General constitutive equations of heat transport at small length scales and high frequencies with extension to mass and electrical charge transport

Hatim Machrafi[a,*], Georgy Lebon[a]
[a]University of Liège, Thermodynamics of Irreversible Processes, Allée du 6-Août, 19, 4000 Liège, Belgium
*Corresponding author: h.machrafi@ulg.ac.be

Abstract
A generalized heat transport equation applicable at small length and short time scales is proposed. It is based on extended irreversible thermodynamics with an infinite number of high-order heat fluxes selected as state variables. Extensions of Fick's and Ohm's laws are also formulated. As a numerical illustration, heat conduction in a rigid body subject to fixed and oscillatory temperature boundary conditions is discussed.

Keywords
Ballistic heat transport; Small length-systems; High-frequency; Extended irreversible thermodynamics

1 Introduction

The increasing interest in nano-technology has led to new insights in the study of heat transport. It is well known that heat transfer at micro and nanoscale behaves differently from that at macroscales [1,2]. At small length scales, transport of heat in complex systems is best quantified by means of the so-called Knudsen number $Kn \equiv \ell/L$ with ℓ denoting the mean free path of the heat carriers, namely phonons, and L, the characteristic dimension of the system under study. The Knudsen number becomes typically comparable or larger than one for micro- and nano-systems, in which case heat transport is referred to as ballistic, i.e. dominated by phonon collisions with the walls. If the Knudsen number is much smaller than one, heat transport is simply diffusive, i.e. dominated by phonon-phonon collisions inside the system and described by Fourier's law. For small-length systems, as well as for high-frequency processes, Fourier's law is no longer valid. Another drawback associated to Fourier's law is that it implies that temperature spreads infinitely fast through the whole body, which is physically unsustainable. These observations justify the need to generalize Fourier's law. This can be achieved by various ways, for instance, via the resolution of Boltzmann's equation [3], making use of the dual time approach [4] or by computer simulations [5]. Here we follow a different route based on Extended Irreversible Thermodynamics (EIT) [6] whose main characteristic is to upgrade the heat flux and higher order heat fluxes to the rank of independent variables. In most applications, for convenience, the analysis is restricted by taking heat flux as single extra variable. In this letter, we go one step further by selecting an infinite number of higher order fluxes. The use of a large number of flux variables finds its justification in the recent progress of nanotechnology and high-frequency processes. The procedure described in this paper is by no means limited to heat transfer but can be easily generalized to other transport phenomena, as electrical conduction and matter diffusion.
As a numerical application, we consider the problem of heat conduction in a one-dimensional rigid body of length L at rest, subject to two different kinds of boundary conditions, made explicit in Section 5. The theoretical model for the general heat transport equation is presented in the next Section 2.

2 A general heat transport equation in terms of high order heat fluxes

At micro and nanoscales, heat transport is mostly influenced by non-local effects and high frequency processes. The classical Fourier law

$$\boldsymbol{q} = -\lambda \nabla T, \tag{1}$$

relating the heat flux vector $\boldsymbol{q}$ to the temperature gradient ∇T, with λ denoting the heat conductivity, is not applicable at short times and small spatial scales. In order to account for high frequencies, Fourier's law has been generalized by Cattaneo [7] under the form

$$\tau \partial_t \boldsymbol{q} + \boldsymbol{q} = -\lambda \nabla T, \tag{2}$$

with τ designating the relaxation time of the heat flux and ∂_t the partial time derivative. Cattaneo's relation is easily derivable from EIT (e.g. [6]) by assuming that the entropy $\eta(e, \boldsymbol{q})$ is not only depending on the internal energy e but besides on the heat flux vector $\boldsymbol{q}$. The corresponding Gibbs equation in a rigid conductor at rest writes as

$$\partial_t \eta(e, \boldsymbol{q}) = T^{-1} \partial_t e - \alpha_1 \boldsymbol{q} . \partial_t \boldsymbol{q}, \tag{3}$$

