Economie appliquée, tome XXXVII - 1984 - N° 1, pp. 215-222

Structural decomposition of multipliers in input-output or social accounting matrix analysis

*Yves Crama, Jacques Defourny and Jules Gazon**

Université de Liège, Belgium

1. THE THEOREM OF INFLUENCE

Both input-output and SAM (social accounting matrix) analysis are concerned with the study of a matrix D^{-1} of multipliers, where $D = (I - A)$ and A denotes a square (n × n) — matrix of technical coefficients or average expenditure propensities. Each element (i, j) of D^{-1} gives the variation in output, or income, of the sector i, due to an increase of one unit of output, or income, in the sector j. So, according to Lantner's (1974) and Gazon's (1976) terminology, we call it the *global influence* $I^G_{(j \to i)}$ *of the pole* j on the *pole* i, while the corresponding element a_{ij} of A is called the *direct influence* of j on i.

Among the main contributions of Lantner's and Gazon's structural approach is the concept of *total influence*, which allows for a meaningful decomposition of the global influence. In order to define this new concept, we consider the graph G = (V, R): here, V denotes a set of n vertices associated with the poles of A, and R is a set of arcs, such that the arc $(j, i) \in R$ if and only if a_{ij} is non-null (see figure 1).

An elementary path from j to i is a sequence $(k_o, \ldots, k_r)$ of vertices, such that $k_o = j$, $k_r = i$, $(k_s, k_{s+1}) \in R$ for every s and $k_t \neq k_s$ if $t \neq s$: it is thus a path which does not pass more than once through a same vertex. In figure 1, there are three elementary paths from j to i: (j, x, y, i), (j, s, i) and (j, v, s, i). Now, the *total influence* of j *on* i $(j \neq i)$ *along an elementary path* $p = (j, k_1, \ldots, k_1, i)$ measures those effects at i of a unit change at j which are directly imputable to the path p, i.e., the combined effect of the direct influence along p

* We thank L. Bragard for his helpful comments.

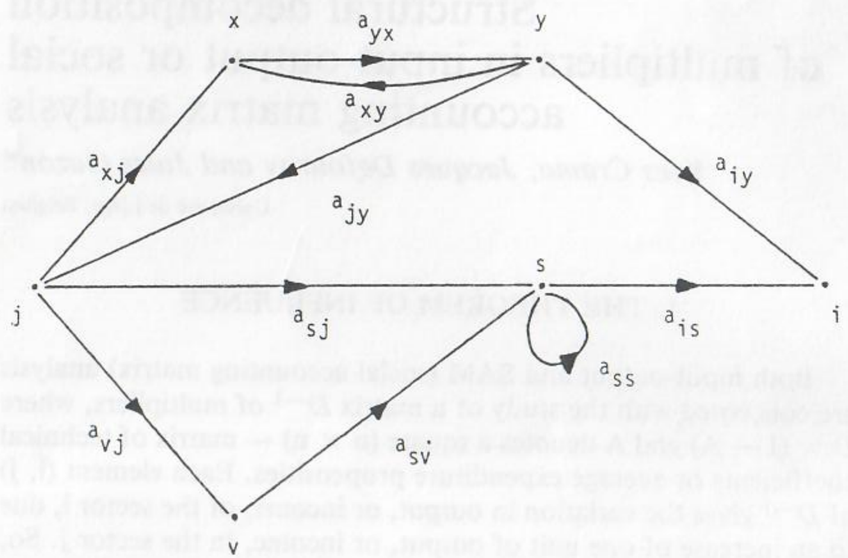

Figure 1. Typical graph associated with a (6 × 6) — matrix of technical coefficients or average expenditure propensities.

(equal to the product $a_{k_1 j}\, a_{k_2 k_1} \ldots a_{ik_r}$) and of the indirect effects induced by the circuits adjacent to p (i.e., those circuits which have one or more vertices in common with p). We denote it by $I^T_{(j \to i)p}$.

Lantner (1974, pp. 242-246) and Gazon (1976, pp. 130-133) have shown that these direct and indirect effects combine multiplicatively, in such a way that the total influence of j on i along p is given by

$$I^T_{(j \to i)p} = a_{k_1 j}\, a_{k_2 k_1} \ldots a_{ik_r}\, \Delta^{(p)} / \Delta \tag{1}$$

where Δ denotes the determinant of *D*, and $\Delta^{(p)}$ denotes the determinant of the submatrix of *D* obtained by removing the rows and the columns associated with the poles j, $k_1, k_2, \ldots, k_r$, i.

The ratio $\Delta^{(p)} / \Delta$ is called the *multiplier of the path* p as it captures the factor by which the direct influence along p is amplified through the effects of adjacent circuits and resulting feedbacks.

