted to sliding, the cracks can open or close. In order to be able to exhibit and compute macroscopic equivalent coefficients, it has been chosen to remain in a linear framework and, consequently, to disregard the complete closing of the crack and the change of permeability that follows. It is possible to take those features into account by considering a stiffness $C$ and a permeability $K$ depending on the relative displacement $[[\vec{u}]]$. It follows that in such a case, the problems of Section 3.4 that yield to the macroscopic constitutive equation become non-linear and have to be solved for each value of the macroscopic strain. In a 3D modeling, the cracks are surfaces (non-plane in general). Eqs. (2a) and (2b) remain unchanged while Eqs. (2c) and (2d) are:

$$\mathrm{div}_S \vec{Q} + [[\vec{q}]] \cdot \vec{n} + \dot{\kappa}^c = 0$$
$$\vec{Q} = -K\overrightarrow{\mathrm{grad}}_S p$$

where the fluid flow on the cracks $\vec{Q}$ is a vector and $\mathrm{div}_S$ and $\overrightarrow{\mathrm{grad}}_S$ denote the surface divergence and gradient.

The possible adsorption of fluid on the solid of the porous medium is not taken into account in this study, the fluid is free and consequently the fluid pressure $p$ is continuous through the cracks. It is then assumed to be differentiable everywhere so the derivative $\frac{dp}{dl}$ can be written as $\frac{dp}{dl} = \overrightarrow{\mathrm{grad}}p \cdot \vec{\tau}$ and Eq. (2d) reads:

$$Q = -K\,\overrightarrow{\mathrm{grad}}p \cdot \vec{\tau} \tag{3}$$

The sets (1) and (2) of equations need to be completed by boundary conditions and fluid mass balance equations at the crack junction points. As the purpose of this paper is the bulk homogenization of the cracked poroelastic medium, the boundary conditions on $\partial\Omega$ are not relevant and are therefore not defined. Concerning the fluid mass balance equation, it is assumed that there is no point fluid source or well at the junction points of cracks, therefore the balance of fluid mass at junction points merely comes down to the (algebraic) sum of the flows coming from the cracks to the junction points, thus tending to zero.

### 2.2. Weak formulations

As stressed in the previous section, the topic of this paper is the bulk homogenization of the cracked poroelastic medium and it is not necessary to precisely define the boundary conditions on $\partial\Omega$ (the boundary of $\Omega$). So, the test fields considered in this section are taken as identically zero on $\partial\Omega$.

#### 2.2.1. Mechanics

The weak formulation (virtual power formulation) of the mechanical equilibrium is obtained in the usual way: first by the scalar multiplication of the balance Eq. (1a) by a test field $\vec{w}$ (a virtual velocity field), second by integrating the product over an intact part of the porous medium $\Omega$ bounded either by a part of the boundary $\partial\Omega$ or cracks and third by modifying the integral $\int \mathrm{div}\,\sigma.\vec{w}\,ds$ by an integration-by-parts and finally by summing all the obtained equations to get:

$$\forall\vec{w}, \vec{w} = 0 \text{ on } \partial\Omega\,,\; -\int_\Omega \sigma : \epsilon(\vec{w})\,\mathrm{d}s - \int_\Gamma \vec{T} \cdot [[\vec{w}]]\mathrm{d}l = 0 \tag{4}$$

The test field $\vec{w}$ is chosen equal to 0 on the boundary $\partial\Omega$ in order to disregard the boundary conditions that, as noted at the end of Section 2.1, are not relevant for the purpose of the study. It has to be stressed out that, contrary to what is usually done but consistently with the discontinuity of the displacement field $\vec{u}$ through the cracks, the integral $\int_\Gamma \vec{T} \cdot [[\vec{w}]]\mathrm{d}l$ over the cracks $\Gamma$ is not eliminated by assuming that the velocity field $\vec{w}$ is continuous through those cracks.

#### 2.2.2. Balance of fluid volume

The fluid volume balance is the same as the fluid mass balance since the fluid is considered as incompressible and it regards the balance of the fluid volume in the porous part (1d) and in the cracks (2c). The weak formulation of the balance of fluid volume in the porous part is obtained in the same way as the virtual power formulation of the equilibrium (4), i.e.:

$$\forall r, r = 0 \text{ on } \partial\Omega\,,\; -\int_\Omega \vec{q} \cdot \overrightarrow{\mathrm{grad}}r\,\mathrm{d}s + \int_\Omega \dot{\kappa}r\,\mathrm{d}s - \int_\Gamma [[\vec{q}]] \cdot \vec{n}r\,\mathrm{d}l = 0 \tag{5}$$

where $r$ represents the pressure test field. Unlike in (4) where $\vec{T} = \sigma \cdot \vec{n}$ is continuous and $\vec{w}$ discontinuous through the cracks, in (5) $\vec{q}$ is discontinuous and $r$ is continuous.

The same procedure along with the balance of flows at the junction points of the cracks yields to:

$$\forall r, r = 0 \text{ on } \partial\Omega, \; -\int_\Gamma Q\frac{dr}{dl}\mathrm{d}l + \int_\Gamma [[\vec{q}]].\vec{n}r\,\mathrm{d}l + \int_\Gamma \dot{\kappa}^c r\,\mathrm{d}l = 0 \quad (6)$$

By adding the two previous weak formulations, we obtain:

$$\forall r, r = 0 \text{ on } \partial\Omega, \; -\int_\Omega \vec{q}\,.\,\overrightarrow{\mathrm{grad}}r\,\mathrm{d}s - \int_\Gamma Q\frac{dr}{dl}\,\mathrm{d}l + \int_\Omega \dot{\kappa} r\,ds + \int_\Gamma \dot{\kappa}^c r\,\mathrm{d}l + \int_{\partial\Omega} Q^b r\,\mathrm{d}l + \sum_i Q^i r\left(\vec{x}^i\right) = 0 \quad (7)$$

which, writing $\frac{dr}{dl} = \overrightarrow{\mathrm{grad}}r\,.\,\vec{\tau}$ reads:

$$\forall r, r = 0 \text{ on } \partial\Omega, \; -\int_\Omega \vec{q}\,.\,\overrightarrow{\mathrm{grad}}r\,\mathrm{d}s - \int_\Gamma Q\vec{\tau}\,.\,\overrightarrow{\mathrm{grad}}r\,\mathrm{d}l + \int_\Omega \dot{\kappa} r\,\mathrm{d}s + \int_\Gamma \dot{\kappa}^c r\,\mathrm{d}l + \int_{\partial\Omega} Q^b r\,\mathrm{d}l + \sum_i Q^i r\left(\vec{x}^i\right) = 0 \quad (8)$$

## 3. Homogenization

As presented in Section 2, the considered method of up-scaling is based on asymptotic expansions, the small parameter $e$ of those expansions being the ratio of the homothety mapping the cell $Y$ onto the periods of the medium. This means that it is not only one medium that is considered but a sequence of media parametrized by $e$. Consequently, all the involved fields $\sigma$, $\vec{u}$, $\kappa$, $\vec{q}$, $p$, $T$, $Q$ and $\kappa^c$ depend on $e$. To underline this dependence, they are denoted by $\sigma^{(e)}$, $\vec{u}^{(e)}$, $\kappa^{(e)}$, $\vec{q}^{(e)}$, $p^{(e)}$, $T^{(e)}$, $Q^{(e)}$ and $\kappa^{c(e)}$.

The behavior of all the mechanical and porous characteristics of the porous matrix with respect to the small parameter $e$ have to be detailed before implementing the asymptotic process since the homogenized modeling depends on this behavior. The mechanical and porous characteristics of the porous matrix, that is to say $c$, $\alpha$, $\beta$ and $k$ are assumed to be locally periodic, i.e. their dependence of the space variable $\vec{x}$ takes the form $f\left(\vec{x}, \frac{\vec{x}}{e}\right)$ where the function $\vec{y} \in Y \longrightarrow f(\vec{x}, \vec{y})$ is $Y$-periodic. On top of the local periodicity of the coefficients of the cracks and according to a similar study presented in Caillerie (1983) for heat conduction, it is consistent to assume for them certain behaviors with respect to the small parameter $e$. As stressed out at the beginning of this section, the use of expansions implies that a sequence of media with thinner and thinner periods is considered. Consequently, these media present more and more cracks and if those cracks are not assumed to be stiffer and stiffer then the whole strain of the media will localize on the cracks. Conversely, if the permeability of the cracks is not assumed to go to zero with $e$ then the whole flow will go through the cracks. In a similar way of thinking, Biot's modulus of the cracks goes to zero like $e$ in order that the macrocopic variation of the porosity should not be concentrated on the cracks. In the asymptotic process, it is then assumed that the modeling (2) of the cracks takes the form:

$$\vec{T}^{(e)} = \frac{C}{e}\,.\,\left[\left[\vec{u}^{(e)}\right]\right] - p^{(e)}\vec{A} \quad (9a)$$

$$\kappa^{c(e)} = \vec{A}\,.\,\left[\left[\vec{u}^{(e)}\right]\right] + eBp^{(e)} \quad (9b)$$

$$\frac{dQ^{(e)}}{dl} + \left[\left[\vec{q}^{(e)}\right]\right].\vec{n} + \dot{\kappa}^{c(e)} = 0 \quad (9c)$$

$$Q^{(e)} = -eK\,\overrightarrow{\mathrm{grad}}p^{(e)}\,.\,\vec{\tau} \quad (9d)$$

### 3.1. Asymptotic expansions

Let $\Gamma^Y$ denote the set of all cracks of the cell $Y$.

We look for the solution $\vec{u}^{(e)}$, $p^{(e)}$ of the consolidation problem of the form of double scale asymptotic expansions:

$$\vec{u}^{(e)} = \vec{u}^{(0)}(\vec{x}) + e\vec{u}^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\vec{u}^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (10a)$$

$$p^{(e)} = p^{(0)}(\vec{x}) + ep^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2p^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (10b)$$

where $\vec{u}^{(k)}(\vec{x}, \vec{y})$ and $p^{(k)}(\vec{x}, \vec{y})$ $k = 1, \ldots$ are functions of the large scale variable $\vec{x} \in \Omega$ and of the small scale variable $\vec{y} \in Y$ and are $Y$-periodic with respect to $\vec{y}$. It can be noted that the first terms $\vec{u}^{(0)}$ and $p^{(0)}$ of those expansions are assumed not to depend on the variable $\vec{y}$. This is consistent with the general idea of the homogenization process which is to smooth out the fine-scale heterogeneities.

According to the expansions (10), those of the gradient of $\vec{u}^{(e)}$ of the strain $\epsilon\left(\vec{u}^{(e)}\right)$, of the gradient of the fluid pressure $p^{(e)}$ and of the jump $\left[\left[\vec{u}^{(e)}\right]\right]$ of $\vec{u}^{(e)}$ on cracks read:

$$\nabla\vec{u}^{(e)} = \nabla^x\vec{u}^{(0)} + \nabla^y\vec{u}^{(1)} + e\left(\nabla^x\vec{u}^{(1)} + \nabla^y\vec{u}^{(2)}\right) + \cdots \quad (11a)$$

$$\epsilon\left(\vec{u}^{(e)}\right) = \epsilon^x\left(\vec{u}^{(0)}\right) + \epsilon^y\left(\vec{u}^{(1)}\right) + e\left(\epsilon^x\left(\vec{u}^{(1)}\right) + \epsilon^y\left(\vec{u}^{(2)}\right)\right) + \cdots \quad (11b)$$

$$\overrightarrow{\mathrm{grad}}p^{(e)} = \overrightarrow{\mathrm{grad}}^x p^{(0)} + \overrightarrow{\mathrm{grad}}^y p^{(1)} + e\left(\overrightarrow{\mathrm{grad}}^x p^{(1)} + \overrightarrow{\mathrm{grad}}^y p^{(2)}\right) + \cdots \quad (11c)$$

$$\left[\left[\vec{u}^{(e)}\right]\right] = e\left[\left[\vec{u}^{(1)}\right]\right]\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\left[\left[\vec{u}^{(2)}\right]\right]\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (11d)$$

where Eqs. (11a), (11b) and (11c) hold true for $\vec{x}$ in the porous parts of $\Omega$ and Eq. (11d) for $\vec{x}$ on the cracks $\Gamma$.

