# Some notions of compactness in Functional Analysis and one related question about diametral dimensions

Loïc Demeulenaere (FRIA-FNRS Grantee)

Comprehensible Seminars - ULiège

October 5, 2017

Introduction

Some notions of compactness

Diametral dimensions

# Mathematical Analysis

# Mathematical Analysis

- **Main topics**: *functions* and related notions, e.g. limits, distributions, measures, etc.

# Mathematical Analysis

- **Main topics**: *functions* and related notions, e.g. limits, distributions, measures, etc.
- After “algebraic” equations, *functional* equations: differentiable equations, integral equations, optimization problems...

# Mathematical Analysis

- **Main topics**: *functions* and related notions, e.g. limits, distributions, measures, etc.
- After "algebraic" equations, *functional* equations: differentiable equations, integral equations, optimization problems...

Study of sets of functions

# Mathematical Analysis

- **Main topics**: *functions* and related notions, e.g. limits, distributions, measures, etc.
- After "algebraic" equations, *functional* equations: differentiable equations, integral equations, optimization problems...

Study of sets of functions
Notions of convergence

# Mathematical Analysis

- **Main topics**: *functions* and related notions, e.g. limits, distributions, measures, etc.
- After "algebraic" equations, *functional* equations: differentiable equations, integral equations, optimization problems...

$$\left.\begin{array}{r}\text{Study of sets of functions}\\ \text{Notions of convergence}\end{array}\right\} \rightsquigarrow \textit{Topological}\text{ study of functional spaces}$$

# Examples

- If $K$ is a compact subset of $\mathbb{R}^n$, if $f$ is a function defined on $K$ and if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $C_0(K)$ s.t.

$$\sup_{x\in K} |f_m(x) - f(x)| \to 0 \text{ if } m \to \infty,$$

then $f \in C_0(K)$.

## Examples

- If $K$ is a compact subset of $\mathbb{R}^n$, if $f$ is a function defined on $K$ and if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $C_0(K)$ s.t.

$$\sup_{x\in K} |f_m(x) - f(x)| \to 0 \text{ if } m \to \infty,$$

then $f \in C_0(K)$.

$\rightsquigarrow$ Convergence in $C_0(K)$ endowed with the metric defined by $\sup_K |.|$.

# Examples

- **Lebesgue's dominated convergence theorem:** if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $\mathrm{L}^1(\mathbb{R})$ which converges pointwise to $f$ and if there exists $F \in \mathrm{L}^1(\mathbb{R})$ with $|f_m| \leq F$ a.e. on $\mathbb{R}$ $\forall m$, then $f \in \mathrm{L}^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} |f_m(x) - f(x)|\, dx \to 0 \text{ if } m \to \infty.$$

# Examples

- **Lebesgue's dominated convergence theorem:** if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $L^1(\mathbb{R})$ which converges pointwise to $f$ and if there exists $F \in L^1(\mathbb{R})$ with $|f_m| \leq F$ a.e. on $\mathbb{R}$ $\forall m$, then $f \in L^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} |f_m(x) - f(x)|\, dx \to 0 \text{ if } m \to \infty.$$

$\leadsto$ Convergence in $L^1(\mathbb{R})$ for the metric defined by $\int_{\mathbb{R}} |.|\, dx$.

# Examples

- **Lebesgue's dominated convergence theorem:** if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $\mathrm{L}^1(\mathbb{R})$ which converges pointwise to $f$ and if there exists $F \in \mathrm{L}^1(\mathbb{R})$ with $|f_m| \leq F$ a.e. on $\mathbb{R}$ $\forall m$, then $f \in \mathrm{L}^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} |f_m(x) - f(x)|\, dx \to 0 \text{ if } m \to \infty.$$

$\rightsquigarrow$ Convergence in $\mathrm{L}^1(\mathbb{R})$ for the metric defined by $\int_{\mathbb{R}} |.|\, dx$.

