# Linearization and quadratization techniques for multilinear 0–1 optimization problems

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# Multilinear 0–1 optimization

## Multilinear 0–1 optimization

$$\min \sum_{e \in E} a_e \prod_{i \in e} x_i + \sum_{i \in V} c_i x_i$$
$$\text{s. t. } x_i \in \{0,1\} \qquad i \in V$$

- $V = \{1, \ldots, n\}$, $E$ = set of subsets $e$ of $V$ with $|e| \geq 2$ and $a_e \neq 0$,
- $V$ and $E$ define a hypergraph $H$.

Example:

$$f(x_1, x_2, x_3) = 9x_1x_2x_3 + 8x_1x_2 - 6x_2x_3 + x_1 - 2x_2 + x_3$$

# Applications: Computer Vision

Image restoration problems modelled as energy minimization

$$E(l) = \sum_{p \in \mathcal{P}} D_p(l_p) + \sum_{S \subseteq \mathcal{P}, |S| \geq 2} \sum_{p_1, \ldots, p_s \in S} V_{p_1, \ldots, p_s}(l_{p_1}, \ldots, l_{p_s}),$$

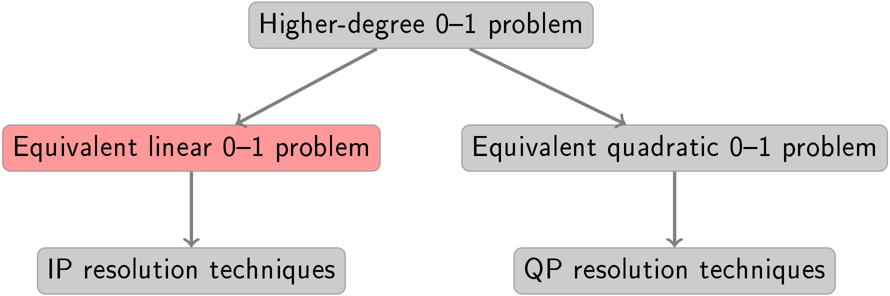
where $l_p \in \{0, 1\} \;\; \forall p \in \mathcal{P}$.

(Image from "Corel database" with additive Gaussian noise.)

# Applications

- Constraint Satisfaction Problem
- Data mining, classification, learning theory...
- Joint supply chain design and inventory management
- Production management
- ...

# General idea



# Standard Linearization (SL)

$$\min \sum_{e \in E} a_e \prod_{i \in e} x_i + \sum_{i \in V} c_i x_i$$

Standard Linearization (Fortet (1959), Glover and Woolsey (1973))

$$y_e = \prod_{i \in e} x_i$$

$$-y_e + x_i \geq 0 \qquad \forall i \in e, \forall e \in E \qquad (1)$$

$$y_e - \sum_{i \in e} x_i \geq 1 - |e| \qquad \forall e \in E \qquad (2)$$

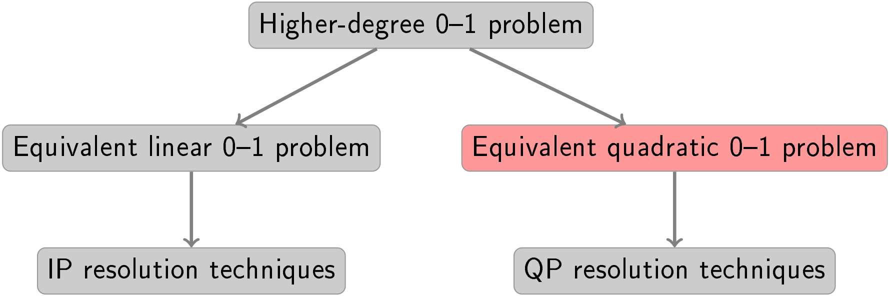
# SL main drawback and contributions

SL drawback: The continuous relaxation given by the SL is very weak!

Contributions:

- Characterization of cases for which SL provides a perfect formulation (Buchheim, Crama, Rodríguez-Heck (2017), discovered independently by Del Pia, Khajavirad (2017)).
- Definition of a class of valid inequalities strengthening the SL formulation, working especially well for simplified computer vision instances (Crama, Rodríguez-Heck (2017)).

# General idea



# Quadratizations definition

### Definition: Quadratization

Given a multilinear polynomial $f(x)$ on $\{0,1\}^n$, we say that $g(x,y)$ is a **quadratization** of $f$ if $g(x,y)$ is a quadratic polynomial depending on $x$ and on $m$ *auxiliary variables* $y_1,\dots,y_m$, such that

$$f(x) = \min\{g(x,y) : y \in \{0,1\}^m\} \ \ \forall x \in \{0,1\}^n.$$

Then,
$\min\{f(x) : x \in \{0,1\}^n\} = \min\{g(x,y) : x \in \{0,1\}^n, y \in \{0,1\}^m\}$.

Which quadratizations are "good"?