As already remarked, $\vec{u}^{(0)}$ of the expansion (10a) is assumed to be a smooth macroscopic displacement field. At this scale the cracks are smoothed out, consequently the expansion of the jump $\left[\left[\vec{u}^{(e)}\right]\right]$ begins at order 1 and, moreover, in the jumps $\left[\left[\vec{u}^{(k)}\right]\right]$, $k = 1, \ldots$ only the small scale variable $\vec{y}$ is concerned.

The constitutive Eq. (1b), the equation governing the variation of porosity (1c), Darcy's law (1e) and Eq. (9) along the cracks entail that the expansions of the stress $\sigma^{(e)}$, the variation of porosity $\delta\eta^{(e)}$ and the fluid flow $\vec{q}^{(e)}$ have to be of the following forms:

$$\sigma^{(e)}(\vec{x}) = \sigma^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e\sigma^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\sigma^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (12a)$$

$$\kappa^{(e)}(\vec{x}) = \kappa^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e\kappa^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\kappa^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (12b)$$

$$\vec{q}^{(e)}(\vec{x}) = \vec{q}^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e\vec{q}^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\vec{q}^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (12c)$$

$$\vec{T}^{(e)}(\vec{x}) = \vec{T}^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e\vec{T}^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (12d)$$

$$\kappa^{c(e)}(\vec{x}) = e\kappa^{c(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2\kappa^{c(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (12e)$$

$$Q^{(e)}(\vec{x}) = eQ^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2Q^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \quad (12f)$$

Eqs. (12a) , (12b) and (12c) hold true for $\vec{x}$ in the porous parts of $\Omega$ and Eqs. (12d), (12e) and (12f) for $\vec{x}$ on the cracks $\Gamma$.

As a consequence of (12c), the jump of the fluid flow through the cracks expands into:

$$[[\vec{q}^{(e)}]](\vec{x}) = [[\vec{q}^{(0)}]]\left(\vec{x}, \frac{\vec{x}^c}{e}\right) + e[[\vec{q}^{(1)}]]\left(\vec{x}, \frac{\vec{x}}{e}\right) + e^2[[\vec{q}^{(2)}]]\left(\vec{x}, \frac{\vec{x}}{e}\right) + \cdots \tag{13}$$

The choices of the order of magnitude of the characteristics of the cracks with respect to $e$ in (9) are such that in the expansions (12), the leading term of the stress vector $\vec{T}^{(e)}$ is of the order of $e^0$ and not of $e$ and that those of $\kappa^{c(e)}$ and $Q^{(e)}$ are of the order of $e$ and not of $e^0$.

### 3.2. Balance equations

#### 3.2.1. Preliminary results

To obtain the macroscopic balance equations (of momentum and of fluid flow) we use a lemma established by Sánchez-Palencia (1980, (pp. 77–78)) that reads:

For a function $F(\vec{x}, \vec{y})$ of $\vec{x}$ and $\vec{y}$, $Y-$periodic in $\vec{y}$, we have:

$$\lim_{e\to 0}\int_\Omega F\left(\vec{x}, \frac{\vec{x}}{e}\right)ds_x = \int_\Omega \langle F\rangle(\vec{x})ds_x$$

where:

$$\langle F\rangle(\vec{x}) = \frac{1}{|Y|}\int_Y F(\vec{x}, \vec{y})ds_y$$

This result is used in the following but it has to be established for integrals of the form $\int_\Gamma F\left(\vec{x}, \frac{\vec{x}}{e}\right)dl_x$ defined on $\Gamma$.

The whole medium is assumed to be periodic and the idea presented in Sánchez-Palencia (1980) consists in splitting the integration domain, here the cracks $\Gamma$, into the union over the periods of the part $e\Gamma^Y$ of the cracks in each of the periods. $e\Gamma^Y$ is the homothetic of $\Gamma^Y$ in the ratio $e$ where $\Gamma^Y$ is the set of cracks of the reference cell $Y$ (see Fig. 1). So the integral $\int_\Gamma F\left(\vec{x}, \frac{\vec{x}}{e}\right)dl_x$ reads:

$$\int_\Gamma F\left(\vec{x}, \frac{\vec{x}}{e}\right)dl_x = \sum_{\text{periods}}\int_{e\Gamma^Y} F\left(\vec{x}, \frac{\vec{x}}{e}\right)dl_x$$

The change of variables $\vec{x} \leftrightarrow e\vec{y}$ in the integral $\int_{e\Gamma_Y} F\left(\vec{x}, \frac{\vec{x}}{e}\right)dl_x$ yields (note that $dl_x = edl_y$):

$$\int_\Gamma F\left(\vec{x}, \frac{\vec{x}}{e}\right)dl_x = \sum_{\text{periods}} e\int_{\Gamma^Y} F(\vec{x}, \vec{y})dl_y$$

so:

$$e\int_\Gamma F\left(\vec{x}, \frac{\vec{x}}{e}\right)dl_x = \sum_{\text{periods}} e^2|Y|\langle F\rangle(\vec{x})$$

where:

$$\langle F\rangle(\vec{x}) = \frac{1}{|Y|}\int_{\Gamma^Y} F(\vec{x}, \vec{y})dl_y$$

now the sum over the periods can be seen as Riemann's sum of the integral $\int_\Omega \langle F\rangle(\vec{x})ds_x$ hence when $e\searrow 0$ we can see that the Sanchez' lemma is valid also for integrals over $\Gamma$:

For a function $F(\vec{x}, \vec{y})$ of $\vec{x}$ and $\vec{y}$, $Y-$periodic in $\vec{y}$, we have:

$$\lim_{e\to 0}\int_\Gamma F\left(\vec{x}, \frac{\vec{x}}{e}\right)dl_x = \int_\Omega \langle F\rangle(\vec{x})ds_x$$

#### 3.2.2. Mechanics

The purpose of this section is to determine the balance equation of the homogenized medium satisfied by the average stress tensor:

$$\langle\sigma\rangle = \frac{1}{|Y|}\int_Y \sigma^{(0)}\,ds_y \tag{14}$$

For that, w consider in the formulation (4) a *macroscopic* virtual velocity field $\vec{w}$ which vanishes on $\partial\Omega$.*Macroscopic* means that this virtual field is smooth (differentiable) all over $\Omega$. Particularly, the field $\vec{w}$ is continuous in $\Omega$ and the jumps $[[\vec{w}]]$ through the cracks are zero. So, using the lemmas of Section 3.2.1, the limit $e\searrow 0$, yields:

$$\forall\vec{w}, \vec{w} = 0 \text{ on } \partial\Omega\,,\; -\int_\Omega \langle\sigma\rangle : \epsilon(\vec{w})\,ds_x = 0 \tag{15}$$

which proves that:

$$\text{div}\,\langle\sigma\rangle = 0 \text{ in } \Omega \tag{16}$$

#### 3.2.3. Fluid flow

Taking $r$ in (8) as a smooth macroscopic function which vanishes on $\partial\Omega$ and making $e\searrow 0$ with the use of lemmas 3.2.1, we get:

$$\forall r, r = 0 \text{ on } \partial\Omega\,,\; -\int_\Omega \frac{1}{|Y|}\left(\int_Y \vec{q}^{(0)}\,ds_y + \int_{\Gamma^Y} Q^{(1)}\vec{\tau}\,dl_y\right)\cdot\overrightarrow{grad}^x r\,ds_x + \int_\Omega \frac{1}{|Y|}\left(\int_Y \dot{\kappa}^{(0)}ds_y + \int_{\Gamma^Y}\left(\dot{\kappa}^{c(1)}\right)dl_y\right)r\,ds_x = 0$$

Defining the average flow vector $\langle\vec{q}\rangle$ and the variation of $\langle\kappa\rangle$ as:

$$\langle\vec{q}\rangle = \frac{1}{|Y|}\left(\int_Y \vec{q}^{(0)}\,ds_y + \int_{\Gamma^Y} Q^{(1)}\vec{\tau}\,dl_y\right) \tag{17}$$

$$\langle\kappa\rangle = \frac{1}{|Y|}\left(\int_Y \kappa^{(0)}\,ds_y + \int_{\Gamma^Y} \kappa^{c(1)}\,dl_y\right) \tag{18}$$

we get:

$$\forall r, r = 0 \text{ on } \partial\Omega\,,\; -\int_\Omega \frac{1}{|Y|}\langle\vec{q}\rangle\cdot\overrightarrow{grad}^x r\,ds_x + \int_\Omega \langle\kappa\rangle r\,ds_x = 0 \tag{19}$$

which entails that:

$$\text{div}^x\langle\vec{q}\rangle + \langle\dot{\kappa}\rangle = 0 \tag{20}$$

### 3.3. Expansions of the constitutive equations

The expansions of Eqs. (1b), (1c), (1e), (9a), (9b) and (9d) yield at the lowest order:

$$\sigma^{(0)} = c : \left(\epsilon^x\left(\vec{u}^{(0)}\right) + \epsilon^y\left(\vec{u}^{(1)}\right)\right) - p^{(0)}\alpha \tag{21a}$$

$$\kappa^{(0)} = \alpha : \left(\epsilon^x\left(\vec{u}^{(0)}\right) + \epsilon^y\left(\vec{u}^{(1)}\right)\right) + \beta p^{(0)} \tag{21b}$$

$$\vec{q}^{(0)} = -k\left(\overrightarrow{\text{grad}}^x p^{(0)} + \overrightarrow{\text{grad}}^y p^{(1)}\right) \tag{21c}$$

$$\vec{T}^{(0)} = C\cdot\left[\left[\vec{u}^{(1)}\right]\right] - p^{(0)}\vec{A} \tag{21d}$$

$$\kappa^{c(1)} = \vec{A}\cdot\left[\left[\vec{u}^{(1)}\right]\right] + Bp^{(0)} \tag{21e}$$

$$Q^{(1)} = -K\left(\overrightarrow{\text{grad}}^x p^{(0)} + \overrightarrow{\text{grad}}^y p^{(1)}\right)\cdot\vec{\tau} \tag{21f}$$

The previous equations represent the expansions of the constitutive equations but they are not the constitutive equations of the homogenized medium. Indeed, they do not involve only the macroscopic strain $\epsilon^x\left(\vec{u}^{(0)}\right)$ and pressure $p^{(0)}$ but also the fields $\vec{u}^{(1)}$ and $p^{(1)}$. Those two fields have to be determined in terms of $\epsilon^x\left(\vec{u}^{(0)}\right)$ and $p^{(0)}$ by solving the so called *self-balanced problems*, presented in the following section.

### 3.4. Self-balanced problems on the cell $Y$

The macroscopic balance Eqs. (16) and (20) or alternatively (15) and (19) are not sufficient to define the macroscopic homogenized modeling of the cracked porous medium, as the macroscopic constitutive equations are needed. Those macroscopic constitutive equations come from the averaging over $Y$ and/or $\Gamma^Y$ of Eq. (21). Those first terms of the expansions of the constitutive equations of the porous medium involve not only the macroscopic fields $\vec{u}^{(0)}$ and $p^{(0)}$ but also $\vec{u}^{(1)}$ and $p^{(1)}$ which depend on $\vec{y}$ and are unknown. Therefore, previous to the averaging of (21), $\vec{u}^{(1)}$ and $p^{(1)}$ must be determined in terms of the macroscopic fields $\vec{u}^{(0)}$ and $p^{(0)}$ (see Sánchez-Palencia, 1980 or Bensoussan et al., 2011 for more details). The starting point to get the so-called self-balanced problems, i.e. the problems allowing to determine $\vec{u}^{(1)}$ and $p^{(1)}$, is, once more, given by the weak formulations (4) and (8) in which the virtual fields can be chosen as needed, accordingly to the aim in view. In Section 3.2, the aim is to determine the homogenized balanced equation, hence the chosen smooth *macroscopic* virtual fields that smear out the heterogeneities. On the contrary, to obtain the self-balanced problems set on the cell $Y$, it is necessary to emphasize the dependence on the variable $\vec{y}$. That is done by taking the suitable test fields of the form $\vec{w}=\theta(\vec{x})\vec{v}\left(\frac{\vec{x}}{e}\right)$ and $r=\theta(\vec{x})t\left(\frac{\vec{x}}{e}\right)$. See e.g. Sánchez-Palencia (1980, page 79) for a more complete presentation.