wherein η and e are measured per unit volume, α_1 is a phenomenological coefficient assumed to be $\boldsymbol{q}$-independent, a dot stands for the scalar product. It will be identified later on and shown to be related to the relaxation time τ and the heat conductivity λ. Combining (3) with the energy conservation law for heat conductors $\partial_t e = -\nabla \cdot \boldsymbol{q}$ and following the usual procedure of EIT, one is led to Cattaneo's relation (2).
However, Cattaneo's relation is not able to cope with non-local effects which are dominant at small length scales. To take them into account, it is suggested in the framework of EIT to introduce a hierarchy of fluxes $\boldsymbol{Q}^{(1)}$, $\boldsymbol{Q}^{(2)}$, ..., $\boldsymbol{Q}^{(n)}$ with $\boldsymbol{Q}^{(1)}$ identical to the heat flux vector $\boldsymbol{q}$, $\boldsymbol{Q}^{(2)}$ (a tensor of rank two) is the flux of $\boldsymbol{q}$, $\boldsymbol{Q}^{(3)}$ the flux of $\boldsymbol{Q}^{(2)}$ and so on. From the kinetic theory point of view, the quantities $\boldsymbol{Q}^{(2)}$and $\boldsymbol{Q}^{(3)}$ represent the higher moments of the velocity distribution. Up to the n^{th}-order flux, the Gibbs equation generalizing relation (3) becomes

$$\partial_t \eta\left(e, \boldsymbol{q}, \boldsymbol{Q}^{(1)}, \ldots, \boldsymbol{Q}^{(n)}\right) = T^{-1} \partial_t e - \alpha_1 \boldsymbol{q} \cdot \partial_t \boldsymbol{q} - \alpha_2 \boldsymbol{Q}^{(2)} \otimes \partial_t \boldsymbol{Q}^{(2)} - \cdots - \alpha_N \boldsymbol{Q}^{(N)} \otimes \partial_t \boldsymbol{Q}^{(N)}, \tag{4}$$

wherein the symbol ⊗ denotes the inner product of the corresponding tensors. Moreover, the time evolution of entropy is governed by a general balance equation which can be given the form

$$\sigma^s = \partial_t \eta + \nabla \cdot \boldsymbol{J}^s \geq 0, \tag{5}$$

wherein $\boldsymbol{J}^s$ stands for the entropy flux and σ^s for the rate of entropy production per unit volume which is positive definite to fulfil the second law of thermodynamics. To derive an expression for σ^s, we need the result (4) for $\partial_t \eta$ and a constitutive relation for $\boldsymbol{J}^s$ in terms of the set of variables.

It is natural to expect that $\boldsymbol{J}^s$it is not simply given by the classical expression $T^{-1}\, \boldsymbol{q}$ [8], but that it will depend on all the higher order fluxes up to order n, namely

$$J^s = T^{-1} \boldsymbol{q} + \beta_1 \boldsymbol{Q}^{(2)} \cdot \boldsymbol{q} + \cdots + \beta_{N-1} \boldsymbol{Q}^{(N)} \otimes \boldsymbol{Q}^{(N-1)}, \tag{6}$$

with β_N designating phenomenological coefficients. It is checked that, after substituting in (5) the expressions of $\partial_t\eta$ and $\boldsymbol{J^s}$ given by (4) and (6), respectively, and eliminating in (4) the time derivative $\partial_t e$ via the energy conservation law, $\partial_t e = -\nabla \cdot \boldsymbol{q}$, one obtains for the entropy production

$$\sigma^s = -\left(-\nabla T^{-1} + \alpha_1 \partial_t \boldsymbol{q} - \beta_1 \nabla \cdot \boldsymbol{Q^{(2)}}\right) \cdot \boldsymbol{q} \dots - \sum_{n=2}^{N} \boldsymbol{Q}^{(n)} \otimes \left(\alpha_n \partial_t \boldsymbol{Q}^{(n)} - \beta_n \nabla \cdot \boldsymbol{Q^{(n+1)}} - \beta_{n-1} \nabla \boldsymbol{Q^{(n-1)}}\right) \geq 0 \tag{7}$$

The above bilinear expression in fluxes and forces (the quantities between parentheses) suggests the following hierarchy of linear flux-force relations

$$\nabla T^{-1} - \alpha_1 \partial_t \boldsymbol{q} + \beta_1 \nabla \cdot \boldsymbol{Q}^{(2)} = \mu_1 \boldsymbol{q} \tag{8}$$

$$\beta_{n-1} \nabla \boldsymbol{Q}^{(n-1)} - \alpha_n \partial_t \boldsymbol{Q}^{(n)} + \beta_n \nabla \cdot \boldsymbol{Q}^{(n+1)} = \mu_n \boldsymbol{Q}^{(n)} \qquad (n = 2,3, \dots N) \tag{9}$$