- **Fourier series**: convergence in $\mathrm{L}^2([a,b])$ for the metric defined by $\sqrt{\int_a^b |.|^2\, dx}$.
- ...

# Examples

- **Lebesgue's dominated convergence theorem:** if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $\mathrm{L}^1(\mathbb{R})$ which converges pointwise to $f$ and if there exists $F \in \mathrm{L}^1(\mathbb{R})$ with $|f_m| \leq F$ a.e. on $\mathbb{R}$ $\forall m$, then $f \in \mathrm{L}^1(\mathbb{R})$ and

$$\int_{\mathbb{R}} |f_m(x) - f(x)| \, dx \to 0 \text{ if } m \to \infty.$$

$\leadsto$ Convergence in $\mathrm{L}^1(\mathbb{R})$ for the metric defined by $\int_{\mathbb{R}} |.| \, dx$.

- **Fourier series**: convergence in $\mathrm{L}^2([a,b])$ for the metric defined by $\sqrt{\int_a^b |.|^2 \, dx}$.
- ...
- Common feature of these metrics: they are defined by *norms*.

# (Semi)norms

## Definition

If $E$ is a vector space on $\mathbb{C}$, a map $p : E \to [0, \infty)$ is a *seminorm* if

1. $p(x + y) \leq p(x) + p(y)\ \forall x, y \in E$;
2. $p(\lambda x) = |\lambda| p(x)\ \forall x \in E,\ \forall \lambda \in \mathbb{C}$.

# (Semi)norms

## Definition

If $E$ is a vector space on $\mathbb{C}$, a map $p : E \to [0, \infty)$ is a *seminorm* if

1. $p(x + y) \leq p(x) + p(y)\ \forall x, y \in E$;
2. $p(\lambda x) = |\lambda| p(x)\ \forall x \in E,\ \forall \lambda \in \mathbb{C}$.

If $p(x) = 0 \Leftrightarrow x = 0$, then $p$ is a *norm*.

# (Semi)norms

## Definition

If $E$ is a vector space on $\mathbb{C}$, a map $p : E \to [0, \infty)$ is a *seminorm* if

1. $p(x + y) \leq p(x) + p(y)$ $\forall x, y \in E$;
2. $p(\lambda x) = |\lambda| p(x)$ $\forall x \in E$, $\forall \lambda \in \mathbb{C}$.

If $p(x) = 0 \Leftrightarrow x = 0$, then $p$ is a *norm*.

## Property

The (semi)norm $p$ defines a vector (pseudo)metric on $E$:

$$d(x, y) = p(x - y).$$

$\rightsquigarrow$ Notions of convergent sequences, Cauchy sequences, etc.

# (Semi)norms

## Definition

If $E$ is a vector space on $\mathbb{C}$, a map $p : E \to [0, \infty)$ is a *seminorm* if

1. $p(x + y) \leq p(x) + p(y)\ \forall x, y \in E$;
2. $p(\lambda x) = |\lambda| p(x)\ \forall x \in E,\ \forall \lambda \in \mathbb{C}$.

If $p(x) = 0 \Leftrightarrow x = 0$, then $p$ is a *norm*.

## Property

The (semi)norm $p$ defines a vector (pseudo)metric on $E$:

$$d(x, y) = p(x - y).$$

$\rightsquigarrow$ Notions of convergent sequences, Cauchy sequences, etc.

## Examples

$C_0(K)$, $L^1(\mathbb{R})$, $L^2([a, b])$ are (complete) normed spaces (i.e. *Banach spaces*).

# In general

In general, one single (semi)norm is not sufficient...

- If $\Omega$ is an open subset of $\mathbb{R}^n$ and if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $C_0(\Omega)$ which uniformly converges to $f$ on every compact of $\Omega$, then $f \in C_0(\Omega)$.

# In general

In general, one single (semi)norm is not sufficient...