#### 3.4.1. Mechanics

Let $\vec{w}$ be the field $\vec{w}=\theta(\vec{x})\vec{v}\left(\frac{\vec{x}}{e}\right)$ where $\vec{v}(y)$ is a periodic function defined on $Y$ and $\theta(\vec{x})$ is a smooth macroscopic function which vanishes on $\partial\Omega$, we have:

$$\epsilon(\vec{w})=\left(\vec{v}\otimes\overrightarrow{\text{grad}}^x\theta\right)^S+\frac{1}{e}\theta\,\epsilon^y(\vec{v})$$

Taking $\vec{w}=\theta(\vec{x})\vec{v}\left(\frac{\vec{x}}{e}\right)$ in (4) and making $e\searrow 0$ with the use of lemmas 3.2.1 and 3.2.1, yield:

$$\forall\theta,\vec{v},\theta=0\text{ on }\partial\Omega\,,\ -\int_\Omega\frac{1}{|Y|}\left(\int_Y\sigma^{(0)}:\epsilon^y(\vec{v})\,\mathrm{d}s_y+\int_{\Gamma^Y}\vec{T}^{(0)}\cdot[[\vec{v}]]\,\mathrm{d}l_y\right)\theta\,\mathrm{d}s_x=0$$

as $\theta$ is any smooth macroscopic field, that entails:

$$\forall\vec{v},Y\text{-periodic}\,,\ \int_Y\sigma^{(0)}:\epsilon^y(\vec{v})\,\mathrm{d}s_y+\int_{\Gamma^Y}\vec{T}^{(0)}\cdot[[\vec{v}]]\,\mathrm{d}l_y=0$$

which is the weak formulation of the mechanical self-balanced of the cell $Y$. Taking into account the constitutive Eqs. (21a) and (21d), yields to the weak formulation of the mechanical self-balanced problem, i.e.:

Given $\epsilon^x\left(\vec{u}^{(0)}\right)$ and $p^{(0)}$, find $\vec{u}^{(1)}(\vec{x},\vec{y})$, $Y$-periodic, such that:

$$\forall\vec{v},Y\text{-periodic}\,,\ \int_Y\left(c:\left(\epsilon^x\left(\vec{u}^{(0)}\right)+\epsilon^y\left(\vec{u}^{(1)}\right)\right)-p^{(0)}\alpha\right):\epsilon^y(\vec{v})\,\mathrm{d}s_y+\int_{\Gamma^Y}\left(C\cdot\left[\left[\vec{u}^{(1)}\right]\right]-p^{(0)}\vec{A}\right)\cdot[[\vec{v}]]\,\mathrm{d}l_y=0 \quad (22)$$

The strong form of the problem reads:

$$\mathrm{div}^y\sigma^{(0)}=0\,,\text{ in }Y \quad (23a)$$

$$\sigma^{(0)}\cdot\vec{n}=\vec{T}^{(0)}\,,\text{ on }\Gamma^Y \quad (23b)$$

$$\sigma^{(0)}=c:\left(\epsilon^x\left(\vec{u}^{(0)}\right)+\epsilon^y\left(\vec{u}^{(1)}\right)\right)-p^{(0)}\alpha \quad (23c)$$

$$\vec{T}^{(0)}=C\cdot\left[\left[\vec{u}^{(1)}\right]\right]-p^{(0)}\vec{A}\,,\text{ on }\Gamma^Y \quad (23d)$$

$$\vec{u}^{(1)},\sigma^{(0)}\ Y\text{-periodic} \quad (23e)$$

It is worth noting that this problem is purely mechanical in the sense that the only unknowns are the displacement field $\vec{u}^{(1)}$ and the stress field $\sigma^{(0)}$, the pressure field $p^{(1)}$ and the flows $\vec{q}^{(0)}$ and $Q^{(1)}$ being not involved.

#### 3.4.2. Fluid flow

Let $r$ be the field $r=\theta(\vec{x})w\left(\frac{\vec{x}}{e}\right)$ where $w(\vec{y})$ is a periodic function defined on $Y$ and $\theta(\vec{x})$ is a smooth macroscopic function which vanishes on $\partial\Omega$. We then have:

$$\overrightarrow{\text{grad}}r=w\,\overrightarrow{\text{grad}}^x\theta+\frac{1}{e}\theta\,\overrightarrow{\text{grad}}^y w$$

Taking $r=\theta(\vec{x})w\left(\frac{\vec{x}}{e}\right)$ in (8) and making $e\searrow 0$, yield:

$$\forall\theta,w\,,\ -\int_\Omega\frac{1}{|Y|}\left(\int_Y\vec{q}^{(0)}\cdot\overrightarrow{\text{grad}}^y w\,\mathrm{d}s_y+\int_{\Gamma^Y}Q^{(1)}\overrightarrow{\text{grad}}^y w\cdot\vec{\tau}\,\mathrm{d}l_y\right)\theta\,\mathrm{d}s_x=0$$

as $\theta$ is any smooth macroscopic field, that entails:

$$\forall w\,Y\text{-periodic}\,,\ \int_Y\vec{q}^{(0)}\cdot\overrightarrow{\text{grad}}^y w\,\mathrm{d}s_y+\int_{\Gamma^Y}Q^{(1)}\overrightarrow{\text{grad}}^y w\cdot\vec{\tau}\,\mathrm{d}l_y=0$$

which is the weak form of the fluid volume self-balanced of the cell $Y$. Taking into account relations (21c) and (21f), yields to the weak formulation of the self-balanced filtration problem that reads:

Given $\overrightarrow{\text{grad}}^x p^{(0)}$, find $p^{(1)}(\vec{x},\vec{y})$, $Y$-periodic, such that:

$$\forall w\,Y\text{-periodic}\,,\ \int_Y k\left(\overrightarrow{\text{grad}}^x p^{(0)}+\overrightarrow{\text{grad}}^y p^{(1)}\right)\cdot\overrightarrow{\text{grad}}^y w\,\mathrm{d}s_y+\int_{\Gamma^Y}\left(K\left(\overrightarrow{\text{grad}}^x p^{(0)}+\overrightarrow{\text{grad}}^y p^{(1)}\right)\cdot\vec{\tau}\right)\left(\overrightarrow{\text{grad}}^y w\cdot\vec{\tau}\right)\mathrm{d}l_y=0 \quad (24)$$

The strong form of the problem reads:

$$\mathrm{div}^y\,\vec{q}^{(0)}=0\,,\text{ in }Y$$

$$\frac{dQ^{(1)}}{dl}+\left[\left[\vec{q}^{(0)}\right]\right]\cdot\vec{n}=0\,,\text{ on }\Gamma^Y$$

$$\vec{q}^{(0)}=-k\left(\overrightarrow{\text{grad}}^x p^{(0)}+\overrightarrow{\text{grad}}^y p^{(1)}\right)$$

$$Q^{(1)}=-K\left(\overrightarrow{\text{grad}}^x p^{(0)}+\overrightarrow{\text{grad}}^y p^{(1)}\right)\cdot\vec{\tau}$$

$$\sum Q^{(1)}=0\text{ at junction points of cracks}$$

$$p^{(1)},\vec{q}^{(0)},Q^{(1)}\ Y\text{-periodic}$$

In a similar way to Section 3.4.1, the problem (24) is a pure filtration one, the displacement field $\vec{u}^{(1)}$ and the stress field $\sigma^{(0)}$ are not involved.

So, at the microscopic scale, i.e. the cell $Y$-scale, the elasticity and the fluid flow problems are completely uncoupled.

### 3.5. Macroscopic constitutive equations

As problems (22) and (24) are linear, it is standard (see Sánchez-Palencia, 1980 or Bensoussan et al., 2011) to prove that $\vec{u}^{(1)}$ linearly depends on $e^x\left(\vec{u}^{(0)}\right)$ and $p^{(0)}$, and $p^{(1)}$ on $\overrightarrow{\text{grad}}^x p^{(0)}$. Consequently, by the constitutive Eq. (21b) and the expression (21b), $\sigma^{(0)}$ and $\kappa^{(0)}$ depend linearly on $e^x\left(\vec{u}^{(0)}\right)$ and $p^{(0)}$ and so does $\langle\sigma\rangle$ and $\langle\kappa\rangle$ defined by (14) and (18). In a similar way, it can be seen that $\langle\vec{q}\rangle$ depends linearly on $\overrightarrow{\text{grad}}^x p^{(0)}$. Therefore, the macroscopic constitutive equations read:

$$\langle\sigma\rangle=c^H:\epsilon^x\left(\vec{u}^{(0)}\right)-p^{(0)}\alpha^H \quad (25a)$$

$$\langle\kappa\rangle=\tilde{\alpha}^H:\epsilon^x\left(\vec{u}^{(0)}\right)+\beta^H p^{(0)} \quad (25b)$$

$$\langle \vec{q} \rangle = -k^H \overrightarrow{\mathrm{grad}}^x p^{(0)} \tag{25c}$$

From those equations and (14), it is obvious that $c^H_{ijkh}$ is the average of $\sigma^{(0)}_{ij}$ for $p^{(0)} = 0$ and $\epsilon^x\left(\vec{u}^{(0)}\right) = (\vec{e}_k \otimes \vec{e}_h)^S$ and that $\alpha^H_{ijk}$ is the average of $-\sigma^{(0)}_{ij}$ for $p^{(0)} = 1$ and $\epsilon^x\left(\vec{u}^{(0)}\right) = 0$. In the same way, $k^H_{ij}$ is equal to $\langle \vec{q} \rangle_i$ (see (17)) for $\overrightarrow{\mathrm{grad}}^x p^{(0)} = \vec{e}_j$.

On the standard energy-based assumptions of symmetries for the elastic tensors $c$ and $C$ namely:

$$c_{ijkl} = c_{klij} \text{ and } C_{ij} = C_{ji}$$

it is classical (see Sánchez-Palencia, 1980 or Bensoussan et al., 2011) to prove that $c^H$, $k^H$, $\alpha^H$ and $\tilde{\alpha}^H$ satisfy the following relations:

$$c^H_{ijkl} = c^H_{klij}$$
$$k^H_{ij} = k^H_{ji}$$
$$\tilde{\alpha}^H = \alpha^H$$

## 4. Damage

In this section, cracks are considered damageable, the purpose being to build the corresponding macroscopic modeling of the cracked porous medium.

### 4.1. Damage parameter and evolution law

Similarly to the approach proposed in Bilbie et al. (2008), the constitutive equations of those cracks (9a) and (9d) are changed to:

$$\vec{T}^{(e)} = \left(1 - d^{(e)}\right)\frac{C}{e} \cdot \left[\left[\vec{u}^{(e)}\right]\right] - p^{(e)}\vec{A} \tag{26}$$

$$Q^{(e)} = -eK\left(d^{(e)}\right)\overrightarrow{\mathrm{grad}}\, p^{(e)} \cdot \vec{\tau} \tag{27}$$

$d^{(e)}$ being the damage parameter of the cracks. It can be stressed that, in the considered damage modeling, the damage parameter modifies only the stiffness $C$ and not $\vec{A}$. This can be seen as purely heuristical but this choice seems consistent with the double scale asymptotic analysis of an elastic saturated porous matrix leading to Biot's modeling, see Auriault and Sanchez-Palencia (1977) and Auriault (2005). On the other hand, the dependence of the permeability $K$ with respect to $d^{(e)}$ is completely phenomenological (e.g. Rastiello et al., 2013).

The damage of the cracks is due to the opening and the shearing of the cracks. For the sake of simplicity (Bilbie et al., 2008), any difference between the opening and the closing of the cracks are disregarded and the evolution of the damage parameter $d$ is given by:

$$d^{(e)}(t) = \sup_{0 \le \tau \le t} f\left(\frac{\left\| \left[\left[\vec{u}^{(e)}\right]\right](\tau) \right\|}{D^{(e)}}\right) \tag{28}$$

where $f$ is the function:

$$z \xrightarrow{f} f(z) = \begin{cases} z(2-z) & 0 \le z < 1 \\ 1 & 1 \le z \end{cases} \tag{29}$$

and $D^{(e)}$ is a length-like feature of the material of the cracks. At initial time $t = 0$, the porous medium is assumed to be unloaded, unstressed, unstrained and undamaged which means that the initial value of the damage parameter is 0.

$d^{(e)}(t)$ is a function of the history $\{[[\vec{u}^{(e)}]](\tau)\,;\,\tau \le t\}$ of $[[\vec{u}^{(e)}]]$ up to time $t$. Its evolution is governed only by the opening or shearing of the cracks and not directly by the fluid pressure.