where we state that for a rigid body $d_t \equiv \partial_t$. Our purpose is to replace the set of relations (8)-(9) by one single constitutive equation taking into account all the n^{th} order fluxes. For the sake of clarity, we will make the development up to the fourth order and generalize afterwards. The fourth order flux equation ($n = 4$ in equation (9)) is given by

$$\beta_3 \nabla \boldsymbol{Q}^{(3)} - \alpha_4 \partial_t \boldsymbol{Q}^{(4)} + \beta_4 \nabla \cdot \boldsymbol{Q}^{(5)} = \mu_4 \boldsymbol{Q}^{(4)} \tag{10}$$

Taking the divergence ($\nabla \cdot$) of (10), substituting it subsequently in (9) with $n = 3$, and omitting any flux $n > 4$, leads to

$$\beta_2 \nabla \boldsymbol{Q}^{(2)} - \alpha_3 \partial_t \boldsymbol{Q}^{(3)} + \frac{\beta_3}{\mu_4}\left(\beta_3 \nabla \cdot \nabla \boldsymbol{Q}^{(3)} - \alpha_4 \nabla \cdot \partial_t \boldsymbol{Q}^{(4)}\right) = \mu_3 \boldsymbol{Q}^{(3)} \tag{11}$$

Apply again operator divergence on (11), substitute it on its turn in (9) but with $n = 2$ and repeat the same operation until arriving at equation (9) with $n = 1$ (which is now equivalent to Eq. (8)), the final result is

$$\nabla T^{-1} - \alpha_1 \partial_t \boldsymbol{q} + \frac{\beta_1}{\mu_2}\Big(\beta_1 \nabla \cdot \nabla \boldsymbol{q} - \alpha_2 \nabla \cdot \partial_t \boldsymbol{Q}^{(2)} + \frac{\beta_2}{\mu_3}\big(\beta_2 \nabla \cdot \nabla \cdot \nabla \boldsymbol{Q}^{(2)} - \alpha_3 \nabla \cdot \nabla \cdot \partial_t \boldsymbol{Q}^{(3)} + \frac{\beta_3}{\mu_4}(\beta_3 \nabla \cdot \nabla \cdot \nabla \cdot \nabla \boldsymbol{Q}^{(3)} - \alpha_4 \nabla \cdot \nabla \cdot \nabla \cdot \partial_t \boldsymbol{Q}^{(4)})\big)\Big) = \mu_1 \boldsymbol{q} \tag{12}$$

This relation may be viewed as a generalized Cattaneo's relation up to the 4th order flux $\boldsymbol{Q}^{(4)}$. However, it is not very convenient from a practical point of view because of the presence of the higher order fluxes which are not directly measurable and whose physical meaning is not clear. This is the reason why we propose in the next section a more suitable expression, wherein all the high-order fluxes have been eliminated.

3 A generalized transport equation in terms of the heat flux $\boldsymbol{q}$

Our purpose is to express equation (12) exclusively in terms of the more physical quantities that are the temperature T and the classical heat flux $\boldsymbol{q}$. The higher order fluxes $\boldsymbol{Q}^{(n)}$ are eliminated by making use of the very definition of the high-order flux, namely

$$\partial_t \boldsymbol{Q}^{(n)} = -\nabla \cdot \boldsymbol{Q}^{(n+1)} \qquad (n = 1,2,3, \dots N) \tag{13}$$

In virtue of (13) and after some rearrangements, equation (12) can be rewritten as

$$\frac{\alpha_1}{\mu_1}\partial_t\boldsymbol{q}+\boldsymbol{q}=-\frac{1}{\mu_1 T^2}\nabla T+\frac{1}{\mu_1}\frac{\beta_1}{\mu_2}\beta_1\nabla^2\boldsymbol{q}+\frac{1}{\mu_1}\frac{\beta_1}{\mu_2}\alpha_2\partial_t\partial_t\boldsymbol{q}-\frac{1}{\mu_1}\frac{\beta_1}{\mu_2}\frac{\beta_2}{\mu_3}\beta_2\partial_t\nabla^2\boldsymbol{q}-\frac{1}{\mu_1}\frac{\beta_1}{\mu_2}\frac{\beta_2}{\mu_3}\alpha_3\partial_t\partial_t\partial_t\boldsymbol{q}+\frac{1}{\mu_1}\frac{\beta_1}{\mu_2}\frac{\beta_2}{\mu_3}\frac{\beta_3}{\mu_4}\beta_3\partial_t\partial_t\nabla^2\boldsymbol{q}+\frac{1}{\mu_1}\frac{\beta_1}{\mu_2}\frac{\beta_2}{\mu_3}\frac{\beta_3}{\mu_4}\alpha_4\partial_t\partial_t\partial_t\partial_t\boldsymbol{q} \quad (14)$$