- If $\Omega$ is an open subset of $\mathbb{R}^n$ and if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $C_0(\Omega)$ which uniformly converges to $f$ on every compact of $\Omega$, then $f \in C_0(\Omega)$.

$\rightsquigarrow$ Convergence defined by a family of seminorms $\sup_K |.|$ ($K$ compact of $\Omega$).

# In general

In general, one single (semi)norm is not sufficient...

- If $\Omega$ is an open subset of $\mathbb{R}^n$ and if $(f_m)_{m\in\mathbb{N}_0}$ is a sequence of $C_0(\Omega)$ which uniformly converges to $f$ on every compact of $\Omega$, then $f \in C_0(\Omega)$.

$\rightsquigarrow$ Convergence defined by a family of seminorms $\sup_K |.|$ ($K$ compact of $\Omega$).

- For holomorphic functions: likewise!

# Locally convex spaces

## Definition

A topological vector space (t.v.s.) $E$ is a *locally convex space* (l.c.s.) if its topology can be defined by a family of seminorms $\mathcal{P}$:

# Locally convex spaces

## Definition

A topological vector space (t.v.s.) $E$ is a *locally convex space* (l.c.s.) if its topology can be defined by a family of seminorms $\mathcal{P}$:
a subset $U$ of $E$ is a neighbourhood (ngbh) of $x \in E$ if
$\exists p \in \mathcal{P}, \varepsilon > 0$ s.t.

$$\{y \in E : p(x - y) \leq \varepsilon\} \subseteq U.$$

# Locally convex spaces

## Definition

A topological vector space (t.v.s.) $E$ is a *locally convex space* (l.c.s.) if its topology can be defined by a family of seminorms $\mathcal{P}$:
a subset $U$ of $E$ is a neighbourhood (ngbh) of $x \in E$ if
$\exists p \in \mathcal{P}, \varepsilon > 0$ s.t.

$$\{y \in E : p(x-y) \leq \varepsilon\} \subseteq U.$$

## Remark

Good definition if $\forall p, q \in \mathcal{P}$, $\exists r \in \mathcal{P}$, $C > 0$ s.t.

$$\sup(p(x), q(x)) \leq Cr(x) \qquad \forall x \in E.$$

# Locally convex spaces

## Definition

A topological vector space (t.v.s.) $E$ is a *locally convex space* (l.c.s.) if its topology can be defined by a family of seminorms $\mathcal{P}$:
a subset $U$ of $E$ is a neighbourhood (ngbh) of $x \in E$ if
$\exists p \in \mathcal{P}, \varepsilon > 0$ s.t.

$$\{y \in E : p(x-y) \leq \varepsilon\} \subseteq U.$$

## Remark

Good definition if $\forall p, q \in \mathcal{P}$, $\exists r \in \mathcal{P}$, $C > 0$ s.t.

$$\sup(p(x), q(x)) \leq Cr(x) \qquad \forall x \in E.$$

$\rightsquigarrow$ *Functional Analysis:* study of l.c.s.

# Examples of topological properties

For a l.c.s. $(E, \mathcal{P})$ ,

- $E$ is Hausdorff iff $\cap_{p\in\mathcal{P}} \ker(p) = \{0\}$;
- every $x \in E$ has a countable basis of nghbs iff $\mathcal{P}$ can be chosen countable;
- $E$ is metrizable iff the two previous points are verified.

# Examples of topological properties

For a l.c.s. $(E, \mathcal{P})$ ,

- $E$ is Hausdorff iff $\cap_{p \in \mathcal{P}} \ker(p) = \{0\}$;
- every $x \in E$ has a countable basis of nghbs iff $\mathcal{P}$ can be chosen countable;
- $E$ is metrizable iff the two previous points are verified.

## Definition

A Fréchet space is a complete, metrizable, l.c.s.

# Some notions linked to compactness

Let $E$ be an l.c.s.

- A *bounded set* of $E$ is a subset $B$ of $E$ s.t., for every 0-ngbh $U$, $\exists \lambda > 0 : B \subseteq \lambda U$.

