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Expansions for the scattering wave function and radiative capture in R-matrix theory

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We derive several expressions for the scattering wave functions in the frame of R-matrix theory. Particular attention is paid to the one-channel situation and to the one-level plus constant background approximation. These results are then applied to the radiative capture process. In particular, we investigate the possible appearance, and the physical interpretation, of asymmetric resonance peaks and of zeros in the capture cross section. The contributions of the external and internal regions are both included. Finally, we show that the resonance parameters of the one-level plus constant background Breit-Wigner approximation are stable against variation of the boundary parameters, the values of which therefore remain largely arbitrary. Correspondingly, the validity of this approximation does not depend critically upon the choice of the boundary parameters.

[NUCLEAR REACTIONS R-matrix theory: various expressions for scattering wave functions with application to radiative capture; dependence of resonance parameters upon boundary conditions.]

I. INTRODUCTION

The proper physical interpretation of the resonance parameters and of the background cross section is a central problem in compound nuclear reaction theory.[1] This topic has recently received renewed attention, stimulated by the experimental observation of background cross sections and of asymmetric resonance peaks, mainly in photonuclear reactions,[2-9] and also in reactions involving only particle channels.[10,11] The type of asymmetry is related to the phase of interference between the background and the resonance contributions, and may therefore yield rather detailed information on the scattering wave function in the vicinity of a resonance. This has been exploited in the analysis of the isobaric analog resonances excited in (p,p) and (p,n) reactions,[11] where the asymmetry can be related to the isospin purity.[12] Here we derive, in the frame of R-matrix theory,[13] expressions for the scattering wave functions and for the radiative capture amplitude which are well suited for the discussion and analysis of this type of experimental data. More specifically, the main purpose of the present paper is threefold.

(i) In Sec. II, we establish several expressions for the scattering wave functions Ψ_E^c at an energy E, both in the internal and external region. Here the upper index c refers to the entrance channel. We show that the result given for Ψ_E^c in Ref. 13 is not valid in the important case when a few levels are treated on a separate footing, as for instance in the one-level plus constant background approximation. We first derive a general expression for Ψ_E^c in the internal region, in terms of the full R matrix. Then we divide the levels into two groups and find a form of Ψ_E^c which contains the level matrix corresponding to the levels of one group only, and is therefore convenient for the discussion of the few-level approximation and of the resonance and background wave functions. We combine the external and internal parts of Ψ_E^c in one single expression. We show that, in the one open channel case, Ψ_E^c becomes real except for an energy-dependent phase factor. This result leads to interesting consequences for the radiative capture amplitude. Finally, we discuss in more detail the one-level plus constant background approximation.

(ii) Section III is devoted to the radiative capture amplitude, in the case when the coupling of the nucleons to the electromagnetic field can be treated in first order perturbation theory. We can then use the expressions obtained in Sec. II for Ψ_E^c and obtain various forms for the radiative capture amplitude. We pay particular attention to the radiative capture at low energy, where the one-channel assumption usually applies. Then we show that the capture cross section must vanish between two consecutive eigenvalues for which the products of the particle by the photon partial width amplitudes have the same sign. A zero always exists in the one-level plus constant background approximation, and may lead to an asymmetric resonance peak if it falls sufficiently close to the resonance energy. These results may be considered as extensions of a theorem due to Wigner.[14,15] We emphasize that we include the contribution of the external region (external capture).

(iii) In Sec. IV, we investigate the influence of the choice of the boundary parameters (channel radii a_c and logarithmic derivatives B_c) on the resonance parameters in the one-level plus constant background approximation. This is related to the problem of the "best" choice of the boundary parameters, i.e., the choice for which the one-level approximation is justified. This problem has been discussed for the one-level approximation without background in Ref. 16 for potential scattering and in Refs. 17 and 18 for the many-channel case. The choice of the boundary parameters was found to be fairly critical in the latter case. Here, we show that this choice is largely arbitrary in the one-level plus constant background approximation. The argument is based upon a set of differential equations which express the dependence of the R-matrix quantities upon the boundary parameters.[19]