By repeating the exercise for higher order fluxes with $n = N$, we obtain the following general equation governing heat conduction in rigid bodies

$$\frac{\alpha_1}{\mu_1}\partial_t\boldsymbol{q}+\boldsymbol{q}=-\frac{1}{\mu_1 T^2}\nabla T+\frac{1}{\mu_1}\sum_{n=1}^{N}\left(\left(\prod_{i=1}^{n}\frac{\beta_i}{\mu_{i+1}}\right)(-1)^{n+1}\beta_n\frac{\partial^{n-1}}{\partial t^{n-1}}\nabla^2\boldsymbol{q}+\left(\prod_{i=1}^{n}\frac{\beta_i}{\mu_{i+1}}\right)(-1)^{n+1}\alpha_{n+1}\frac{\partial^{n+1}}{\partial t^{n+1}}\boldsymbol{q}\right)$$

$(n = 1,2,3,\dots,N)$, with $N \in \mathbb{N}^+$ (15)

Note that for $n = 0$, (15) reduces to Cattaneo's law by setting $\beta_0 = 0$, i.e. by omitting non-local contributions. It remains to relate the various phenomenological coefficients to physical quantities. Setting $\alpha_1 = \beta_1 = 0$, equation (8) reduces to Fourier's law from which follows that $\mu_1 = \frac{1}{\lambda T^2}$. Comparing unity's in (8) indicate that the dimension of the ratio $\alpha_1/\mu_1$is that of a time (t) so that $[\alpha_1] = [t][\mu_1]$. Defining the time unit as the heat flux $(\boldsymbol{q})$ relaxation time τ_1 one has $\alpha_1 = \tau_1\mu_1 = \frac{\tau_1}{\lambda T^2}$. As the heat flux $\boldsymbol{q}$ is expressed in W/m^2, the unity of $\boldsymbol{Q}^{(2)}$ is W/(m*s) according to (13): $[\boldsymbol{Q}^{(n+1)}] = \frac{[L]}{[t]}[\boldsymbol{Q}^{(n)}]$, where $[L]$ is a unity in length. As a direct consequence of the property that $\boldsymbol{Q}^{(2)}$ is the flux of $\boldsymbol{q}$, one has in (8) that $\frac{\beta_1}{\alpha_1} = -1$ or $\beta_1 = -\alpha_1 = -\tau_1\mu_1$. Let us move to the second order approximation. Defining the length scale as given by the mean free path of the phonons ℓ and the unit time by τ (making no longer a difference between the higher orders of the mean free paths and the relaxation times) leads to $\mu_2 = -\frac{\tau}{\ell^2}\beta_1 = \frac{\tau^2}{\ell^2}\mu_1$ and further to $\beta_2 = -\alpha_2 = -\tau\mu_2$. At the n$^{\text{th}}$-order, it is directly inferred that

$$\begin{aligned}\alpha_n &= \tau\left(\frac{\tau}{\ell}\right)^{2n-2}\mu_1\\ \beta_n &= -\tau\left(\frac{\tau}{\ell}\right)^{2n-2}\mu_1\\ \mu_n &= \left(\frac{\tau}{\ell}\right)^{2n-2}\mu_1\end{aligned} \quad (16)$$

Substituting (16) in (15) and performing some rearrangements one obtains the final result

$$\tau\partial_t\boldsymbol{q}+\boldsymbol{q}=-\lambda\nabla T+\sum_{n=1}^{N}\left(\ell^2\tau^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\nabla^2\boldsymbol{q}-\tau^{n+1}\frac{\partial^{n+1}}{\partial t^{n+1}}\boldsymbol{q}\right) \quad (17)$$