### 4.2. Homogenization of the damageable cracked porous medium

The damage parameter is sought in the form of the double scale asymptotic expansion:

$$d^{(e)}(\vec{x}, t) = d^{(0)}\left(\vec{x}, \frac{\vec{x}}{e}, t\right) + ed^{(1)}\left(\vec{x}, \frac{\vec{x}}{e}, t\right) + e^2 d^{(2)}\left(\vec{x}, \frac{\vec{x}}{e}, t\right) + \cdots$$

Almost all the analysis and equations of Section 3 remain valid. Eq. (23d) is slightly modified to:

$$\vec{T}^{(0)} = \left(1 - d^{(0)}\right)C \cdot \left[\left[\vec{u}^{(1)}\right]\right] - p^{(0)}\vec{A}$$

but all the other equations of (23) remain unmodified. The main change lies in the nature of the mechanical self-balanced problem (23) on the cell $Y$ which, in Section 3.4 is a purely time independent elastic problem. In this section, due to damage, it becomes a non-linear quasi-static evolution, involving an evolution law for the damage parameter $d^{(0)}$ which comes from the expansion (28).

As, according to (11d), the first term of the expansion of $[[\vec{u}^{(e)}]]$ is of the order of $e$, it is consistent to assume that $D^{(e)}$ is proportional to $e$:

$$D^{(e)} = eD$$

With this assumption, it can be proved (see Appendix B) that the evolution law for $d^{(0)}$ reads:

$$d^{(0)}(t) = \sup_{0 \le \tau \le t} f\left(\frac{\left\| \left[\left[\vec{u}^{(1)}\right]\right](\tau) \right\|}{D}\right) \tag{30}$$

In the case of damageable cracks, the weak formulation of the mechanical self-balanced problem reads:

$$\begin{aligned}
&\text{Given the histories } \left\{\epsilon^x\left(\vec{u}^{(0)}\right)(\tau)\,;\, 0 \le \tau \le t\right\} \\
&\quad \text{and } \left\{p^{(0)}(\tau)\,;\, 0 \le \tau \le t\right\}, \text{find } \vec{u}^{(1)}(\vec{x}, \vec{y}, \tau) \\
&\quad \text{and } d^{(0)}(\vec{x}, \vec{y}, \tau),\ Y\text{-periodic}, \tau \in [0, t], \text{ such that:} \\
&\forall \tau \in [0, t], \forall \vec{v}, Y\text{-periodic}, \int_Y \left(c : \left(\epsilon^x\left(\vec{u}^{(0)}(\tau)\right)\right.\right. \\
&\left.\left. + \epsilon^y\left(\vec{u}^{(1)}(\tau)\right)\right) - p^{(0)}(\tau)\alpha\right) : \epsilon^y(\vec{v})\, ds_y \\
&+ \int_{\Gamma^Y} \left(\left(1 - d^{(0)}(\tau)\right)C \cdot \left[\left[\vec{u}^{(1)}(\tau)\right]\right] - p^{(0)}(\tau)\vec{A}\right) \cdot [[\vec{v}]]\, dl_y = 0 \\
&\text{with } d^{(0)}(\tau) = \sup_{0 \le \rho \le \tau} f\left(\frac{\left\| \left[\left[\vec{u}^{(1)}(\rho)\right]\right] \right\|}{D}\right)
\end{aligned} \tag{31}$$

The solution of this quasi-static evolution problem, allows the stress $\sigma^{(0)}(t)$ and by (14) the macroscopic stress $\langle \sigma \rangle(t)$ to be determined. Thus, the macroscopic constitutive equation reads:

$$\left(\left\{\epsilon^x\left(\vec{u}^{(0)}\right)(\tau)\,;\, 0 \le \tau \le t\right\}, \left\{p^{(0)}(\tau)\,;\, 0 \le \tau \le t\right\}\right) \longrightarrow \langle \sigma \rangle(t)$$

The strong form of problem (31) can be obtained in the same way as (23) is obtained from (22).

The solution of the evolution problem (31) yields to $d^{(0)}(t)$, as a functional of the histories $\left\{\epsilon^x\left(\vec{u}^{(0)}\right)(\tau)\,;\, 0 \le \tau \le t\right\}$ and $\{p^{(0)}(\tau)\,;\, 0 \le \tau \le t\}$. The self-balanced filtration problem (24) remains the same with $K(d^{(0)})$ instead of $K$.

## 5. Numerical implementation of the linear case

The identification and the use of the macroscopic constitutive equations of the cracked porous medium require the solution of the (weak formulation) problems (22) for the mechanical part - or (31) in the case of damageable cracks - and (24) for the flow part. However, even for simple configurations of the elementary cell, those problems cannot be solved analytically and a numerical FEM computation is needed. For the sake of simplicity, the numerical computation is carried out in 2D.

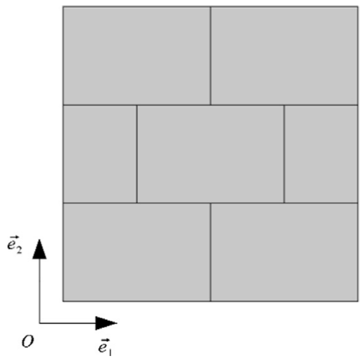



**Fig. 2.** Elementary cell geometry, see Marinelli (2013).

### 5.1. Cell and material properties

In order to carry out the numerical computation, an elementary cell (or REV) has to be defined. The considered cell is depicted in Fig. 2; the same cell as Marinelli (2013) has been adopted, the dimensions are the unity for both sides. The grey parts represent the poroelastic, homogeneous isotropic matrix (Lame constants: $\lambda = 1,442 \cdot 10^9$ Pa and $\mu = 0,961 \cdot 10^9$ Pa , i.e. Young's modulus $E = 2,5 \cdot 10^9$ Pa and Poisson's ratio $\nu = 0,3$), corresponding to a typical clay rock material. Biot's tensor of the poroelastic matrix is proportional to the identity: $\alpha = 0,4\,\mathbb{I}$; its transmissivity $K$ is assumed to be $10^{-7}\mathrm{m}^2/\mathrm{Pa}$ i.e. this is equivalent to assume a permeability $k^h$ equal to $10^{-10}\mathrm{m}/\mathrm{Pa}$ with a crack thickness 0,001 times the cell (unit) length.[1]

Assuming that the material of the cracks is isotropic and that its Biot's tensor is proportional to the identity (see (A.6) and (A.7)), the stiffness tensor of the crack and its Biot's vector respectively read $C = C_T \vec{\tau} \otimes \vec{\tau} + C_N \vec{n} \otimes \vec{n}$ and $\vec{A} = A\vec{n}$. In the following, a parametric study on the stiffnesses $C_T$ and $C_N$, Biot's coefficient $A$ and the permeability of the crack $k^h$ is proposed in order to study the influence of the characteristics of the cracks on the equivalent macroscopic poroelastic medium (note that $C_T = C_N = C$).

**Remark 1.** Due to the symmetries of the elementary cell with respect to the two axes and the homogeneity and isotropy of the poroelastic medium and cracks, some properties of the homogenized medium are expected to be isotropic. So, following Léné and Duvaut (1981) or Caillerie (1984) or Appendix C for the quasi-static non-linear case of damageable cracks, the homogenized medium is orthotropic, the stiffnesses $c^H_{1112}$, $c^H_{2212}$ (and all those ensued by the usual symmetries of indices) are zero, and Biot's matrix $\alpha^H$ is diagonal ($\alpha^H_{12} = \alpha^H_{21} = 0$). It has to be noted that, in general, the homogenized material is just orthotropic and not isotropic. In the same way, the homogenized permeability matrix is diagonal.

In this following section, the self-equilibrium problem of the cell (22) is numerically solved . The solutions of a 3+1 boundary value problem (three for the elasticity problem and one for the water pressure), are integrated and the homogenized coefficients are obtained in the macroscale[2].

### 5.2. Numerical validation: mechanics

#### 5.2.1. Macroscopic elastic stiffnesses

According to Section 3.5, the macroscopic stiffnesses are obtained through the solution of problem (22) for $p^{(0)} = 0$ and for $\epsilon^x(u^{(0)})$, i.e. pure elongations in the direction 1 and 2 and a simple shear, thus imposing a strain matrix as follows :

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ; \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For those computations, the crack stiffnesses are $C_T = C_N = C = 10^{12}\mathrm{Pa}/\mathrm{m}$.

Results of the displacement and Von Mises stress field at the micro-scale are given in Fig. 3. As expected, despite the isotropic elasticity at the micro-level, the homogenized solution is not isotropic due to the cell geometry.

This result is quantitatively consistent with Marinelli (2013).

A parametric study on the influence of the cracks stiffness $C$ in the range from $10^6\mathrm{Pa/m}$ to $10^{15}\mathrm{Pa/m}$ is here proposed. The elastic characteristics of the porous matrix remain unchanged.

Results are presented in Fig. 4.

Consistently with the remark 1 about the orthotropy of the homogenized medium, the coefficients $c^H_{1112}$, $c^H_{1211}$, $c^H_{1222}$ and $c^H_{2212}$[3] are found numerically null. For a small enough crack stiffness $C$, the dependence of the homogenized coefficients on $C$ is almost linear, i.e. the overall stiffness is essentially controlled by that of the cracks and is independent from the porous material properties. It is obvious that the cracked porous medium is weaker than the porous medium itself since when a stress is applied, the overall strain is larger than what it would be in the intact porous medium. It is obvious too that the overall stiffness is close to that of the porous medium if the cracks are very stiff; indeed, in that case, the porous medium is almost continuous. That is numerically verified, for a crack stiffness $G$ higher than $1 \cdot 10^{15}$Pa, the homogenized medium is isotropic, its Young's modulus and Poisson's ratio being those of the porous medium.

$$\underline{\underline{c}}^H = \begin{pmatrix} 3,36 \cdot 10^9 & 1,44 \cdot 10^9 & 0 \\ 1,44 \cdot 10^9 & 3,36 \cdot 10^9 & 0 \\ 0 & 0 & 1,92 \cdot 10^9 \end{pmatrix}$$
$$Pa = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix} \tag{33}$$

#### 5.2.2. Macroscopic Biot's tensor

Similarly to the computation of the homogenized stiffnesses, the macroscopic Biot's tensor $\alpha^H$ is obtained through the solution of problem (22), for $p^{(0)} = 1$ and $\epsilon^x\left(u^{(0)}\right) = 0$. As in the previous case, a parametric analysis showing the influence of the crack properties on the global, homogenized answer is proposed. The parametric analysis covers the range: $0 \le \vec{A} \le 1$, $\vec{A}$ being dimensionless. Fig. 5 (left) illustrates a quasi-linear relationship between the homogenized Biot matrix $\alpha^H$ and the Biot vector $\vec{A}$ in the thin elastic layer. The difference of slopes between $\alpha^H_{11}$ and $\alpha^H_{22}$ in Fig. 5 (left) shows, as expected, that a dependence exists between the homogenized Biot matrix $\alpha^H$ and the crack stiffness $G$; this is shown in Fig. 5 (right).

[1] More details are given in Appendix A. Very thin layer of a deformable porous medium.