II. SCATTERING WAVE FUNCTION

A. Notation

We denote by Ψ_E^c the scattering wave function with an incoming wave of unit flux in channel c, and by v_c and k_c the relative velocity and relative wave number, respectively. We call ψ_c the surface function, which contains the external wave functions of the two fragments. We have in the external region $(r_{c'} \geq a_{c'})$

$$\Psi_E^c = \sum_{c'} \psi_{c'}[v_c^{-1/2} I_c(r_c, k_c)\delta_{cc'} - v_{c'}^{-1/2} U_{cc'} O_{c'}(r_{c'}, k_{c'})] . \quad \text{(II.1)}$$

Here, I_c and O_c denote the incoming and outgoing waves, respectively. As much as possible we adopt the notation of Ref. 13. In the internal region $(r_{c'} \leq a_{c'})$, we expand Ψ_E^c in terms of the eigenstates $\{X_\lambda\}$, which have a specified logarithmic derivative B_c at $r_c = a_c$:

$$\Psi_E^c = \sum_\lambda A_\lambda^c X_\lambda . \quad \text{(II.2)}$$

The coefficients A_λ^c fulfill the linear system of equations[13,20]

$$\sum_\mu D_{\lambda\mu} A_\mu^c = -i\hbar^{1/2}\Omega_c \Gamma_{\lambda c}{}^{1/2} , \quad \text{(II.3)}$$

where

$$D_{\lambda\mu} = (\underline{A}^{-1})_{\lambda\mu} = (E_\lambda - E)\delta_{\lambda\mu} + \sum_c (\Delta_{\lambda\mu c} - \tfrac{1}{2} i \Gamma_{\lambda\mu c}) , \quad \text{(II.4)}$$

$$\Delta_{\lambda\mu c} = (B_c - S_c)\gamma_{\lambda c}\gamma_{\mu c} ,$$
$$\Gamma_{\lambda\mu c} = 2P_c \gamma_{\lambda c}\gamma_{\mu c} , \quad \text{(II.5)}$$
$$\Gamma_{\lambda c} = \Gamma_{\lambda\lambda c} .$$

The quantities E_λ, Ω_c, S_c, and P_c are the familiar eigenvalues, hard sphere functions, shift and penetration factors of R-matrix theory.[13] Equating (II.1) and (II.2) at $r_{c'} = a_{c'}$, we find

$$U_{c'c} = \Omega_c{}^2 \delta_{cc'} - \Omega_{c'} \hbar^{1/2} \sum_\lambda A_\lambda^c \Gamma_{\lambda c'}{}^{1/2} . \quad \text{(II.6)}$$

B. Wave function in the internal region

The relation (IX.1.31) of Ref. 13 for Ψ_E^c in the internal region is obtained by solving Eqs. (II.3) and by inserting the result in Eq. (II.2). The resulting expression is not convenient in practice since it involves the infinite matrix $\underline{A}$. Hence it is useful to consider few-level approximations. We use the standard notation[13]

$$L_c^0 = S_c - B_c + iP_c = S_c^0 + iP_c ,$$
$$L_{cc'}^0 = L_c^0 \delta_{cc'} , \quad \text{(II.7)}$$

and write Eq. (II.3) in the form

$$A_\omega^c + \sum_{c'} L_{c'}^0 (E_\omega - E)^{-1} \gamma_{\omega c'} \sum_\eta A_\eta^c \gamma_{\eta c'} = -i\hbar^{1/2}\Omega_c \Gamma_{\omega c}{}^{1/2} (E_\omega - E)^{-1} . \quad \text{(II.8)}$$

By multiplying Eq. (II.8) by $\gamma_{\omega c'}$ and summing over ω, we obtain a system of linear equations for the quantities $\sum_\eta A_\eta^c \gamma_{\eta c'}$, whose solution reads

$$\sum_\eta A_\eta^c \gamma_{\eta c'} = -i\hbar^{1/2}\Omega_c (2P_c)^{1/2} \times \sum_{c''} [(\underline{1} - \underline{R}\underline{L}^0)^{-1}]_{c'c''} R_{c''c} , \quad \text{(II.9)}$$

where

$$R_{cc'} = \sum_\omega \frac{\gamma_{\omega c}\gamma_{\omega c'}}{E_\omega - E} \quad \text{(II.10)}$$

are the elements of the R matrix. From (II.8) and (II.9), we obtain

$$A_\omega^c = -i\hbar^{1/2}\Omega_c (2P_c)^{1/2} (E_\omega - E)^{-1} \times \left\{ \gamma_{\omega c} + \sum_{c'} \gamma_{\omega c'} L_{c'}^0 \sum_{c''} [(\underline{1} - \underline{R}\underline{L}^0)^{-1}]_{c'c''} R_{c''c} \right\} . \quad \text{(II.11)}$$