Within the perspective of practical applications, it is interesting to consider the three first order approximations of (17) given by

$$\tau\partial_t\boldsymbol{q} + \boldsymbol{q} = -\lambda\nabla T + \ell^2\nabla^2\boldsymbol{q} - \tau^2\partial_t^2\boldsymbol{q} \tag{18}$$
$$\tau\partial_t\boldsymbol{q} + \boldsymbol{q} = -\lambda\nabla T + \ell^2\nabla^2\boldsymbol{q} - \tau^2\partial_t^2\boldsymbol{q} + \ell^2\tau\partial_t\nabla^2\boldsymbol{q} - \tau^3\partial_t^3\boldsymbol{q} \tag{19}$$
$$\tau\partial_t\boldsymbol{q} + \boldsymbol{q} = -\lambda\nabla T + \ell^2\nabla^2\boldsymbol{q} - \tau^2\partial_t^2\boldsymbol{q} + \ell^2\tau\partial_t\nabla^2\boldsymbol{q} - \tau^3\partial_t^3\boldsymbol{q} + \ell^2\tau^2\partial_t^2\nabla^2\boldsymbol{q} - \tau^4\partial_t^4\boldsymbol{q} \tag{20}$$

Let us briefly comment Eq. (18). Letting τ and ℓ tend to zero, one finds back Fouriers' law $\boldsymbol{q} = -\lambda\nabla T$. With $\ell = 0$ and omitting the second order derivatives in space and time, one recovers Cattaneo's relation, while setting only $\partial_t^2\boldsymbol{q} = 0$, expression (18) reduces to Guyer-Krumhansl's relation [9].
It may be convenient to reformulate the above results in dimensionless form. Let us, therefore, rescale the heat flux $\boldsymbol{q}$ with $\frac{\lambda\,(T_H - T_L)}{L}$, the space coordinate with the characteristic length L, the time t with τ and define the dimensionless temperature by $T \to (T - T_L)/(T_H - T_L)$. Here, T_H and T_L are the high and low temperatures at the respective hot and cold sides of the system. Introducing the Knudsen number $Kn \equiv \frac{\ell}{L}$ and using the same notation for the non-dimensional quantities, equation (17) becomes

$$\partial_t\boldsymbol{q} + \boldsymbol{q} = -\nabla T + \sum_{n=1}^{N}\left(Kn^2\frac{\partial^{n-1}}{\partial t^{n-1}}\nabla^2\boldsymbol{q} - \frac{\partial^{n+1}}{\partial t^{n+1}}\boldsymbol{q}\right) \tag{21}$$

4. A simplified expression of (17)

Equation (21) describes both non-local effects and high frequency processes. Solving it implies the determination of the high order time derivatives $\partial^n\boldsymbol{q}/\partial t^n$ at $t = 0$. However, if we consider only non-local processes, it is then desirable to eliminate the terms $\partial^n/\partial t^n$ $(n > 2)$ from relation (17). This is easily achieved by assuming that $\alpha_n \ll \alpha_1$, which amounts to neglect the higher relaxation times τ_n $(n \geq 2)$ in comparison to τ in equation (12). Under these assumptions and limiting the developments to $n = 3$, the latter reads as

$$\nabla T^{-1} - \alpha_1\partial_t\boldsymbol{q} + \frac{\beta_1}{\mu_2}\left(\beta_1\nabla\cdot\nabla\boldsymbol{q} + \frac{\beta_2}{\mu_3}\left(\beta_2\nabla\cdot\nabla\cdot\nabla\boldsymbol{Q}^{(2)} + \frac{\beta_3}{\mu_4}\left(\beta_3\nabla\cdot\nabla\cdot\nabla\cdot\nabla\boldsymbol{Q}^{(3)}\right)\right)\right) = \mu_1\boldsymbol{q} \tag{22}$$

Moreover, relation (9) wherein $n = 2$ and $\alpha_2 = 0$, may be written as

$$\boldsymbol{Q}^{(2)} = \frac{\beta_1}{\mu_2}\nabla\boldsymbol{q} + \frac{\beta_2}{\mu_2}\nabla\cdot\boldsymbol{Q}^{(3)} \tag{23}$$

Substituting (23) in (22), making use of (16) and omitting high order flux terms of $O(\boldsymbol{Q}^{(3)})$ yields

$$\nabla T^{-1} - \alpha_1\partial_t\boldsymbol{q} + \frac{\beta_1}{\mu_2}\beta_1\nabla^2\boldsymbol{q} - \frac{\ell^2}{\tau}\frac{\beta_1}{\mu_2}\frac{\beta_2}{\mu_3}\beta_2\nabla^4\boldsymbol{q} = \mu_1\boldsymbol{q}\,, \tag{24}$$

which, in virtue of the relation (16) leads to

$$\tau\partial_t\boldsymbol{q} + \boldsymbol{q} = -\lambda\nabla T + \ell^2\nabla^2\boldsymbol{q} + \ell^4\nabla^4\boldsymbol{q}\,. \tag{25}$$

The advantage of the above expression with respect to (17) is that it is written exclusively in terms of the physical heat flux vector $\boldsymbol{q}$.