# Some notions linked to compactness

Let $E$ be an l.c.s.

- A *bounded set* of $E$ is a subset $B$ of $E$ s.t., for every 0-ngbh $U$, $\exists \lambda > 0 : B \subseteq \lambda U$.
- If $U, V$ is 2 subsets of $E$, $V$ is *precompact with respect to* $U$ if, $\forall \varepsilon > 0$, $\exists P$ finite subset of $E$ s.t.

$$V \subseteq \varepsilon U + P.$$

# Some notions linked to compactness

Let $E$ be an l.c.s.

- A *bounded set* of $E$ is a subset $B$ of $E$ s.t., for every 0-ngbh $U$, $\exists\lambda > 0 : B \subseteq \lambda U$.
- If $U, V$ is 2 subsets of $E$, $V$ is *precompact with respect to $U$* if, $\forall\varepsilon > 0$, $\exists P$ finite subset of $E$ s.t.

$$V \subseteq \varepsilon U + P.$$

- $K \subseteq E$ is *precompact* if it is precompact with respect to each 0-ngbh.

# Links

## Proposition

$$\text{Compact} \implies \text{Precompact} \implies \text{Bounded.}$$

# Links

## Proposition

$$\text{Compact} \implies \text{Precompact} \implies \text{Bounded.}$$

## Remarks

- A precompact set is not always compact, but, in *complete spaces*, a set is precompact iff its closure is compact.

# Links

## Proposition

$$\text{Compact} \implies \text{Precompact} \implies \text{Bounded.}$$

## Remarks

- A precompact set is not always compact, but, in *complete spaces*, a set is precompact iff its closure is compact.
- The closed unit ball of an infinite-dimensional normed space is never precompact. In particular, a closed bounded set is not always (pre)compact.

# Two important classes of Fréchet spaces

Let $E$ be a Fréchet space.

- $E$ is *Montel* if every bounded set is precompact.

# Two important classes of Fréchet spaces

Let $E$ be a Fréchet space.

- $E$ is *Montel* if every bounded set is precompact.
- $E$ is *Schwartz* if, for every 0-ngbh $U$, there exists a 0-ngbh $V$ which is precompact with respect to $U$.

# Two important classes of Fréchet spaces

Let $E$ be a Fréchet space.

- $E$ is *Montel* if every bounded set is precompact.
- $E$ is *Schwartz* if, for every 0-ngbh $U$, there exists a 0-ngbh $V$ which is precompact with respect to $U$.

## Examples

- If $\Omega$ is an open set of $\mathbb{C}$, $H(\Omega)$ is Schwartz and Montel.

## Two important classes of Fréchet spaces

Let $E$ be a Fréchet space.

- $E$ is *Montel* if every bounded set is precompact.
- $E$ is *Schwartz* if, for every 0-ngbh $U$, there exists a 0-ngbh $V$ which is precompact with respect to $U$.

### Examples

- If $\Omega$ is an open set of $\mathbb{C}$, $H(\Omega)$ is Schwartz and Montel.
- An infinite-dimensional Banach space is neither Montel, nor Schwartz.

# Two important classes of Fréchet spaces

Let $E$ be a Fréchet space.

- $E$ is *Montel* if every bounded set is precompact.
- $E$ is *Schwartz* if, for every 0-ngbh $U$, there exists a 0-ngbh $V$ which is precompact with respect to $U$.

## Examples

- If $\Omega$ is an open set of $\mathbb{C}$, $H(\Omega)$ is Schwartz and Montel.
- An infinite-dimensional Banach space is neither Montel, nor Schwartz.

## Warning!

A Schwartz space is always Montel, but the converse is false!

# Isomorphisms and topological invariants

Sometimes, we have to compare the structures of l.c.s./t.v.s.

# Isomorphisms and topological invariants

Sometimes, we have to compare the structures of l.c.s./t.v.s.