On this basis, Lantner (1974, pp. 246-247) and Gazon (1976, pp. 134-135) propose the following «*theorem of influence*»: *the global influence of a pole j on a pole i is equal to the sum of the total influences of j on i along all elementary paths joining j to i.*

More formally, if we denote by P_{ji} the set of all elementary paths from j to i:

$$I^G_{(j \to i)} = \sum_{p \in P_{ji}} I^T_{(j \to i)p} \tag{2}$$

This appealing result, which allows fruitful investigations beyond the traditional scope of input-output or SAM analysis (see, e.g., Defourny [1982], Defourny and Thorbecke [1983]), has been established in Gazon (1976), by proving that the total influence along an elementary path takes into account *all* the effects due to that path, and *only* those ones. The proof relies on a careful inspection of the transmission of the influence within the structure G, and may thus seem rather involved.

We propose hereunder a more classical, though less intuitive, algebraic proof of the theorem of influence.

2. THE PROOF

Let us first introduce a useful notation: in the sequel, we denote by

$$\Delta^{k_1, \ldots, k_r}_{j_1, \ldots, j_r}$$

the determinant of the submatrix we obtain after removing the rows $k_1, \ldots, k_r$ and the columns $j_1, \ldots, j_r$ from *D*. So, for instance, Δ^j_i denotes the minor of the element (j, i) of *D*, and Δ^p_p is identical to the determinant $\Delta^{(p)}$ defined above.

By usual matrix algebra rules, we know that:

$$I^G_{(j \to i)} = (-1)^{i+j}\, \Delta^j_i / \Delta \tag{3}$$

Hence, if we assume (without restriction) that i = n and j = 1, and if p denotes a path $(k_o, \ldots, k_r)$, (1) and (3) entail that the thesis (2) is equivalent to:

$$(-1)^{1+n}\, \Delta^1_n = \sum_{p \in P_{1n}} \Big(\prod_{k_i \in p} a_{k_{i+1} k_i} \Big)\, \Delta^{(p)} \tag{4}$$

The proof proceeds in two steps: first, we show that Δ_n^1 can be split into a sum of terms:

$$d^{(p)} = (-1)^m \prod_{k_i \in p} a_{k_{i+1}k_i} \Delta^{(p)} \tag{5}$$

where p denotes an elementary path from 1 to n, and the sum extends over all those paths. Then, we prove that $(-1)^m = (-1)^{1+n}$, which completes the proof.

Step 1. Let us expand Δ_n^1 along the first column of *D*. We get:

$$\Delta_n^1 = \sum_{k=2}^{n} (-1)^{\alpha_k} (-a_{k1}) \Delta_{1,n}^{1,k} \tag{6}$$

(we are not concerned now with the value of α_k). The term corresponding to k = n in (6) can be rewritten under the form given in (5).

So, let us now consider:

$$\Delta_{1,n}^{1,l}$$

with $2 \leqslant l < n$, and let us expand it along the column l. Then, we obtain:

$$\Delta_{1,n}^{1,l} = \sum_{\substack{k=2 \\ k \neq l}}^{n} (-1)^{\beta_k} (-a_{kl}) \Delta_{1,l,n}^{1,l,k} \tag{7}$$

Again, the term corresponding to k = n presents the desired structure, and we can expand

$$\Delta_{1,l,n}^{1,l,j} (2 \leqslant j < n, j \neq 1)$$

along the column j of *D*.

By iterating the procedure, we see that it is possible to decompose Δ_n^1 into a sum of terms $d^{(p)}$.

Due to the rule we apply in order to select the column along which we expand the sub-determinants, it is straightforward to check that the list of vertices appearing in the definition of $d^{(p)}$ always determines an elementary path from 1 to n. Moreover, all such paths are taken into consideration in our decomposition.

Step 2. Let us consider a particular path $p = (1, k_1, \ldots, k_r, n)$, and the corresponding term:

$$d^{(p)} = (-1)^\alpha (-a_{k_1 1})(-a_{k_2 k_1}) \ldots (-a_{nk_r}) \Delta^{(p)}$$
$$= (-1)^m a_{k_1 1} a_{k_2 k_1} \ldots a_{nk_r} \Delta^{(p)} \tag{8}$$

Let $k_o = 1$, $k_{r+1} = n$ and $n_i = \#\{j \mid j < i \text{ and } k_j < k_i\}$ (n_i is the number of columns (resp. rows) which have been selected before the column (resp. the row) k_i and which are on the left of k_i (resp. above k_i) in *D*).

In particular, $n_o = 0$, $n_1 = 1$ and $n_{r+1} = r + 1$. Now, when we expand Δ_n^1 and obtain $d^{(p)}$, the coefficient of $(-a_{k_{i+1}k_i})$ is equal to $(-1)^{\nu_i}$, with $\nu_i = (k_i - n_i) + (k_{i+1} - n_{i+1})$.

So, the numerical value of m in the formula (8) is given by:

$$m = \sum_{i=0}^{r} (k_i - n_i + k_{i+1} - n_{i+1}) + (r + 1)$$

and

$$(-1)^m = (-1)^{1+n}$$

This completes the second step and the proof.

2. AN EXAMPLE

In order to illustrate this result, let us come back to the particular case of the structure represented by the network in figure 1.