[2] The following Cauchy stress and Biot problem notation is considered:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{1111} & C_{1122} & C_{1112} \\ C_{1122} & C_{2222} & C_{2212} \\ C_{1112} & C_{2212} & C_{1212} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{pmatrix} \text{ and } \vec{A}^H = \begin{pmatrix} A_{1111} \\ A_{2222} \\ A_{1212} \end{pmatrix}$$

[3] Notation in the Cauchy relation:

$$\begin{pmatrix} \langle\sigma_{11}\rangle \\ \langle\sigma_{22}\rangle \\ \langle\sigma_{12}\rangle \end{pmatrix} = \begin{pmatrix} c^H_{1111} & c^H_{1122} & c^H_{1112} \\ c^H_{2211} & c^H_{2222} & c^H_{2212} \\ c^H_{1211} & c^H_{1222} & c^H_{1212} \end{pmatrix} \cdot \begin{pmatrix} \epsilon^x_{11} \\ \epsilon^x_{22} \\ \epsilon^x_{12} \end{pmatrix} \tag{32}$$

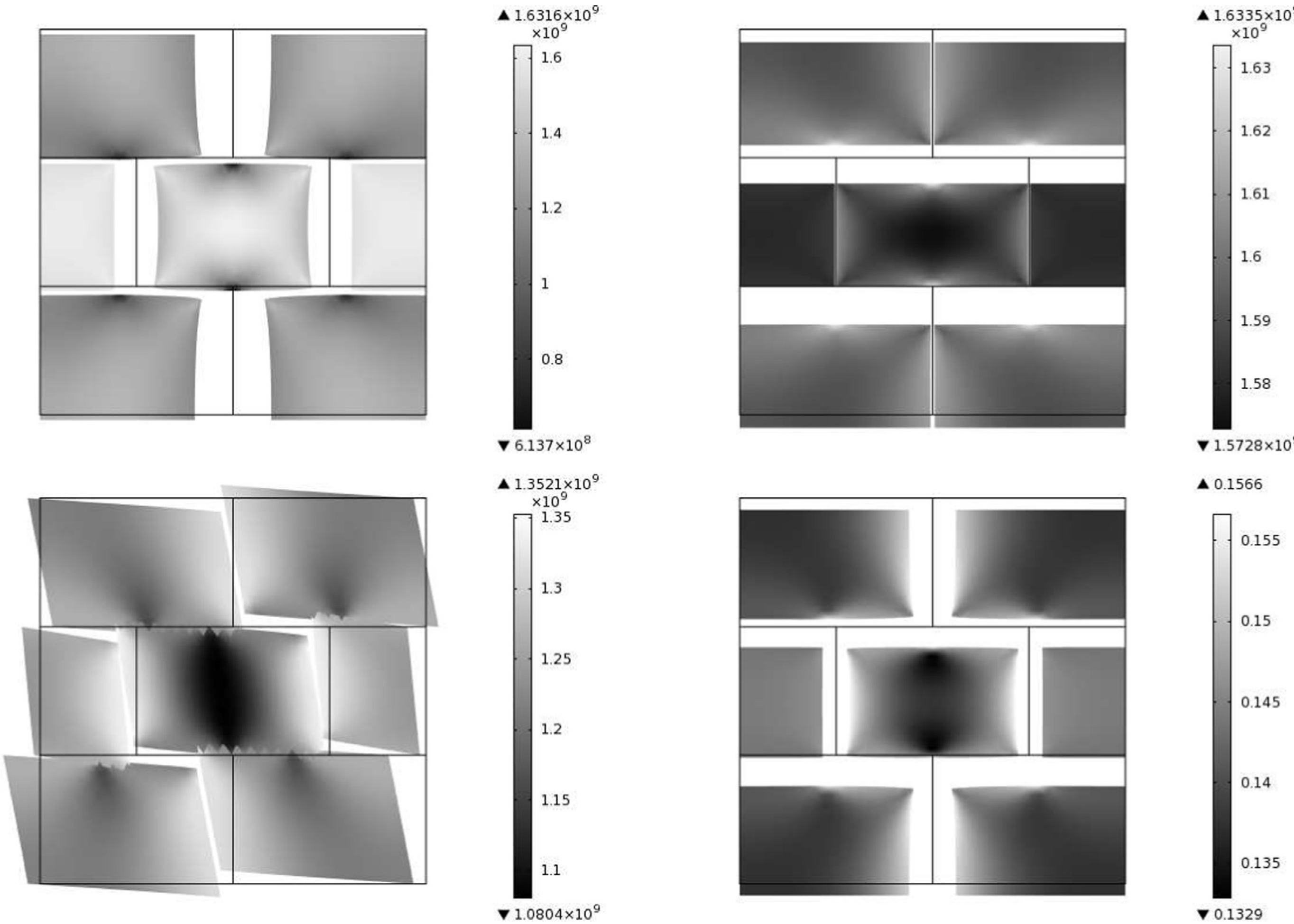



**Fig. 3.** Von Mises stress fields N/mm$^2$ in the cell for loadings in the 4 degrees of freedom: $lm = 11, 22, 12$ and $p^{(0)}$.

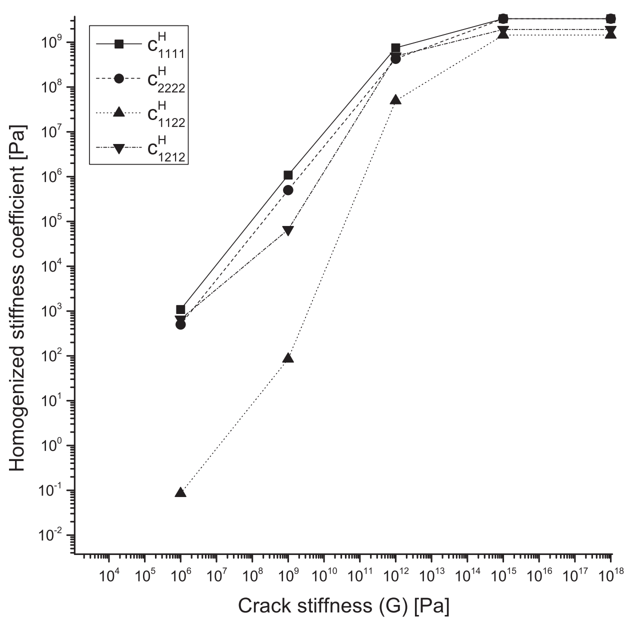



**Fig. 4.** Homogenized coefficients vs. crack stiffness evolution.

### 5.3. Numerical validation: permeability

Similarly to the computation of the macroscopic mechanical characteristics of the homogenized poroelastic medium and according to Section 3.5, the macroscopic permeability matrix is obtained through the solution of problem (24), for a macroscopic pressure gradient $\overrightarrow{\text{grad}}^x p^{(0)}$ successively parallel to the directions 1 and 2, i.e. respectively $\vec{e}_1$ and $\vec{e}_2$. For those computations, the permeability of the cracks is assumed homogeneous and equal to $K = 1 \cdot 10^{-7}$m/s with a crack thickness equal to 0.001 times the cell size.

Due to the shape of the considered cell and the homogeneity of the permeabilities of the porous matrix and of the cracks, it can be seen that the solution $p^{(1)}$ of the problem (24) for $\overrightarrow{\text{grad}}^x p^{(0)} = \vec{e}_1$ is constant and, according to (25c), gives:

$$k^H_{11} = k + \frac{K}{Y_2}$$

Consistently with the remark 1, $k^H_{21}$ is found to be equal to 0. The computation of $k^H_{22}$ needs the solution of problem (24) for $\overrightarrow{\text{grad}}^x p^{(0)} = \vec{e}_2$. Finally the macroscopic permeability matrix reads:

$$\underline{\underline{k}}^H = \begin{pmatrix} 2,986 \cdot 10^{-10} & 0 \\ 0 & 2,324 \cdot 10^{-10} \end{pmatrix}$$

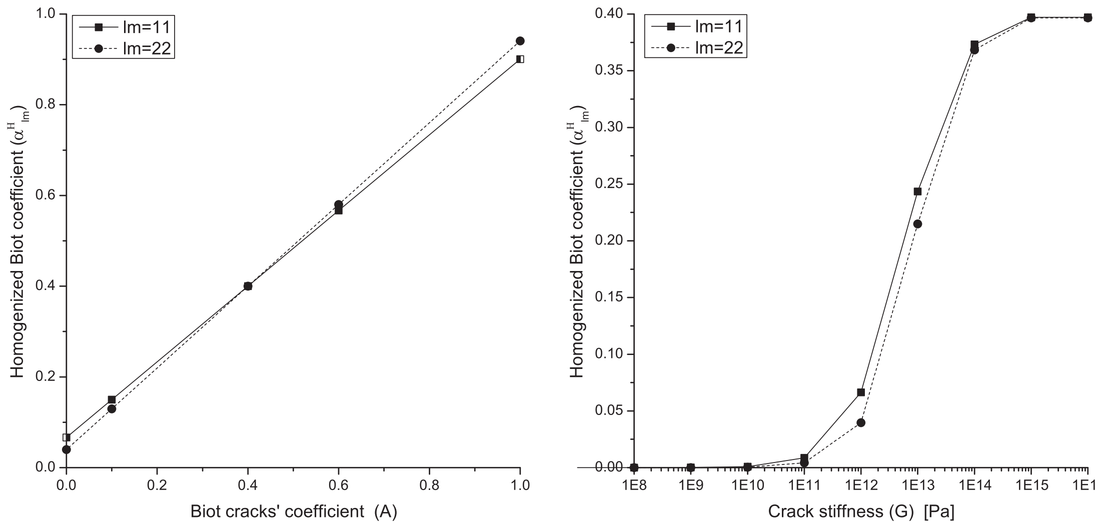



**Fig. 5.** Left: relation between Biot coefficient in the cracks and homogenized Biot coefficient, crack stiffness $(G) = 10^{12}\mathrm{Pa/m}$. Right: relation between Crack stiffness (G) and homogenized Biot coefficient for $\vec{A} = \vec{0}$ and $\vec{\alpha} = 0.4 \cdot \vec{I}$.

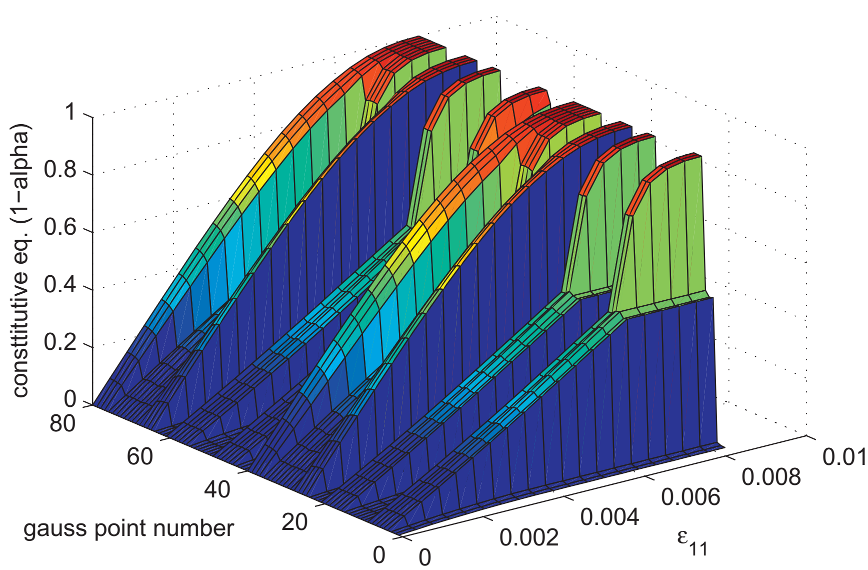



**Fig. 6.** Oedometric test: evolution of the damage law for all the Gauss points of the cracks. $D = 0.008$.

## 6. Numerical results: damage

The purpose of this section is to – partially – study the macroscopic behavior of the cracked porous medium when the cracks are damageable. As in the previous section, the macroscopic behaviour is obtained through the solution of a self-balanced problem set on the elementary cell, namely problem (31). The self-balanced filtration problem (24) remains essentially unchanged and is not considered in this section.

The main differences with the computations of Section 5 are that the problem is not linear and that the data of problem (31) are the histories of the macroscopic strain and the pressure. Newton's method and a time stepping procedure is then required.

The constitutive equation is numerically determined for two typical geomechanics examples, an oedometric and a biaxial tests in drained condition. Atmospheric pressure is neglected, that means that $p^{(0)} = 0$, so the medium is in fact dry.

The geometry of the cell as well Young's modulus and Poisson's ratio of the porous matrix and the elastic stiffness of the undamaged cracks are given in Section 5.

### 6.1. Oedometric loading

Damage model is first applied using a strain controlled path, i.e. an oedometric test. An arbitrary uniaxial macrostrain $\epsilon^x\left(u^{(0)}\right) = -0,01\vec{e}_1 \otimes \vec{e}_1$ is applied in 20 steps and the damage parameter is $D = 0.008$ times the size of the cell. All the elastic coefficients are the same used in previous sections.

The evolution of the damage law for all the Gauss points in the crack network is depicted in Fig. 6.

The stress-strain curve is given in Fig. 7 as well as the number of iterations needed for convergence in Newton's method.