Another form for A_ω^c can be found by using in Eq. (II.11) the relation (IX.1.13) of Ref. 13:

$$A_\omega^c = -i\hbar^{1/2}\Omega_c (2P_c)^{1/2} (E_\omega - E)^{-1} \times \left(\gamma_{\omega c} + \sum_{c'} L_{c'}^0 \gamma_{\omega c'} \sum_{\eta\eta'} \gamma_{\eta c}\gamma_{\eta' c'} A_{\eta\eta'} \right) . \quad \text{(II.12)}$$

We now divide the levels X_ω into two groups. We reserve the indices λ, μ, ... for the retained levels, and call ν, τ, ... the other ones. The quantities ω, η, ... refer generically to both types of levels. We write

$$R_{cc'} = R^0_{cc'} + R'_{cc'} = \sum_\nu \frac{\gamma_{\nu c}\gamma_{\nu c'}}{E_\nu - E} + \sum_\lambda \frac{\gamma_{\lambda c}\gamma_{\lambda c'}}{E_\lambda - E}. \quad \text{(II.13)}$$

The following relation holds[13]:

$$[(\underline{1} - \underline{R}\,\underline{L}^0)^{-1}\underline{R}]_{cc'} = [(\underline{1} - \underline{R}^0\underline{L}^0)^{-1}\underline{R}^0]_{cc'} + \sum_{\lambda\mu} \alpha_{\mu c}\alpha_{\mu c'}\bar{A}_{\lambda\mu}, \quad \text{(II.14)}$$

where

$$\bar{A}_{\lambda\mu} = (\underline{\bar{D}}^{-1})_{\lambda\mu}, \quad \bar{D}_{\lambda\mu} = (E_\lambda - E)\delta_{\lambda\mu} - \sum_c \beta_{\lambda c}\gamma_{\mu c}, \quad \text{(II.15)}$$

$$\beta_{\lambda c} = L^0_c \alpha_{\lambda c}, \quad \alpha_{\lambda c} = \sum_{c'} [(\underline{1} - \underline{R}^0\underline{L}^0)^{-1}]_{cc'}\gamma_{\lambda c'}. \quad \text{(II.16)}$$

Equations (II.11) and (II.14) yield

$$A^c_\omega = -i\hbar^{1/2}\Omega_c(2P_c)^{1/2}(E_\omega - E)^{-1} \times \left(\alpha_{\omega c} + \sum_{c'}\sum_{\lambda\mu} \bar{A}_{\lambda\mu}\gamma_{\omega c'}\beta_{\lambda c'}\alpha_{\mu c}\right). \quad \text{(II.17)}$$

This is the result we were aiming at, since the dimension of the matrix $\bar{A}_{\lambda\mu}$ equals the number of selected levels. The right-hand side of Eq. (II.17) can be given a simple form when ω equals one of the selected values μ. Then we can use the following relation, obtained from Eqs. (II.15):

$$(E_\lambda - E)\bar{A}_{\lambda\mu} - \delta_{\lambda\mu} = \sum_c \sum_{\mu'} \beta_{\lambda c}\gamma_{\mu' c}\bar{A}_{\mu'\mu}. \quad \text{(II.18)}$$

We find

$$A^c_\mu = -i\hbar^{1/2}\Omega_c(2P_c)^{1/2}\sum_\lambda \bar{A}_{\mu\lambda}\alpha_{\lambda c}. \quad \text{(II.19)}$$

This is the expression given in Ref. 13 [Eq. (IX.1.27)] where, however, it was overlooked that it holds *only* to the selected indices. In the general case, the more complicated expression (II.17) must be used. The expression for Ψ^c_E in the internal region reads

$$\Psi^c_E = -i\hbar^{1/2}(2P_c)^{1/2}\Omega_c\left[\sum_\nu \frac{\alpha_{\nu c}}{E_\nu - E}X_\nu + \sum_{\lambda\mu}\bar{A}_{\lambda\mu}\alpha_{\mu c}\left(X_\lambda + \sum_{c'}\beta_{\lambda c'}\sum_\nu \frac{\gamma_{\nu c'}}{E_\nu - E}X_\nu\right)\right]. \quad \text{(II.20)}$$