It is easy to verify that, performing the same exercise for $n = N$, using the same rescaling notation for the non-dimensional quantities as for (21), results into

$$\partial_t \boldsymbol{q} + \boldsymbol{q} = -\nabla T + \sum_{n=1}^{N} (Kn^{2n} \nabla^{2n} \boldsymbol{q}) \tag{26}$$

5 One-dimensional numerical illustration

To assess the importance of the high order fluxes, we calculate the temperature distribution in a 1D rigid body subject to a temperature difference $\Delta T = T_H - T_L$ between the hot and cold faces located at $z = 1$ and $z = 0$, respectively. The governing equations are expression (21) for $N = 2$ and the dimensionless energy balance written as

$$\partial_t T = -\tau \frac{\kappa_T}{L^2} \nabla \cdot \boldsymbol{q} = -M \nabla \cdot \boldsymbol{q} \tag{27}$$

wherein $\kappa_T (= \lambda / c_v)$ is the thermal diffusivity, c_v the heat capacity per unit volume and $M \equiv \frac{\tau}{\tau_T}$ a dimensionless number, with $\tau_T = L^2/\kappa_T$ the thermal characteristic time. The quantity M allows to compare ballistic to diffusion heat transfer, for $M \gg 1$, heat propagation is mainly of diffusive rather than ballistic nature, while the opposite holds for $M \ll 1$. Here, we will select $\tau \equiv \frac{1}{2} \tau_T$, to visualize the relative importance of both heat transfer mechanisms. The results provided by Eq. (21) will be compared to those obtained from Fourier's law through two different boundary conditions. Figure 1 presents a 3D plot of the temperature as a function of the dimensionless space-coordinate z and time t. The cold side $z = 0$ is at a fixed temperature T_L and at the hot side $z = 1$, two different boundary conditions are imposed: first, a fixed temperature $T = T_H$ and second, a periodic one $T_H = \frac{1}{2}(1 - \mathrm{Cos}[P\pi t])$ with $P = 2/\tau$ Hz. Initially at $t = 0$, $T = T_L$, while the heat flux and its time derivatives are assumed to be zero.