## Definition

An *isomorphism* of t.v.s. is a linear, bijective, continuous, and open map between 2 t.v.s.

# Isomorphisms and topological invariants

Sometimes, we have to compare the structures of l.c.s./t.v.s.

## Definition

An *isomorphism* of t.v.s. is a linear, bijective, continuous, and open map between 2 t.v.s.

- **Question:** when are 2 t.v.s. non-isomorphic?

# Isomorphisms and topological invariants

Sometimes, we have to compare the structures of l.c.s./t.v.s.

## Definition

An *isomorphism* of t.v.s. is a linear, bijective, continuous, and open map between 2 t.v.s.

- **Question:** when are 2 t.v.s. non-isomorphic?

$\rightsquigarrow$ *Topological (linear) invariants:* properties preserved by isomorphisms

## Isomorphisms and topological invariants

Sometimes, we have to compare the structures of l.c.s./t.v.s.

### Definition

An *isomorphism* of t.v.s. is a linear, bijective, continuous, and open map between 2 t.v.s.

- **Question:** when are 2 t.v.s. non-isomorphic?

$\rightsquigarrow$ *Topological (linear) invariants:* properties preserved by isomorphisms

### Examples

Dimension in linear algebra, being Hausdorff in topological spaces, etc.

# Isomorphisms and topological invariants

Sometimes, we have to compare the structures of l.c.s./t.v.s.

## Definition

An *isomorphism* of t.v.s. is a linear, bijective, continuous, and open map between 2 t.v.s.

- **Question:** when are 2 t.v.s. non-isomorphic?

$\rightsquigarrow$ *Topological (linear) invariants:* properties preserved by isomorphisms

## Examples

Dimension in linear algebra, being Hausdorff in topological spaces, etc.

And *diametral dimension(s)* in t.v.s.!

# Kolmogorov's diameters

Let $E$ be a vector space, $V, U \subseteq E$, $V \subseteq \lambda U$ $(\lambda > 0)$.

# Kolmogorov's diameters

Let $E$ be a vector space, $V, U \subseteq E$, $V \subseteq \lambda U$ $(\lambda > 0)$.

## Definition

The $n^{\text{th}}$ Kolmogorov's diameters of $V$ with respect to $U$ is

$$\delta_n(V,U) = \inf \{\delta > 0 : \exists L \subseteq E, \dim(L) \leq n, \text{ s.t. } V \subseteq \delta U + L\}.$$

# Kolmogorov's diameters

Let $E$ be a vector space, $V, U \subseteq E$, $V \subseteq \lambda U$ $(\lambda > 0)$.

## Definition

The $n^{\text{th}}$ Kolmogorov's diameters of $V$ with respect to $U$ is

$$\delta_n(V, U) = \inf \{\delta > 0 : \exists L \subseteq E, \dim(L) \leq n, \text{ s.t. } V \subseteq \delta U + L\}.$$

## Important property

If $U$ is a ball centred at 0 and associated to a seminorm, then

$$V \text{ is precompact with respect to } U \Leftrightarrow \delta_n(V, U) \to 0.$$

# Diametral dimension

Let $E$ be a t.v.s. and $\mathcal{U}$ be a basis of 0-nghbs.

## Definition

The *diametral dimension of E* is

$$\Delta(E) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, V \subseteq U, \text{ s.t. } \xi_n \delta_n(V, U) \to 0\right\}.$$

# Diametral dimension

Let $E$ be a t.v.s. and $\mathcal{U}$ be a basis of 0-nghbs.

## Definition

The *diametral dimension of $E$* is

$$\Delta(E) = \left\{\xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, V \subseteq U, \text{ s.t. } \xi_n \delta_n(V, U) \to 0\right\}.$$

## Properties

1. $\Delta$ is a topological invariant (if $E \cong F$, then $\Delta(E) = \Delta(F)$).

# Diametral dimension

Let $E$ be a t.v.s. and $\mathcal{U}$ be a basis of 0-nghbs.