- Small number of auxiliary variables.
- Good optimization properties: submodularity (intuitive measure: small number of positive quadratic terms).

# Termwise quadratizations

Multilinear expression of a pseudo-Boolean function:

$$f(x) = -35x_1x_2x_3x_4x_5 + 50x_1x_2x_3x_4 - 10x_1x_2x_4x_5 + 5x_2x_3x_4 + 5x_4x_5 - 20x_1$$

**Idea:** quadratize monomial by monomial, using different sets of auxiliary variables for each monomial.

- Negative case well solved (one auxiliary variable, submodular quadratization).
- Positive monomials much more difficult: just improved the best bound for number of variables!

# Negative monomial

Kolmogorov and Zabih (2004), Freedman and Drineas (2005).

$$-\prod_{i=1}^{n} x_i = \min_{y\in\{0,1\}} -y\left(\sum_{i=1}^{n} x_i - (n-1)\right).$$

Why is this a quadratization? $f(x) = -x_1x_2x_3x_4$

- If $x_i = 1$ for all $i$, then $\min_{y\in\{0,1\}} -y$, reached for $y = 1$, value $-1$.
- If there is an $i$ with $x_i = 0$, then $y$ has a nonnegative coefficient, minimum reached for $y = 0$.

# Positive monomial: Literature

## Ishikawa (2011)

$$\prod_{i=1}^{n} x_i = \min_{y_1,\dots y_k \in \{0,1\}} \sum_{i=1}^{k} y_i(c_{i,n}(-S_1 + 2i) - 1) + aS_2,$$

$S_1, S_2$: elementary linear and quadratic symmetric polynomials in $n$ variables,

$\mathbf{k} = \lfloor \frac{\mathbf{n-1}}{\mathbf{2}} \rfloor$ and $c_{i,n} = \begin{cases} 1, \text{if } n \text{ is odd and } i = k, \\ 2, \text{otherwise.} \end{cases}$

- **Number of variables:** best published bound for positive monomials.
- **Submodularity:** $\binom{n}{2}$ positive quadratic terms, but very good computational results.

# 1st improvement: $\lceil \frac{n}{4} \rceil$ variables

Theorem 1 (E. Boros, Y. Crama, E. R-H)

For all integers $n, m$, if $n \geq 2$, $\frac{n}{4} \leq m \leq \frac{n}{2}$, and $N = n - 2m$ then

$$g(x, y) = \frac{1}{2}\left(X - Ny_1 - 2Y\right)\left(X - Ny_1 - 2Y - 1\right)$$

is a quadratization of the positive monomial $P_n = \prod_{i=1}^{n} x_i$ using $m$ auxiliary variables, where $X = \sum_{i=1}^{n} x_i$ and $Y = \sum_{j=2}^{m} y_j$.

# 2nd improvement: $\lceil log(n) \rceil - 1$ variables

**Theorem 2 (E. Boros, Y. Crama, E. R-H)**

Let $n \leq 2^{k+1}$, $K = 2^{k+1} - n$ and $X = \sum_{i=1}^{n} x_i$. Then,

$$g(x,y) = \frac{1}{2}(K + X - \sum_{i=1}^{k} 2^i y_i)(K + X - \sum_{i=1}^{k} 2^i y_i - 1)$$

is a quadratization of the positive monomial $f(x) = P_n(x) = \prod_{i=1}^{n} x_i$ using $k$ auxiliary variables.

Proof idea:

- $g(x,y) \geq 0$ (half-product of consecutive integers).
- If $X \leq n-1$: $K+X$ even: make 1st factor zero, $K+X$ odd: make 2nd factor zero.
- If $X = n$: 1st factor is at least 2, 2nd factor is at least 1.

# Positive monomial: new quadratizations

Smallest number of variables known until now:

- $\lceil log(n) \rceil - 1$ variables

Two other quadratizations, more variables but maybe better optimization properties (?)

- $\lceil log(n-1) \rceil$ variables.
- $\lceil \frac{n}{4} \rceil$ variables.

→ Quadratizations being tested by a group at Cornell University.

# Current work: computational

Instance sets:

- random polynomials,
- computer vision inspired polynomials,
- supply chain & inventory management.

Methods to compare:

- Standard linearization
- Termwise quadratizations

| Pos. Mon. ($P_n$) | Neg. Mon. ($N_n$) |
|---|---|
| Ishikawa | 1-var. quadrat. |
| $\lceil \frac{n}{4} \rceil$ | 1-var. quadrat. |
| $\lceil log(n-1) \rceil$ | 1-var. quadrat. |
| $\lceil log(n) \rceil - 1$ | 1-var. quadrat. |

# Current work: theoretical

Open questions:

## Conjecture 1

We need at least $m = \lceil log(n) \rceil - 1$ variables to quadratize the positive monomial.

## Conjecture 2

There is a trade-off between having small number of variables and good optimization properties, more precisely, the "most submodular" quadratizations of the positive monomial have $n-1$ positive quadratic terms and use $m = n-2$ variables.

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