In the analysis of an isolated resonance, it appears natural to treat explicitly only one level (X_λ) and to take the other ones into account in a global way. Equation (II.20) yields

$$\Psi^c_E = -i\hbar^{1/2}(2P_c)^{1/2}\Omega_c\left[\sum_\nu \frac{\alpha_{\nu c}}{E_\nu - E}X_\nu + \frac{\alpha_{\lambda c}}{E_\lambda - E - \sum_{c'}\beta_{\lambda c'}\gamma_{\lambda c'}}\left(X_\lambda + \sum_{c'}\beta_{\lambda c'}\sum_\nu \frac{\gamma_{\nu c'}}{E_\nu - E}X_\nu\right)\right]. \quad \text{(II.21)}$$

The one-level approximation plus constant background is obtained by assuming that the matrix $\underline{R}^0$ is independent of energy, in the region of interest. The corresponding Breit-Wigner form of the collision matrix is given by[13]

$$U_{cc'} = U^0_{cc'} + i\Omega_c\Omega_{c'}\frac{\bar{\Gamma}_{\lambda c}^{1/2}\bar{\Gamma}_{\lambda c'}^{1/2}}{\bar{E}_\lambda - E - \frac{1}{2}i\bar{\Gamma}_\lambda}, \quad \text{(II.22)}$$

where

$$\bar{E}_\lambda = E_\lambda - \text{Re}\sum_{c'}\beta_{\lambda c'}\gamma_{\lambda c'} = E_\lambda + \bar{\Delta}_\lambda, \quad \text{(II.23)}$$

$$\bar{\Gamma}_{\lambda c}^{1/2} = (2P_c)^{1/2}\alpha_{\lambda c}, \quad \bar{\Gamma}_\lambda = \sum_c |\bar{\Gamma}_{\lambda c}|, \quad \text{(II.24)}$$

$$U^0_{cc'} = \Omega_c^2\delta_{cc'} + 2i\Omega_c P_c^{1/2}\Omega_{c'}P_{c'}^{1/2}[(\underline{1} - \underline{R}^0\underline{L}^0)^{-1}\underline{R}^0]_{cc'}. \quad \text{(II.25)}$$

These expressions will be used in Sec. IV. For simplicity, we neglect here the energy dependence of the surface functions L^0_c. Then the quantities $\alpha_{\lambda c}$ and $\beta_{\lambda c}$ are independent of energy. Equation (II.22) shows that the quantities $U^0_{cc'}$, $\bar{\Gamma}_{\lambda c}^{1/2}$, $\bar{E}_\lambda$, and $\bar{\Gamma}_\lambda$ can then be identified with the background, partial width, resonance energy, and total width, respectively. Equation (II.21) can now be written in the form

$$\Psi^c_E = -i\hbar^{1/2}\Omega_c\left(\xi^0_c + \frac{\bar{\Gamma}_{\lambda c}^{1/2}}{\bar{E}_\lambda - E + \frac{1}{2}i\bar{\Gamma}_\lambda}Y_\lambda\right), \quad \text{(II.26)}$$

where

$$\xi^0_c = \sum_\nu \frac{\bar{\Gamma}_{\nu c}^{1/2}}{E_\nu - E}X_\nu, \qquad Y_\lambda = X_\lambda + \sum_{c'}\beta_{\lambda c'}\sum_\nu \frac{\gamma_{\nu c'}}{E_\nu - E}X_\nu. \quad \text{(II.27)}$$

The state Y_λ has the properties which are expected for a resonance state: it is independent of the entrance channel and appears to be excited with a Breit-Wigner "probability amplitude." We note that it contains admixtures from all the eigenstates X_ω. Moreover, Ψ^c_E contains a background contribution (ξ_c^0) which depends upon the entrance channel. The states Y_λ and ξ_c^0 are not orthogonal. It is therefore not quite justified to identify the coefficient of Y_λ with an "excitation probability amplitude." We shall see below that $\Psi_{E_\lambda}^c$ reduces to X_λ in the internal region in the one-channel case.