## Definition

The *diametral dimension of E* is

$$\Delta(E) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, V \subseteq U, \text{ s.t. } \xi_n \delta_n(V, U) \to 0 \right\}.$$

## Properties

1. $\Delta$ is a topological invariant (if $E \cong F$, then $\Delta(E) = \Delta(F)$).
2. If $E$ is Fréchet,
   - if $E$ is not Schwartz, $\Delta(E) = c_0$;
   - if $E$ is Schwartz, $l_\infty \subseteq \Delta(E)$.

# Another diametral dimension...

**Definition**

$$\Delta_b(E) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded}, \xi_n \delta_n(B, U) \to 0 \right\}.$$

# Another diametral dimension...

## Definition

$$\Delta_b(E) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded}, \xi_n \delta_n(B, U) \to 0 \right\}.$$

## Open question

Do we have

$$\Delta(E) = \Delta_b(E)$$

if $E$ is Fréchet?

## Proposition

If $E$ is Fréchet,

- if $E$ is not Montel, $\Delta_b(E) = c_0$;
- if $E$ is Montel, $l_\infty \subseteq \Delta_b(E)$.

## Proposition

If $E$ is Fréchet,

- if $E$ is not Montel, $\Delta_b(E) = c_0$;
- if $E$ is Montel, $l_\infty \subseteq \Delta_b(E)$.

## Consequences

- If $E$ is not Montel, $\Delta(E) = \Delta_b(E) = c_0$.
- If $E$ is Montel but not Schwartz, then $\Delta(E) = c_0 \subsetneq \Delta_b(E)$.

## Proposition

If $E$ is Fréchet,

- if $E$ is not Montel, $\Delta_b(E) = c_0$;
- if $E$ is Montel, $l_\infty \subseteq \Delta_b(E)$.

## Consequences

- If $E$ is not Montel, $\Delta(E) = \Delta_b(E) = c_0$.
- If $E$ is Montel but not Schwartz, then $\Delta(E) = c_0 \subsetneq \Delta_b(E)$.

## New open question

Do we have

$$\Delta(E) = \Delta_b(E)$$

if $E$ is Fréchet-Schwartz?

# A positive partial result

Slight variations of diametral dimensions...

$$\Delta^{\infty}(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, \text{ s.t. } (\xi_n \delta_n(V,U))_n \in \ell_\infty \right\},$$

$$\Delta_b^{\infty}(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded}, (\xi_n \delta_n(B,U))_n \in \ell_\infty \right\}.$$

# A positive partial result

Slight variations of diametral dimensions...

$$\Delta^{\infty}(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, \text{ s.t. } (\xi_n \delta_n(V,U))_n \in \ell_\infty \right\},$$

$$\Delta_b^{\infty}(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded}, (\xi_n \delta_n(B,U))_n \in \ell_\infty \right\}.$$

Theorem (2016, L.D., L. Frerick, J. Wengenroth)

If $E$ is Fréchet-Schwartz, then

$$\Delta^{\infty}(E) = \Delta_b^{\infty}(E).$$

# A positive partial result

Slight variations of diametral dimensions...

$$\Delta^{\infty}(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, \text{ s.t. } (\xi_n \delta_n(V, U))_n \in \ell_\infty \right\},$$

$$\Delta_b^{\infty}(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded}, (\xi_n \delta_n(B, U))_n \in \ell_\infty \right\}.$$

**Theorem** (2016, L.D., L. Frerick, J. Wengenroth)
If $E$ is Fréchet-Schwartz, then

$$\Delta^{\infty}(E) = \Delta_b^{\infty}(E).$$

In particular, if $\Delta(E) = \Delta^{\infty}(E)$, then $\Delta(E) = \Delta_b(E)$.