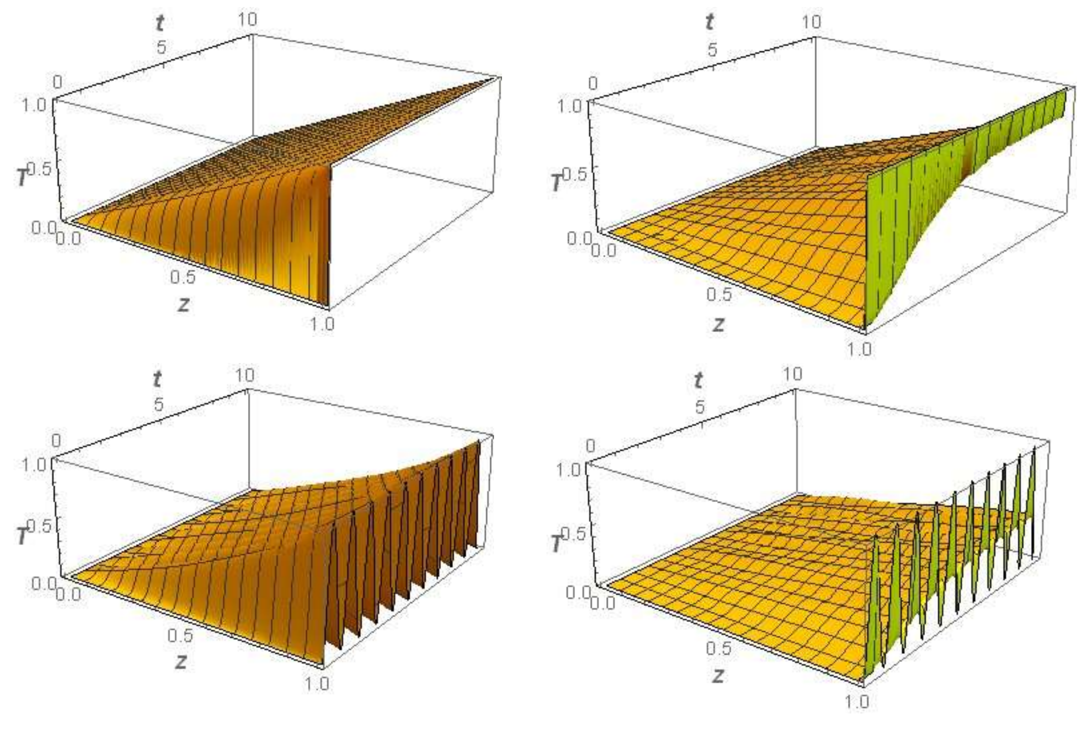

Fig. 1. 3D-temperature distribution as a function of the space and time. The upper figures refer to fixed temperatures at both boundaries: at $z = 0, T = T_L = 0$, at $z = 1, T_H = T_L + \Delta T$ with

$\Delta T = 1$ The results in the two lower figures are obtained for $T = T_L = 0$ at $z = 0$. and an oscillating boundary condition $T = T_H = \frac{1}{2}(1 - \mathrm{Cos}[P\pi t])$, $(P = 2)$ at $z = 1$. The results based on Fourier equation are shown on the left figures and the results from EIT ($N = 2$) are on the right ones, the Knudsen number is taken as $Kn = 1$.

By imposing an oscillatory temperature at one boundary, one observes that, with Fourier's law, the heat signal reaches instantaneously the opposite boundary of the system, indicating that heat propagates at infinite speed. In contrast, with the EIT model, much of the oscillations are strongly attenuated (should M have been chosen much smaller than 1 or Kn much larger than one, the oscillations would have been dimmed instantaneously) and are not reminiscent of the oscillations imposed at the border, reflecting the property that heat propagates at finite speed. The same behavior is noticed in the case of fixed temperatures at both sides. With Fourier's law, the temperature is felt everywhere in the bulk after application of the temperature difference at the boundaries, while with EIT, the temperature is increasing gradually in the course of time and space. Typically, one also observes that EIT predicts a small persisting temperature jump, which is a property that is pointed out by several authors, e.g [1, 10].

6. Extension to other constitutive laws

The previous sections were devoted to the problem of heat rigid conductors and general constitutive equations for the heat flux were derived in presence of strong non-localities and high frequencies. Formally, the same procedure can be extended to any kind of phenomena whose behaviour is governed by constitutive relations having the structure of Fourier's law, such as Fick's or Ohm's laws. Table 1 gives some examples of how such classical constitutive equations can be extended to small-length-scales systems and high-frequency processes. Note that thermoelectric effects like the Seebeck effect (expressing that a temperature gradient ∇T gives raise to an electric current density $\boldsymbol{I}$) or the Peltier effect (an electrical potential gradient ∇V produces a heat flux $\boldsymbol{q}$) can be treated in the same way by combining the expressions of the first and last equations of Table 1. Of course, similar generalizations can also be formulated from equation (26).

Table 1: Generalized Fourier, Fick and Ohm laws

Fourier	$\boldsymbol{q} = -\lambda\nabla T$	$\tau\partial_t\boldsymbol{q} + \boldsymbol{q} = -\lambda\nabla T + \sum_{n=1}^{N}\left(\ell^2\tau^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\nabla^2\boldsymbol{q} - \tau^{n+1}\frac{\partial^{n+1}}{\partial t^{n+1}}\boldsymbol{q}\right)$
Fick	$\boldsymbol{J} = -\rho D\nabla C$	$\tau\partial_t\boldsymbol{J} + \boldsymbol{J} = -\rho D\nabla C + \sum_{n=1}^{N}\left(\ell^2\tau^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\nabla^2\boldsymbol{J} - \tau^{n+1}\frac{\partial^{n+1}}{\partial t^{n+1}}\boldsymbol{J}\right)$
Ohm	$\boldsymbol{I} = -\sigma\nabla V$	$\tau\partial_t\boldsymbol{I} + \boldsymbol{I} = -\sigma\nabla V + \sum_{n=1}^{N}\left(\ell^2\tau^{n-1}\frac{\partial^{n-1}}{\partial t^{n-1}}\nabla^2\boldsymbol{I} - \tau^{n+1}\frac{\partial^{n+1}}{\partial t^{n+1}}\boldsymbol{I}\right)$