### 6.2. Biaxial test loading

The second considered example corresponds to a mixed stress/strain controlled path. A confining pressure and a gradually increasing longitudinal strain are applied, thus reproducing the loading path of a biaxial test. The solution of the self-balance problem (31) gives the stress in terms of the history of the strain: a

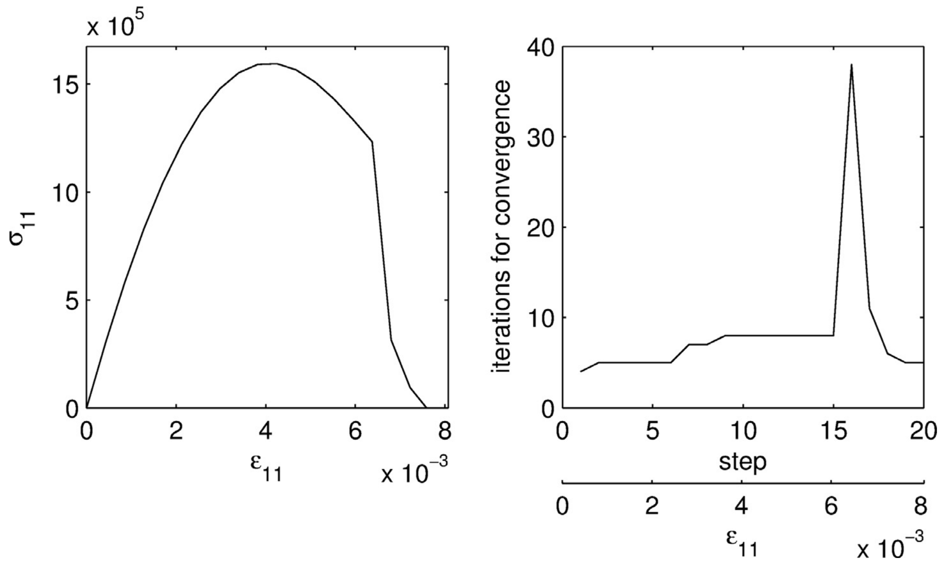



**Fig. 7.** Oedometric test. Left: stress-strain 11 axis, right: number of iterations for convergence of Newton method. .

suitable procedure, presented in what follows, is needed to simulate the biaxial test.

In the time stepping scheme, the finite element solution of problem (31) gives the stress $\sigma^n$ at the end of the step $n$ in terms of the strain $\epsilon^n$ at step $n$:

$$\sigma^n = \mathcal{T}^n(\epsilon^n)$$

The stress $\sigma^n$ and strain $\epsilon^n$ are then decomposed in two supplementary subspaces $E_1$ and $E_2$ of the space of second order tensors:

$$\sigma^n = \sigma_1^n + \sigma_2^n$$
$$\epsilon^n = \epsilon_1^n + \epsilon_2^n$$

where $\sigma_1^n$ and $\epsilon_2^n$ are given and $\sigma_2^n$ and $\epsilon_1^n$ are unknown. The problem to be solved reads:

$$\text{Given } \sigma_1^n \in E_1 \text{ and } \epsilon_2^n \in E_2, \text{ find } \sigma_2^n \in E_2 \text{ and } \epsilon_1^n \in E_1 \text{ such that :}$$
$$\sigma_1^n + \sigma_2^n - \mathcal{T}^n(\epsilon_1^n + \epsilon_2^n) = 0 \tag{34}$$

$\mathcal{T}^n$ is non-linear, so Newton's method is used to determine the solution of the problem. The linearized equation to be solved in $\delta\epsilon_1^{n(k)}$ and $\delta\sigma_2^{n(k)}$ at iteration $(k)$ of Newton's method reads:

$$\delta\sigma_2^{n(k)} - C^n : \delta\epsilon_1^{n(k)} + \sigma_1^n + \sigma_2^{n(k)} - \mathcal{T}^n\left(\epsilon_1^{n(k)} + \epsilon_2^n\right) = 0$$

where $C^n = \frac{d\mathcal{T}^n}{\delta\epsilon^n}$ is computed at $\left(\epsilon_1^{n(k)} + \epsilon_2^n\right)$. The approximated solution at step $k$ is then updated into:

$$\epsilon_1^{n(k+1)} = \epsilon_1^{n(k)} + \delta\epsilon_1^{n(k)}$$
$$\sigma_2^{n(k+1)} = \sigma_2^{n(k)} + \delta\sigma_2^{n(k)}$$

The procedure is then applied to the case of a biaxial test using the same elastic coefficients as in the previous cases. As a first loading sequence, a confining pressure $p_c = 1 \cdot 10^7$Pa is gradually applied (time steps from 1 to 16), then a strain $\epsilon_{22} = 0.04$ (time steps from 17 to 40) is gradually imposed, the confining pressure remaining constant. Similarly to the previous section, the macroscale values $p_c = 1 \cdot 10^7$Pa and $\epsilon_{22} = 0.04$ are chosen in order to create a significant amount of damage.

In the first part of the loading, the total stress $\sigma^n$ is given ($\sigma_1^n = \sigma^n$, $\sigma_2^n \equiv 0$):

$$\underline{\underline{\sigma}}^n = \frac{n}{16}\begin{pmatrix} p_c & 0 \\ 0 & p_c \end{pmatrix}$$

where $1 \leq n \leq 16$ and the whole strain $\epsilon^n$ is unknown ($\epsilon_1^n = \epsilon^n$, $\epsilon_2^n \equiv 0$).

In the second part of the loading, $17 \leq n \leq 40$, the data are:

$$\underline{\underline{\sigma}}_1^n = \begin{pmatrix} p_c & 0 \\ 0 & 0 \end{pmatrix}, \quad \underline{\underline{\epsilon}}_2^n = \frac{n-16}{24}\begin{pmatrix} 0 & 0 \\ 0 & \epsilon_{22} \end{pmatrix}$$

and the unknowns are:

$$\underline{\underline{\sigma}}_2^n = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{22}^n \end{pmatrix}, \quad \underline{\underline{\epsilon}}_1^n = \begin{pmatrix} \epsilon_{11}^n & \epsilon_{12}^n \\ \epsilon_{12}^n & 0 \end{pmatrix}$$

Results are given in Fig. 8.

### 6.3. Oedometric and biaxial tests: discussion of the results

The above presented cases (oedometer and biaxial tests) show that the constitutive law for the given REV can be obtained for any loading path.

Microscale damage implies softening of the homogenized stress tensor. In accordance with the combination of the applied macrostrains on the time-history, cracks are affected in a different manner and, consequently, the stress tensor evolves depending on these macrostrain paths.

The homogenized stresses $\sigma_{1111}^H$, $\sigma_{1122}^H$ and $\sigma_{1212}^H$ are similarly affected but $\sigma_{2222}^H$ has significantly less degradation due to the non-symmetric loading. Accordingly to the lemma of Appendix C the stresses $\sigma_{1112}^H$ and $\sigma_{2212}^H$ and their symmetric $\sigma_{1211}^H$ and $\sigma_{1222}^H$ are equal to zero.

It should be noted that for a large strain, which also implies high damage, some homogenized stresses become zero. In other terms, the REV can be considered as broken after a given iteration. Moreover, the damaged (but not completely broken) cracks close, leading to a non-unique solution (Chambon et al., 2004). An example of the deformed configuration and the Von Mises stress field is depicted in Fig. 9.

It is worth noting that when the solution bifurcates, Newton's method require a larger number of iterations to converge. In Fig. 7, 38 iterations are required to fulfill the convergence criterion when localization occurs (at step 16, the macrostrain is about $\varepsilon_{11} = 6.4 \times 10^{-3}$). The same phenomenon is observed in the biaxial test in Fig. 8. Some localization features are explored in the next section.

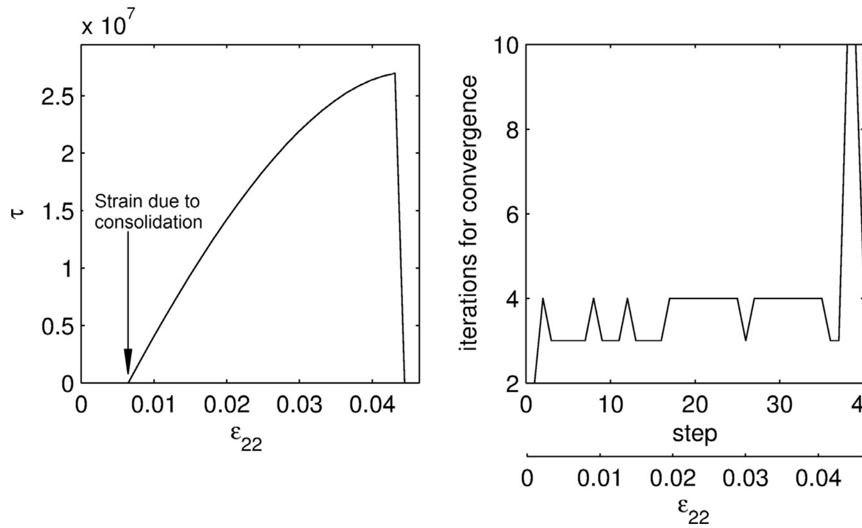



**Fig. 8.** Biaxial compression test. Left: stress-strain 22 axis, right: number of iterations for convergence of Newton method.

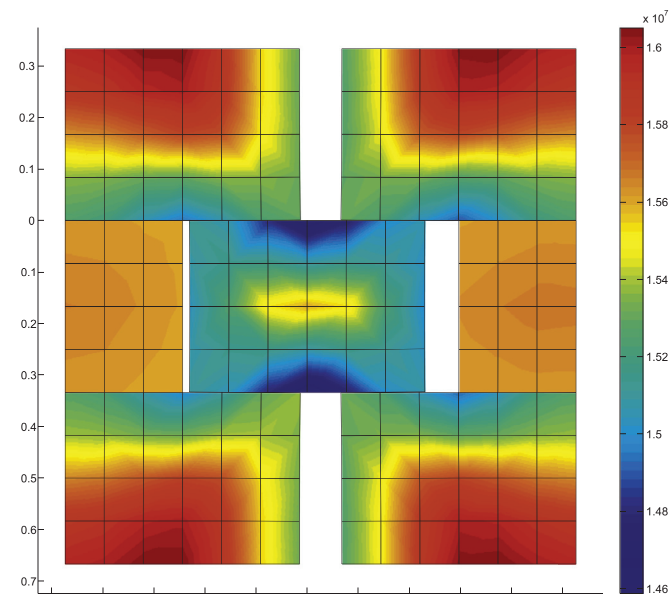



**Fig. 9.** Loss of solution uniqueness (deformed configuration and Von Mises stress field).

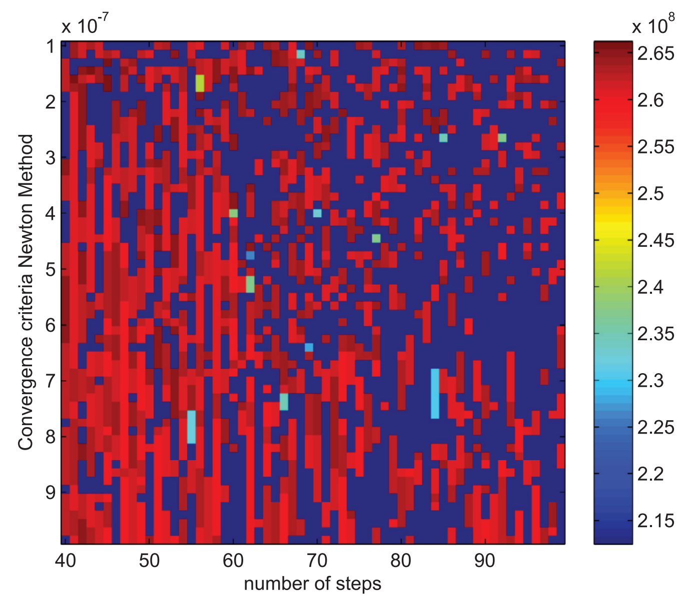



**Fig. 10.** Bifurcation study, value of $\sigma^H_{2222}$ at the end of an oedometric loading, parametric study changing the number of time steps and convergence criteria.

### 6.4. Bifurcation and loss of solution uniqueness

Uniqueness cannot be assured in problems presenting softening (e.g. Chambon et al., 2004). It can be proved that when the stress decreases the discharge branch can have any slope, and the solution is not known a-priori. From the point of view of a PDE formulation, when some of the eigenvalues of the stiffness tensor become zero, the problem loses its ellipticity in those points, so any strain applied in the direction of the vanishing eigenvalues will fulfill the equilibrium condition, thus making the solution non-unique.