# Schwartz spaces with $\Delta(E) = \Delta^{\infty}(E)$

# Schwartz spaces with $\Delta(E) = \Delta^\infty(E)$

- Classic sequence spaces ("*Köthe echelon spaces*") (2017, F. Bastin, L.D.);

# Schwartz spaces with $\Delta(E) = \Delta^\infty(E)$

- Classic sequence spaces ("*Köthe echelon spaces*") (2017, F. Bastin, L.D.);
- Hilbertizable spaces (in particular *nuclear* spaces) (2016, L.D., L. Frerick, J. Wengenroth).

# Another sufficient condition

## Definition (2013, T. Terzioğlu)

A bounded set $B$ of a Fréchet space $E$ is *prominent* if, for every 0-ngbh $U$, there exist a 0-nghb $V$ and $C > 0$ s.t. $\forall n$

$$\delta_n(V, U) \leq C\delta_n(B, V).$$

# Another sufficient condition

## Definition (2013, T. Terzioğlu)

A bounded set $B$ of a Fréchet space $E$ is *prominent* if, for every 0-ngbh $U$, there exist a 0-nghb $V$ and $C > 0$ s.t. $\forall n$

$$\delta_n(V, U) \leq C\delta_n(B, V).$$

## Proposition (2013, T. Terzioğlu)

If $E$ has a prominent bounded set, then $\Delta(E) = \Delta_b(E)$, but the converse is false.

# Another sufficient condition

**Definition (2013, T. Terzioğlu)**

A bounded set $B$ of a Fréchet space $E$ is *prominent* if, for every 0-ngbh $U$, there exist a 0-nghb $V$ and $C > 0$ s.t. $\forall n$

$$\delta_n(V, U) \leq C\delta_n(B, V).$$

**Proposition (2013, T. Terzioğlu)**

If $E$ has a prominent bounded set, then $\Delta(E) = \Delta_b(E)$, but the converse is false.

**Spaces with prominent bounded sets (2016, L.D., L.F., J.W.)**

Fréchet spaces with **Property** $(\overline{\Omega})$: if $\mathcal{U} = (U_k)_k$,

$$\forall m, \exists k, \forall j, \exists C > 0, \forall r > 0, U_k \subseteq rU_m + \frac{C}{r}U_j$$

(the converse is false).

# And for non-Fréchet/non-metrizable spaces?

# And for non-Fréchet/non-metrizable spaces?

**Theorem** (2017, F. Bastin, L.D.)

There exists a family of Schwartz (and/or nuclear), non-metrizable, l.c.s. $E$ with

$$\Delta(E) \neq \Delta_b(E).$$

# And for non-Fréchet/non-metrizable spaces?

**Theorem** (2017, F. Bastin, L.D.)

There exists a family of Schwartz (and/or nuclear), non-metrizable, l.c.s. $E$ with

$$\Delta(E) \neq \Delta_b(E).$$

**Main idea:** considering spaces for which the linear span of each bounded set is finite-dimensional.

Thank you for your attention!

# References I

F. Bastin and L. Demeulenaere.
On the equality between two diametral dimensions.
*Functiones et Approximatio, Commentarii Mathematici*, 56(1):95–107, 2017.

L. Demeulenaere.
Dimension diamétrale, espaces de suites, propriétés $(DN)$ et $(\Omega)$.
Master's thesis, University of Liège, 2014.

L. Demeulenaere, L. Frerick, and J. Wengenroth.
Diametral dimensions of Fréchet spaces.
*Studia Math.*, 234(3):271–280, 2016.

H. Jarchow.
*Locally Convex Spaces*.
Mathematische Leitfäden. B.G. Teubner, Stuttgart, 1981.

# References II

R. G. Meise and D. Vogt.
*Introduction to Functional Analysis*.
Number 2 in Oxford Graduate Texts in Mathematics.
Clarendon Press, Oxford, 1997.
Translated from German by M.S. Ramanujan.

T. Terzioğlu.
Quasinormability and diametral dimension.
*Turkish J. Math.*, 37(5):847–851, 2013.