Some notions of compactness in Functional Analysis and one related question about diametral dimensions

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Introduction

Some notions of compactness

Diametral dimensions
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Diametral dimensions
Mathematical Analysis
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- **Main topics**: functions and related notions, e.g. limits, distributions, measures, etc.
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Study of sets of functions
Notions of convergence
Mathematical Analysis

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\[ \text{Study of sets of functions} \quad \left\{ \begin{array}{c} \text{Notions of convergence} \\ \sim \text{Topological study of functional spaces} \end{array} \right. \]
Examples

- If $K$ is a compact subset of $\mathbb{R}^n$, if $f$ is a function defined on $K$ and if $(f_m)_{m \in \mathbb{N}_0}$ is a sequence of $C_0(K)$ s.t.

  $$\sup_{x \in K} |f_m(x) - f(x)| \to 0 \text{ if } m \to \infty,$$

then $f \in C_0(K)$. 
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  Convergence in $C_0(K)$ endowed with the metric defined by $\sup_K |.|$. 

Examples

- **Lebesgue’s dominated convergence theorem:** if \((f_m)_{m \in \mathbb{N}_0}\) is a sequence of \(L^1(\mathbb{R})\) which converges pointwise to \(f\) and if there exists \(F \in L^1(\mathbb{R})\) with \(|f_m| \leq F\) a.e. on \(\mathbb{R}\) \(\forall m\), then \(f \in L^1(\mathbb{R})\) and

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\int_{\mathbb{R}} |f_m(x) - f(x)| \, dx \to 0 \text{ if } m \to \infty.
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\[\leadsto\] Convergence in $L^1(\mathbb{R})$ for the metric defined by $\int_{\mathbb{R}} |.| \, dx$. 

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- **Fourier series**: convergence in \(L^2([a, b])\) for the metric defined by \(\sqrt{\int_a^b |.|^2 \, dx}\).

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- Convergence in \(L^1(\mathbb{R})\) for the metric defined by \(\int_{\mathbb{R}} |.| \, dx\).
- **Fourier series:** convergence in \(L^2([a, b])\) for the metric defined by \(\sqrt{\int_a^b |.|^2 \, dx}\).
- ...
- Common feature of these metrics: they are defined by norms.
(Semi)norms

Definition
If $E$ is a vector space on $\mathbb{C}$, a map $p : E \to [0, \infty)$ is a seminorm if

1. $p(x + y) \leq p(x) + p(y) \ \forall x, y \in E$;
2. $p(\lambda x) = |\lambda|p(x) \ \forall x \in E, \forall \lambda \in \mathbb{C}$.

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$C_0(K)$, $L_1(\mathbb{R})$, $L_2([a, b])$ are (complete) normed spaces (i.e. Banach spaces).
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The (semi)norn $p$ defines a vector (pseudo)metric on $E$:

$$d(x, y) = p(x - y).$$

~ Notions of convergent sequences, Cauchy sequences, etc.
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**Examples**

$C_0(K), \ L^1(\mathbb{R}), \ L^2([a, b])$ are (complete) normed spaces (i.e. Banach spaces).
In general, one single (semi)norm is not sufficient...

- If \( \Omega \) is an open subset of \( \mathbb{R}^n \) and if \( (f_m)_{m \in \mathbb{N}_0} \) is a sequence of \( C_0(\Omega) \) which uniformly converges to \( f \) on every compact of \( \Omega \), then \( f \in C_0(\Omega) \).
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\[ \sim \text{Convergence defined by a family of seminorms } \sup_{K} |.| (K \text{ compact of } \Omega). \]
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Convergence defined by a family of seminorms $\sup_K ||.||$ ($K$ compact of $\Omega$).

- For holomorphic functions: likewise!
Locally convex spaces

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A topological vector space (t.v.s.) $E$ is a *locally convex space* (l.c.s.) if its topology can be defined by a family of seminorms $\mathcal{P}$:
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Remark
Good definition if $\forall p, q \in \mathcal{P}, \exists r \in \mathcal{P}, C > 0 \text{ s.t.}$

$$\sup(p(x), q(x)) \leq Cr(x) \quad \forall x \in E.$$
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*Functional Analysis*: study of l.c.s.
Examples of topological properties

For a l.c.s. \((E, \mathcal{P})\),

- \(E\) is Hausdorff iff \(\cap_{p \in \mathcal{P}} \ker(p) = \{0\}\);
- every \(x \in E\) has a countable basis of nghbs iff \(\mathcal{P}\) can be chosen countable;
- \(E\) is metrizable iff the two previous points are verified.
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Definition

A Fréchet space is a complete, metrizable, l.c.s.
Introduction

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Diametral dimensions
Some notions linked to compactness

Let $E$ be an l.c.s.

- A *bounded set* of $E$ is a subset $B$ of $E$ s.t., for every 0-neighborhood $U$, $\exists \lambda > 0 : B \subseteq \lambda U$. 
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$$V \subseteq \varepsilon U + P.$$
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- $K \subseteq E$ is *precompact* if it is precompact with respect to each 0-neighborhood.
Proposition

Compact $\implies$ Precompact $\implies$ Bounded.
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Remarks

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Remarks

- A precompact set is not always compact, but, in *complete spaces*, a set is precompact iff its closure is compact.
- The closed unit ball of an infinite-dimensional normed space is never precompact. In particular, a closed bounded set is not always (pre)compact.
Two important classes of Fréchet spaces

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- $E$ is *Montel* if every bounded set is precompact.
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- $E$ is *Schwartz* if, for every 0-neighborhood $U$, there exists a 0-neighborhood $V$ which is precompact with respect to $U$. 