The loss of uniqueness, or bifurcation, in numerical modeling is an interesting phenomenon that agrees with experimental observations; nevertheless, it represents a numerical challenge, i.e. decrease of efficiency of Newton's method, and a practical issue as some of the solutions may not be found, representing a main concern in e.g. reliability or failure analysis.

In this section, some different possible solutions for a biaxial compression test are explored. This attempt should not be considered as an exhaustive procedure to find all the possible solutions but rather a demonstration of the existence of bifurcation.

The procedure consists in solving the same problems presented in the previous sections using slightly different numerical parameters, i.e. the number of steps of the applied loading path and the convergence criterion for Newton's method. 3600 computations are performed and the results of the asymptotic post-peak value of the homogenized stress $\sigma^H_{2222}$ for a strain $\epsilon_{11} = 0.85\%$ is shown in (Fig. 10). The results put in evidence the existence of two solution attractors, one around $\sigma^H_{2222} = 2.1 \times 10^8$Pa and another around $\sigma^H_{2222} = 2.6 \times 10^8$Pa. Two different localization modes of the crack network exist, thus representing two possible solutions of the same problem. Moreover, other solutions (0.6% of the total) between the two previous attractors exist (see also Fig. 11).

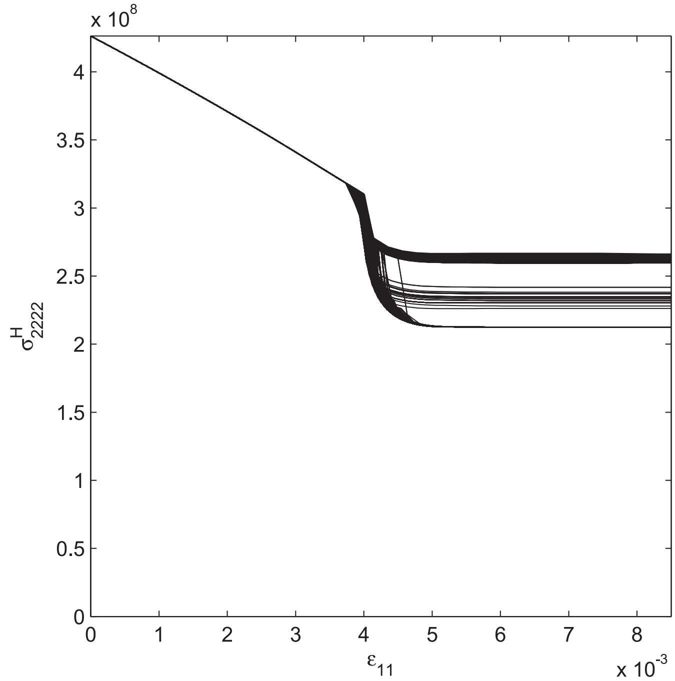



**Fig. 11.** Bifurcation study: evolution of $\sigma^H_{2222}$ vs $\epsilon_{11}$.

## 7. Conclusions

A theoretical and numerical approach for describing the macroscopic poroelastic properties of a saturated, deformable, cracked porous medium has been presented in this paper.

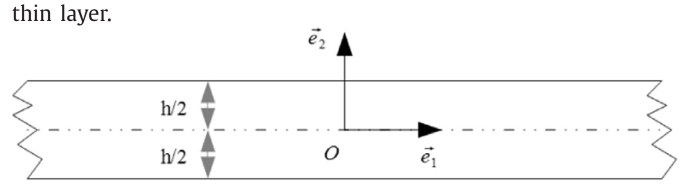
The first part of the paper is devoted to the formulation of the equations describing the hydro-mechanical behavior of the cracked fully-saturated porous medium. Under the classical assumption of well-separated scales, the asymptotic homogenization method has allowed to obtain the macroscopic description of the whole porous medium, considered formed by a porous matrix and a periodic network of interconnected infilled discontinuities, containing a poroelasto-damageable material. The homogenized coefficients (elastic tensor, Biot's coefficient and modulus, permeability tensor) embody all the information about the microstructure by means of the characteristic functions that describe the small oscillations of the primary variables of the problem. The numerical solution of the problem over the elementary cell has allowed to obtain the homogenized coefficients in the linear case. The macroscale reflects the anisotropy coming from the microscale configuration. The damage associated to the cracks' opening implies a degradation of the homogenized stresses and results in a non-linear problem requiring a Newton-like procedure. The methodology has been applied to two different cases, a strain controlled path, i.e. an oedometric test and a biaxial test This latter problem correspond to a general case as it requires the development of a controllability scheme so that the constitutive law for the given REV and any loading path can be obtained. At some point of the time-history (as expected for a mechanical damage problem), for high enough values of damage, bifurcation is observed, leading to a quick degradation of the homogenized stresses and to a loss of solution uniqueness. This latter aspect has been discussed but future works should explore other aspects such as the influence of the cell size.

The proposed method allows to numerically obtain the material constitutive behavior for an REV of a porous medium characterized by infilled discontinuities. The obtained law can be therefore adopted in any multiscale approach (e.g. Kouznetsova et al., 2001) so that real-scale geomechanics or engineering problems can be treated , e.g. Nguyen et al. (2014), Guo and Zhao (2014) or Guo and Zhao (2015).

## Appendix A. Very thin layer of a deformable porous medium

The purpose of this appendix is to heuristically justify the poroelastic modeling (see Eq. (2)) of a very thin layer of poroelastic medium of weak stiffness and large permeability. A more rigorous approach for a partly similar but simpler case can be found in Caillerie (1983).

Let's consider a very thin 2D deformable porous medium of thickness $h$ in the $x_2$ direction governed by Biot's equations. This thin layer is embedded in another deformable porous medium the poroelastic characteristics do not depend on the thickness of the thin layer.




The porosity of this medium is assumed to be close to 1 so that its elastic coefficients are very small and its permeability very high. To make that precise, we consider Eq. (1) where the elastic stiffness tensor $c^h$, Biot's tensor and coefficient $\alpha^h$ and $\beta^h$ and the permeability $k^h$ depend on $h$ in the following manner:

$$c^h = h\tilde{C}\ ;\ \alpha^h = \alpha\ (\alpha^h \text{ is constant})\ ;\ \beta^h = \frac{\beta}{h}\ ;\ k^h = \frac{K}{h}$$

Taking into account the symmetries $\tilde{C}_{ijkl} = \tilde{C}_{ijlk}$ and $\alpha_{ij} = \alpha_{ji}$, Biot's equation can be rewritten, using the index notation:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{A.1a}$$

$$\sigma_{ij} = h\tilde{C}_{ijkl}\frac{\partial u_k}{\partial x_l} - p\alpha_{ij} \tag{A.1b}$$

$$\kappa = \alpha_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\beta}{h}p \tag{A.1c}$$

$$\frac{\partial q_i}{\partial x_i} + \dot{\kappa} = 0 \tag{A.1d}$$

$$q_i = -\frac{K}{h}\frac{\partial p}{\partial x_i} \tag{A.1e}$$

Due to the continuity conditions on the two sides of the layer, it is consistent to assume that the displacement and the pressure in the layer are of order 1 with respect to $h$:

$$\vec{u} = O(1) \tag{A.2a}$$

$$p = O(1) \tag{A.2b}$$

Consistently with Eq. (A.1), the stress tensor $\sigma$ and the relative flow of fluid $\vec{q}$ depend on $h$ as :

$$\sigma = O(1) \tag{A.3a}$$

$$q_1 = O(h^{-1}) \tag{A.3b}$$

$$q_2 = O(1) \tag{A.3c}$$

moreover, according to (A.1c), the variation of porosity depends on $h$ as:

$$\kappa = 0\left(h^{-1}\right) \tag{A.4}$$

The integration of the balance equation (A.1a) in $x_2$ over $[-\frac{h}{2}, \frac{h}{2}]$ yields:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{i1}}{\partial x_1} dx_2 + [[\sigma_{i2}]] = 0 \tag{A.5}$$

where the jump $[[f]]$ of a function $f$ is:

$$[[f]] = f\left(\frac{h}{2}\right) - f\left(-\frac{h}{2}\right)$$

as $\sigma = O(1)$, for $h \to 0$, Eq. (A.5) reads:

$$[[\sigma_{i2}]] = 0$$

Which means that $\sigma_{i2}$ is continuous.

The integration of (A.1c) yields:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \kappa \, dx_2 = \alpha_{i1} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u_i}{\partial x_1} dx_2 + \alpha_{i2}[[u_i]] + \frac{\beta}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} p \, dx_2$$

which, taking into account the order of magnitude of $p$ and $\kappa$ with respect to $h$ (see (A.2a) and (A.4)), yields in the limit $h \to 0$:

$$\kappa^c = A_i[[u_i]] + \beta p$$

where the components of the vector $\vec{A}$ are $A_i = \lim_{h\to 0} \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha_{ij} \, dx_2$ and:

$$\kappa^c = \lim_{h\to 0} \int_{-\frac{h}{2}}^{\frac{h}{2}} \kappa \, dx_2$$

In the same way, the integration of (A.1d) yields:

$$\frac{d}{dx_1} \int_{-\frac{h}{2}}^{\frac{h}{2}} q_1 dx_2 + [[q_2]] + \int_{-\frac{h}{2}}^{\frac{h}{2}} \dot{\kappa} \, dx_2 = 0$$

according to the order of magnitude of $q_1$, $q_2$ and $\kappa$ (see (A.3) and (A.4)), for $h \to 0$ that reads:

$$\frac{d}{dx_1} Q + [[q_2]] + \dot{\kappa}^c = 0$$

where $\eta$ is assumed independent of $x_2$ and where:

$$Q = \lim_{h\to 0} \int_{-\frac{h}{2}}^{\frac{h}{2}} q_1 dx_2$$

The integration of the constitutive Eq. (A.1b) yields:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} \, dx_2 = h\tilde{C}_{ijk1} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial u_k}{\partial x_1} dx_2 + h\tilde{C}_{ijk2}[[u_k]] - \int_{-\frac{h}{2}}^{\frac{h}{2}} p\alpha_{ij} \, dx_2$$

Passing to the limit $h \to 0$ after dividing by $h$, that yields:

$$\sigma_{i2} = C_{ik}[[u_k]] - pA_i$$

where the components of the matrix of the linear application $C$ and of the vector $\vec{A}$ are respectively $C_{ij} = \tilde{C}_{i2j2}$ and $A_i = \lim_{h\to 0} \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \alpha_{i2} \, dx_2$. It can be seen that according to the usual symmetries of elastic stiffnesses the crack stiffness tensor $C$ is symmetrical.

**Remark 2.** If the material of the crack is isotropic then $\tilde{C}_{1222} = 0$ and $C_{12} = C_{21} = 0$. In the same way, if Biot's tensor $\alpha$ of the material of the crack is proportional to the identity $\mathbb{I}$ then $A_1 = 0$, that is to say that $\vec{A}$ is normal to the crack.