Generalizing Fick and Ohm equations requires to specify the corresponding set of state variables and their evolution equations. In case of matter diffusion in a two-component system, the state variables are $\boldsymbol{V_F} = (C, \boldsymbol{J})$, where C stands for the mass fraction of, say, the first constituent and $\boldsymbol{J}$ for the flux of matter. In case of electrical conduction, one has as state variables $\boldsymbol{V_O} = (z, \boldsymbol{I})$, with z the electrical charge per unit mass (related to the electrical potential V by Poisson's equation $\rho z = -\varepsilon_0\nabla^2 V$, where ε_0 is the electrical permittivity) and $\boldsymbol{I}$ the electrical current density. Finally, in Table 1, D denotes the coefficient of diffusion and σ the electrical conductivity. The balance equations for C and z are respectively $\rho\partial_t C = -\nabla\cdot\boldsymbol{J}$ and $\rho\partial_t z = -\nabla\cdot\boldsymbol{I}$.

7 Conclusions

The aim of this work is to generalize the usual transport equations of continuum physics by considering the problem of heat conduction in rigid bodies as a case study. This is achieved in the framework of EIT by using an infinite set of flux variables. The approach is original and the two most important results are embodied in equations (17) and (25). Including an infinite number of fluxes into the description is not merely a pure academic game but is of importance to describe very high frequency processes and low scale phenomena as those occurring at micro and nanoscales. The main advantage of relations (17) and (25) is that they do not ignore a priori the influence of the high order fluxes. However for convenience and within the perspective of practical applications, the relevant constitutive laws are not formulated in terms of the whole set of high order fluxes but instead in terms of the sole heat flux $\boldsymbol{q}$, which is directly accessible to measurements, contrary to the higher order fluxes. The analysis is also easily transposable to other situations involving electrical current (Ohm's law) and matter diffusion (Fick's law), keeping in mind the corresponding balance equations. As an application, we have calculated heat propagation in a 1D rigid rod submitted to a fixed temperature at one side, the other side being subject to either an oscillatory or a fixed temperature. The numerical results show explicitly that the proposed model predicts a finite velocity of propagation of heat signals, in opposition to Fourier's law which leads to temperature disturbances propagating with boundless speed within the limit of high frequencies.

Acknowledgements

H.M. acknowledges BelSPo for financial support. G.L. acknowledges the support of the 9th "Projet de collaboration Wallonie-Bruxeles-Québec 2015-2017". We have appreciated fruitful discussions with our colleagues of Ecole Polytechnique de Montreal, particularly M. Grmela, Ch. Dubois and A. Behrang.

References

[1] A.A. Joshi, A. Majumdar, Transient ballistic and diffusive phonon heat transport in thin films, J. Appl. Phys., 74 (1993), 31–39.

[2] T.L. Hill, Thermodynamics of Small Systems, Dover, New York (1994).

[3] Y.K. Koh, D.G. Cahill, B. Sun, Nonlocal theory for heat transport at high frequencies, Phys. Rev. B, 90, 205412 (2014).

[4] D.Y. Tzou, Macro to Microscale Heat Transfer. The Lagging Behaviour, Taylor and Francis, New York (1997).

[5] C.M. Sabliov, D.A. Salvi, D. Boldor, High frequency electromagnetism, heat transfer and fluid flow coupling in ANSYS multiphysics, J. Microw. Power Electromagn. Energy, 41 (2007) 5-17.

[6] D. Jou, J. Casas-Vazquez, G. Lebon, Extended Irreversible Thermodynamics, fourth ed., Springer-Verlag, Berlin, 2010.

[7] C. Cattaneo, Sulla conduzione del calore, Atti Seminario Mat. Fis. Univ. Modena, 3 (1948), 83-101.

[8] S.R. de Groot, P. Mazur, Non-Equilibrium Thermodynamics, North- Holland , Amsterdam, 1962.

[9] R.A. Guyer, J.A. Krumhansl, Solution of the linearized Boltzmann phonon equation, Phys. Rev., 148 (1966), 766–778.

[10] E.T. Swartz, R.O. Pohl, Thermal boundary resistance, Rev. Mod. Phys. 61 (1989) 605–668.