*Warning!* A Schwartz space is always Montel, but the converse is false!
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- An infinite-dimensional Banach space is neither Montel, nor Schwartz.
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Isomorphisms and topological invariants

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Dimension in linear algebra, being Hausdorff in topological spaces, etc.
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*And diametral dimension(s) in t.v.s.!*
Kolmogorov’s diameters

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**Definition**
The $n^{th}$ Kolmogorov’s diameters of $V$ with respect to $U$ is

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\delta_n(V, U) = \inf \{ \delta > 0 : \exists L \subseteq E, \text{dim}(L) \leq n, \text{s.t. } V \subseteq \delta U + L \}.
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**Important property**
If $U$ is a ball centred at 0 and associated to a seminorm, then

$$V \text{ is precompact with respect to } U \iff \delta_n(V, U) \to 0.$$
Diametral dimension

Let $E$ be a t.v.s. and $\mathcal{U}$ be a basis of 0-nghbs.

Definition

The *diametral dimension* of $E$ is

$$\Delta(E) = \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, V \subseteq U, \text{ s.t. } \xi_n \delta_n(V, U) \to 0 \right\}.$$
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**Properties**

1. $\Delta$ is a topological invariant (if $E \cong F$, then $\Delta(E) = \Delta(F)$).
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**Properties**

1. $\Delta$ is a topological invariant (if $E \cong F$, then $\Delta(E) = \Delta(F)$).
2. If $E$ is Fréchet,
   - if $E$ is not Schwartz, $\Delta(E) = c_0$;
   - if $E$ is Schwartz, $l_\infty \subseteq \Delta(E)$. 
Another diametral dimension...

Definition

\[ \Delta_b(E) = \left\{ \xi \in C^{N_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded}, \xi_n \delta_n(B, U) \to 0 \right\}. \]
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Open question

Do we have

\[ \Delta(E) = \Delta_b(E) \]

if \( E \) is Fréchet?
Proposition

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Consequences

- If $E$ is not Montel, $\Delta(E) = \Delta_b(E) = c_0$.
- If $E$ is Montel but not Schwartz, then $\Delta(E) = c_0 \subsetneq \Delta_b(E)$.
Proposition

If $E$ is Fréchet,

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- If $E$ is not Montel, $\Delta(E) = \Delta_b(E) = c_0$.
- If $E$ is Montel but not Schwartz, then $\Delta(E) = c_0 \subset \Delta_b(E)$.

New open question

Do we have

$$\Delta(E) = \Delta_b(E)$$

if $E$ is Fréchet-Schwartz?
A positive partial result

Slight variations of diametral dimensions...

$$\Delta^\infty(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, \text{s.t. } (\xi_n \delta_n(V, U))_n \in \ell_\infty \right\},$$

$$\Delta_b^\infty(E) := \left\{ \xi \in \mathbb{C}^{\mathbb{N}_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded}, (\xi_n \delta_n(B, U))_n \in \ell_\infty \right\}.$$
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\[ \Delta_b^{\infty}(E) := \left\{ \xi \in \mathbb{C}^{N_0} : \forall U \in \mathcal{U}, \forall B \text{ bounded}, (\xi_n \delta_n(B, U))_n \in \ell_{\infty} \right\}. \]

**Theorem (2016, L.D., L. Frerick, J. Wengenroth)**

If \( E \) is Fréchet-Schwartz, then

\[ \Delta_{\infty}(E) = \Delta_b^{\infty}(E). \]
A positive partial result

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$$\Delta^\infty(E) := \left\{ \xi \in C^{N_0} : \forall U \in \mathcal{U}, \exists V \in \mathcal{U}, \text{ s.t. } (\xi_n \delta_n (V, U))_n \in \ell_\infty \right\},$$

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**Theorem** (2016, L.D., L. Frerick, J. Wengenroth)

If $E$ is Fréchet-Schwartz, then

$$\Delta^\infty(E) = \Delta^\infty_b(E).$$

In particular, if $\Delta(E) = \Delta^\infty(E)$, then $\Delta(E) = \Delta_b(E)$. 
Schwartz spaces with $\Delta(E) = \Delta^\infty(E)$
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- Classic sequence spaces ("Köthe echelon spaces") (2017, F. Bastin, L.D.);
Schwartz spaces with $\Delta(E) = \Delta^\infty(E)$

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Another sufficient condition

**Definition (2013, T. Terzioğlu)**

A bounded set $B$ of a Fréchet space $E$ is *prominent* if, for every 0-neighborhood $U$, there exist a 0-neighborhood $V$ and $C > 0$ s.t. $\forall n$

$$\delta_n(V, U) \leq C\delta_n(B, V).$$
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**Proposition (2013, T. Terzioğlu)**
If $E$ has a prominent bounded set, then $\Delta(E) = \Delta_b(E)$, but the converse is false.
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If $E$ has a prominent bounded set, then $\Delta(E) = \Delta_b(E)$, but the converse is false.


Fréchet spaces with Property ($\overline{\Omega}$): if $\mathcal{U} = (U_k)_k$,

$$\forall m, \exists k, \forall j, \exists C > 0, \forall r > 0, U_k \subseteq rU_m + \frac{C}{r}U_j$$

(the converse is false).
And for non-Fréchet/non-metrizable spaces?
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**Theorem (2017, F. Bastin, L.D.)**
There exists a family of Schwartz (and/or nuclear), non-metrizable, l.c.s. $E$ with

\[ \Delta(E) \neq \Delta_b(E). \]
And for non-Fréchet/non-metrizable spaces?

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There exists a family of Schwartz (and/or nuclear), non-metrizable, l.c.s. $E$ with

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**Main idea:** considering spaces for which the linear span of each bounded set is finite-dimensional.
Thank you for your attention!
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