The integration of Darcy's law (A.1b) yields:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} q_1 \, dx_2 = -K\frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial p}{\partial x_1} dx_2$$

that yields, passing to the limit $h \to 0$:

$$Q = -K\frac{\partial p}{\partial x_1}$$

It can be noticed that, in the considered case, $\sum_i \sigma_{i2}\vec{e}_i = \sigma \,.\, \vec{n} = \vec{T}$ and $q_2 = \vec{q} \,.\, \vec{n}$, where $\vec{n} = \vec{e}_2$ is the normal to the thin layer, and moreover that $x_1$ is the curvilinear coordinate $s$ along the crack so the equations along the crack can be written:

$$\vec{T} = C \,.\, [[\vec{u}]] - p\vec{A}$$
$$\kappa^c = \vec{A}.[[\vec{u}]] + \beta p$$
$$\frac{dQ}{dl} + [[\vec{q}]].\vec{n} + \dot{\kappa}^c = 0$$
$$Q = -K\frac{dp}{dl}$$

which are the interface conditions to be considered along any crack even not a straight one. Moreover, if the material of the crack is isotropic and its Biot's tensor $\alpha$ is proportional to the identity then the stiffness tensor of the crack and its Biot's vector read:

$$C = C_T\vec{\tau} \otimes \vec{\tau} + C_N\vec{n} \otimes \vec{n} \tag{A.6}$$

$$\vec{A} = A\vec{n} \tag{A.7}$$

## Appendix B. Proof of the damage evolution law (30) at order zero

According to (11d), the expansion of $\| [[\vec{u}^{(e)}]] \|$ reads:

$$\| [[\vec{u}^{(e)}]] \| = e\| [[\vec{u}^{(1)}]] \| + e^2(\cdots)$$

therefore, that of $f\left(\frac{\| [[\vec{u}^{(e)}]] \|}{D^{(e)}}\right)$ reads:

$$f\left(\frac{\| [[\vec{u}^{(e)}]] \|}{D^{(e)}}\right) = f\left(\frac{\| [[\vec{u}^{(1)}]] \|}{D}\right) + e^2(\cdots)$$

Using that expansion, the evolution law (28) entails that:

$$\forall e, t, \tau, 0 \le \tau \le t \,,\, d^{(0)}(t) + ed^{(1)}(t) + e^2d^{(2)}(t) + \cdots$$
$$\ge f\left(\frac{\| [[\vec{u}^{(1)}]](\tau) \|}{D}\right) + e^2(\cdots)$$

then necessarily:

$$\forall t, \tau, 0 \le \tau \le t \,,\, d^{(0)}(t) \ge f\left(\frac{\| [[\vec{u}^{(1)}]](\tau) \|}{D}\right) \tag{B.1}$$

otherwise for any small enough $e$, $d^{(e)}(t)$ would be less than $f\left(\frac{\| [[\vec{u}^{(e)}]](\tau) \|}{D}\right)$ for some $\tau$ comprised between 0 and $t$. Eq. (B.1) is clearly equivalent to:

$$\forall t \ge 0 \,,\, d^{(0)}(t) \ge \sup_{0\le\tau\le t} f\left(\frac{\| [[\vec{u}^{(1)}]](\tau) \|}{D}\right)$$

Moreover, if for some $\tau$, $0 \le \tau \le t$, $d^{(0)}(t)$ is such that:

$$d^{(0)}(t) > f\left(\frac{\| [[\vec{u}^{(1)}]](\tau) \|}{D}\right)$$

then for small enough $e$'s we have:

$$d^{(e)}(t) > f\left(\frac{\| [[\vec{u}^{(e)}]](\tau) \|}{D^{(e)}}\right) \tag{B.2}$$

That means that $d^{(0)}(t) > \sup_{0\le\tau\le t} f(\frac{\| [[\vec{u}^{(1)}]](\tau) \|}{D})$ entails that, or small enough $e's$:

$$d^{(e)}(t) > \sup_{0\le\tau\le t} f\left(\frac{\| [[\vec{u}^{(e)}]](\tau) \|}{D^{(e)}}\right)$$

which is not possible for $d^{(e)}(t)=\sup_{0\le\tau\le t} f\left(\frac{\|[[\vec{u}^{(e)}]](\tau)\|}{D^{(e)}}\right)$, consequently $d^{(0)}(t)$ is exactly the maximum of $f\left(\frac{\|[[\vec{u}^{(1)}]](\tau)\|}{D}\right)$:

$$\forall t\ge 0\,,\; d^{(0)}(t)=\sup_{0\le\tau\le t} f\left(\frac{\left\|\left[\left[\vec{u}^{(1)}\right]\right](\tau)\right\|}{D}\right)$$

which is the evolution law (30).

## Appendix C. Symmetries

Let $\vec{u}(\tau)$, defined and periodic on $Y$, and $\Sigma(\tau)$ belonging to $\mathcal{L}^S$ (space of the symmetric second order tensors), $0\le\tau\le t$, be the solution of the problem:

Given the histories $\{E(\tau)\,;\,0\le\tau\le t\}$ and $\{P(\tau)\,;\,0\le\tau\le t\}$,
find $\vec{u}(\vec{x},\vec{y},\tau)$, $d(\vec{x},\vec{y},\tau)$, $Y$-periodic and $\Sigma(\tau)$, $\tau\in[0,t]$,
such that:

$$\begin{aligned}&\forall\,\tau\in[0,t]\forall\vec{v}, Y\text{-periodic}, \forall E^*\in\mathcal{L}^S\,,\;\int_Y \left(c:(E(\tau)+\epsilon^y(\vec{u}(\tau)))\right.\\&\left.-P(\tau)\alpha\right):\epsilon^y(\vec{v})\,\mathrm{d}s_y+\int_{\Gamma^Y}\left((1-d(\tau))C\,.\,[[\vec{u}(\tau)]]-P(\tau)\vec{A}\right)\\&.[[\vec{v}]]\,\mathrm{d}l_y-|Y|\Sigma(\tau):E^*=0\\&\text{with } d(\tau)=\sup_{0\le\rho\le\tau} f\left(\frac{\|[[\vec{u}(\rho)]]\|}{D}\right)\end{aligned}\tag{C.1}$$

where $\forall\tau$, $E(\tau)\in\mathcal{L}^S$ and $P(\tau)\in\mathbb{R}$.

Taking $E^*=0$, $E(\tau)=\epsilon^x\left(\vec{u}^{(0)}(\tau)\right)$ and $P(\tau)=p^{(0)}(\tau)$, the previous problem comes down to (31). Posing $\sigma=c:\epsilon^y(E(\tau).\vec{y}+\vec{u}(\tau))-P\alpha$ and taking $\vec{v}=0$ yield $\Sigma(\tau)=\langle\sigma(\tau)\rangle=\frac{1}{|Y|}\int_Y\sigma(\tau)\,\mathrm{d}s_y$.

Now, let's assume that there exists an isometric-valued function $\tau\to R(\tau)$ defined in [0, $t$] ($R^{-1}(\tau)=R^T(\tau)$) such that, for all $\tau$, $R(\tau)$ leaves the cell $Y$ and the cracks $\Gamma^Y$ unchanged and such that the tensors $c$, $\alpha$, $C$ and the vector $\vec{A}$ satisfy:

$$\begin{aligned}&\forall M,N\in\mathcal{L}^S,\left(R(\tau)\circ M\circ R^T(\tau)\right):\left(c(R(\tau).\vec{y}):\left(R(\tau)\circ N\circ R^T(\tau)\right)\right)\\&\quad=M:(c(\vec{y}):N)\end{aligned}\tag{C.2a}$$

$$\left(R^T(\tau)\circ\alpha(R(\tau).\vec{y})\circ R(\tau)\right)=\alpha(\vec{y})\tag{C.2b}$$

$$\left(R^T(\tau)\circ C(R(\tau).\vec{y})\circ R\right)=C(\vec{y})\tag{C.2c}$$

$$\forall\vec{y}\in Y\,,\;R^T(\tau).\vec{A}(R(\tau).\vec{y})=\vec{A}(\vec{y})\tag{C.2d}$$

Under those conditions, we have the following lemma:

**Lemma.** *Let $\vec{u}(\tau)$ and $\Sigma(\tau)$ be the solution of (C.1) for the data $E(\tau)$ and $P(\tau)$, $0\le\tau\le t$ then $R(\tau).\vec{u}(\tau)$ and $R^T(\tau)\circ\Sigma(\tau)\circ R(\tau)$ are solution of (C.1) for the data $R(\tau)\circ E\circ R^T(\tau)$ and $P(\tau)$.*

**Proof.** All the following algebraic calculi being performed for any time $\tau$ and the sake of simplicity, the variable $\tau$ is omitted.

Taking into account the relations (C.2), (C.1) reads:

$$\begin{aligned}&\forall\vec{v}, Y\text{-periodic}, \forall E^*\in\mathcal{L}^S\,,\;\int_Y\left[c(R.\vec{y}):\left(R\circ\epsilon^y(E.\vec{y}+\vec{u})\circ R^T\right)\right]:\\&\left(R\circ\epsilon^y(E^*.\vec{y}+\vec{v})\circ R^T\right)\mathrm{d}s_y\\&-\int_Y P\left(R^T\circ\alpha(R.\vec{y})\circ R\right):(\epsilon^y(E^*.\vec{y}+\vec{v})\circ)\,\mathrm{d}s_y\\&+\int_{\Gamma^Y}\left(\left(R^T\circ C(R.\vec{y})\circ R\right).[[\vec{u}]]-PR^T.\vec{A}(R.\vec{y})\right)\\&.[[\vec{v}]]\,\mathrm{d}l_y-|Y|\Sigma:E^*=0\end{aligned}$$

that is too:

$$\begin{aligned}&\forall\vec{v}, Y\text{-periodic}, \forall E^*\in\mathcal{L}^S\,,\;\int_Y\left[c(R.\vec{y}):\left(\epsilon^y(R\circ E.\vec{y}+R.\vec{u})\circ R^T\right)\right]:\\&\left(\epsilon^y(R\circ E^*.\vec{y}+R.\vec{v})\circ R^T\right)\mathrm{d}s_y\\&-\int_Y P\alpha(R.\vec{y}):\left(\epsilon^y(R\circ E^*.\vec{y}+R.\vec{v})\circ R^T\right)\mathrm{d}s_y\\&+\int_{\Gamma^Y}\left((C(R.\vec{y})).(R.[[\vec{u}]])-P\vec{A}(R.\vec{y})\right)\\&.(R.[[\vec{v}]])\,\mathrm{d}l_y-|Y|\Sigma:E^*=0\end{aligned}$$

In the change of variables $\vec{y}\leftrightarrow\tilde{\vec{y}}=R.\vec{y}$, the cell $Y$ and the cracks $\Gamma^Y$ remained unchanged. Moreover, $\nabla^y\vec{v}=\nabla^{\tilde{y}}\vec{v}\circ R$ and, as $R$ is an isometry, $\mathrm{d}s_{\tilde{y}}=\mathrm{d}s_y$ and $\mathrm{d}l_{\tilde{y}}=\mathrm{d}l_y$ on $\Gamma^Y$. So, by this change of variables, the previous formulation reads:

$$\begin{aligned}&\forall\vec{v}, Y\text{-periodic}, \forall E^*\in\mathcal{L}^S\,,\\&\int_Y\left(c\left(\tilde{\vec{y}}\right):\left(\epsilon^{\tilde{y}}\left(R\circ E\circ R^T.\tilde{\vec{y}}+R.\vec{u}\right)\right)-P\,\alpha\left(\tilde{\vec{y}}\right)\right):\\&\left(\epsilon^{\tilde{y}}\left(R\circ E^*\circ R^T.\tilde{\vec{y}}+R.\vec{v}\right)\right)\mathrm{d}s_{\tilde{y}}+\int_{\Gamma^Y}\left(\left(C\left(\tilde{\vec{y}}\right)\right).(R.[[\vec{u}]])-P\vec{A}\left(\tilde{\vec{y}}\right)\right)\\&.(R.[[\vec{v}]])\,\mathrm{d}l_{\tilde{y}}-|Y|\Sigma:E^*=0\end{aligned}$$

that is too, posing $\tilde{\vec{v}}=R.\vec{v}$ and $\tilde{E}^*=R\circ E^*\circ R^T$:

$$\begin{aligned}&\forall\tilde{\vec{v}}, Y\text{-periodic}, \forall\tilde{E}^*\in\mathcal{L}^S\,,\;\int_Y\left(c\left(\tilde{\vec{y}}\right):\right.\\&\left.\left(\epsilon^{\tilde{y}}\left(R\circ E\circ R^T.\tilde{\vec{y}}+R.\vec{u}\right)\right)-P\,\alpha\left(\tilde{\vec{y}}\right)\right):\left(\epsilon^{\tilde{y}}\left(\tilde{E}^*.\tilde{\vec{y}}+\tilde{\vec{v}}\right)\right)\mathrm{d}s_{\tilde{y}}\\&+\int_{\Gamma^Y}\left(\left(C\left(\tilde{\vec{y}}\right)\right).(R.[[\vec{u}]])-P\vec{A}\left(\tilde{\vec{y}}\right)\right)\\&.\left(\left[\left[\tilde{\vec{v}}\right]\right]\right)\mathrm{d}l_{\tilde{y}}-|Y|\left(R^T\circ\Sigma\circ R\right):\tilde{E}^*=0\end{aligned}$$

The comparison of that formulation with (C.1) shows that $R(\tau).\vec{u}$ and $R^T(\tau)\circ\Sigma\circ R(\tau)$ are the solution of the problem (C.1) for the data $R(\tau)\circ E\circ R^T(\tau)$ and $P$. □

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