In this specific case, the matrix (I — A) is the following:

	j	x	y	v	s	i
j..........	1		$-a_{jy}$			
x..........	$-a_{xj}$	1	$-a_{xy}$			
y..........		$-a_{yx}$	1			
v..........	$-a_{vj}$			1		
s..........	$-a_{sj}$			$-a_{sv}$	$1-a_{ss}$	
i..........			$-a_{iy}$		$-a_{is}$	1

and Δ is the determinant of matrix $(I - A)$

From the «theorem of influence», it follows that the global influence $I^G_{(j\to i)}$, which is the element (i, j) of $(I - A)^{-1}$, is equal to the sum of the total influences of the three elementary paths spanning poles i and j. If we call (j, x, y, i) path 1, (j, s, i) path 2 and (j, v, s, i) path 3, we can thus write:

$$I^G_{(j\to i)} = I^T_{(j\to i)1} + I^T_{(j\to i)2} + I^T_{(j\to i)3}$$

Or, given the definition of the total influence in (1):

$$I^G_{(j\to i)} = a_{xj}\, a_{yx} a_{iy} \frac{\Delta^{(1)}}{\Delta} + a_{sj}\, a_{is} \frac{\Delta^{(2)}}{\Delta} + a_{vj}\, a_{sv} a_{is} \frac{\Delta^{(3)}}{\Delta}$$

where:

$$\Delta^{(1)} = \begin{vmatrix} 1 & 0 \\ -a_{sv} & 1-a_{ss} \end{vmatrix} = 1 - a_{ss}$$

$$\Delta^{(2)} = \begin{vmatrix} 1 & -a_{xy} & 0 \\ -a_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 - a_{xy}a_{yx}$$

$$\Delta^{(3)} = \begin{vmatrix} 1 & -a_{xy} \\ -a_{yx} & 1 \end{vmatrix} = 1 - a_{xy}a_{yx}$$

Let us verify that we recover the global influence as expressed in (3) which is the usual matrix algebra result: if we expand Δ^j_i by minors according to the elements of its first column, we obtain:

$$\Delta^j_i = (-a_{xj}) \begin{vmatrix} -a_{yx} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_{sv} & 1-a_{ss} \\ 0 & -a_{iy} & 0 & -a_{is} \end{vmatrix}$$

$$\Delta^j_i + (-a_{vj}) \begin{vmatrix} 1 & -a_{xy} & 0 & 0 \\ -a_{yx} & 1 & 0 & 0 \\ 0 & 0 & -a_{sv} & 1-a_{ss} \\ 0 & -a_{iy} & 0 & -a_{is} \end{vmatrix}$$

$$-(-a_{sj}) \begin{vmatrix} 1 & -a_{xy} & 0 & 0 \\ -a_{yx} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -a_{iy} & 0 & -a_{is} \end{vmatrix}$$

The first minor above can be further expanded by suppressing colum x, the second by suppressing colum v and the third one by suppressing column s:

$$\Delta^j_i = (-a_{xj})(-a_{yx}) \begin{vmatrix} 0 & 1 & 0 \\ 0 & -a_{sv} & 1-a_{ss} \\ -a_{iy} & 0 & -a_{is} \end{vmatrix}$$

$$+ (-a_{vj})(-a_{sv}) \begin{vmatrix} 1 & -a_{xy} & 0 \\ -a_{yx} & 1 & 0 \\ 0 & -a_{iy} & -a_{is} \end{vmatrix}$$

$$-(-a_{sj})(-a_{is}) \begin{vmatrix} 1 & -a_{xy} & 0 \\ -a_{yx} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Or finally:

$$(-1)^{1+6}\,\Delta_i^j = a_{xi}a_{yx}a_{iy}\,\Delta^{(1)} + a_{vj}a_{sv}a_{is}\,\Delta^{(3)}$$

$$+ a_{sj}a_{is}\,\Delta^{(2)}$$

Then dividing by Δ we recover:

$$I^G_{(j\to i)} = \frac{(-1)^{i+j}\cdot\Delta_i^j}{\Delta} = I^T_{(j\to i)1} + I^T_{(j\to i)2} + I^T_{(j\to i)3}$$

RÉSUMÉ

Cette note fournit une démonstration algébrique du «théorème de l'influence», lequel permet une décomposition particulièrement intéressante des multiplicateurs obtenus par l'analyse input-output ou par l'analyse de matrices sociales comptables. Ce théorème a été établi par R. Lantner (1974) et démontré par J. Gazon (1976) dans le cadre de leur analyse structurale.

ABSTRACT

We provide an algebraic proof of the so-called theorem of influence, which yields a useful decomposition of multipliers in input-output or SAM analysis. This theorem was previously stated by Lantner (1974) and proved by Gazon (1976) in the framework of structural analysis.

REFERENCES

Defourny, J., 1982, «Une approche structurale pour l'analyse input-output: un premier bilan», Economie Appliquée, tome XXXVI, 203-230.

Defourny, J. and Thorbecke, E., «Structural path analysis and multiplier decomposition within a social accounting matrix framework», to appear in the *Economic Journal*, March 1984.

Gazon, J., 1976, *Transmission de l'influence économique — Une approche structurale*, Collection de l'I.M.E. (n° 13) (Sirey, Paris).

Lantner, R., 1974, *Théorie de la dominance économique* (Dunod, Paris).