C. Wave function in full space

Following Lane and Lynn,[21] we extend the basis states X_ω in the external region by introducing the quantities

$$\tilde{X}_\omega = X_\omega \quad \text{for } r_{c'} \le a_{c'},$$
$$= \left(\frac{2M_{c'}a_{c'}}{\hbar^2}\right)^{1/2} \gamma_{\omega c'} \frac{O_{c'}(r_{c'}, k_{c'})}{O_{c'}(a_{c'}, k_{c'})} \psi_{c'} \quad \text{for } r_{c'} \ge a_{c'}. \qquad \text{(II.28)}$$

We note that the value of $\tilde{X}_\omega$ in the external region is energy-dependent, and that the radial derivative of $\tilde{X}_\omega$ is discontinuous at $r_{c'} = a_{c'}$. From Eqs. (II.1) and (II.6), we find, in the whole space,

$$\Psi_E^c = (v_c)^{-1/2}\Omega_c[\Omega_c^{-1}I_c - \Omega_c O_c]_{\text{ext}}\psi_c\delta_{cc'} + \sum_\omega A_\omega^c \tilde{X}_\omega, \qquad \text{(II.29)}$$

where the index "ext" indicates that the quantity contained inside the square brackets differs from zero in the external region only; since $\Omega_c^* = \Omega_c^{-1}$, this quantity is, moreover, purely imaginary. The one-level form is obtained from Eqs. (II.26) and (II.29)

$$\Psi_E^c = v_c^{-1/2}\Omega_c[\Omega_c^* I_c - \Omega_c O_c]_{\text{ext}}\psi_c - i\hbar^{1/2}\Omega_c\left(\tilde{\xi}_c^0 + \frac{\overline{\Gamma}_{\lambda c}{}^{1/2}}{\overline{E}_\lambda - E + \frac{1}{2}i\overline{\Gamma}_\lambda}\tilde{Y}_\lambda\right), \qquad \text{(II.30)}$$

where $\tilde{\xi}_c^0$ and $\tilde{Y}_\lambda$ are obtained by replacing X_ω by $\tilde{X}_\omega$ in the expressions (II.27) of ξ_c^0 and Y_λ. The quantity $\tilde{Y}_\lambda$ is purely outgoing in the external region and is therefore also a possible candidate for the wave function of the resonance.

D. One-channel case

The definition of $\alpha_{\lambda c}$ involves the matrix $(\underline{1} - \underline{R}^0\underline{L}^0)^{-1}$, which takes a simple form only when $\underline{R}^0$ is diagonal. Most of the expressions given below can easily be extended to that case. In the present section, however, we restrict the discussion to the one-channel case, which applies to low energy neutron scattering and to radiative capture on even-even nuclei, as we shall discuss in Sec. III. Eq. (II.11) yields

$$A_\omega^c = -i\hbar^{1/2}\Omega_c\gamma_{\omega c}(1 - R_{cc}L_c^0)^{-1}. \qquad \text{(II.31)}$$

From Eqs. (II.28) and (II.29), we then find that, in all space,

$$\Psi_E^c = 2iv_c^{-1/2}\Omega_c(1 - R_{cc}L_c^0)^{-1} \times \left\{\text{Im}[\Omega_c^*(1 - R_{cc}L_c^0)I_c]_{\text{ext}}\psi_c - \tfrac{1}{2}v_c^{1/2}\hbar^{1/2}(2P_c)^{1/2}\sum_\omega \frac{\gamma_{\omega c}}{E_\omega - E}X_\omega\right\}. \qquad \text{(II.32)}$$

This expression shows that Ψ_E^c is in the one-channel case a real function of the coordinates except for an energy dependent phase factor. Since this result is not based on any approximation, we expect that it can be proved in the frame of general scattering theory. This is shown in the Appendix.

Using Eqs. (II.18) and (II.21), we find the following expression for Ψ_E^c in the internal region:

$$\Psi_E^c = -i\hbar^{1/2}\Omega_c(E_\lambda - E - \beta_{\lambda c}\gamma_{\lambda c})^{-1} \times \left[\Gamma_{\lambda c}{}^{1/2}X_\lambda + (E_\lambda - E)\sum_\nu \frac{\overline{\Gamma}_{\nu c}{}^{1/2}}{E_\nu - E}X_\nu\right]. \qquad \text{(II.33)}$$

This result shows that $A_\nu^c(E_\nu) = 0$ for $\nu \ne \lambda$. Hence, the one-level approximation without background is exact at $E = E_\lambda$. In other words, ξ_c^0 is exactly compensated by the components of the resonance state Y_λ along the background eigenstates X_ν ($\nu \ne \lambda$) at the energy E_λ. It can be checked that the closed channels do not modify qualitatively these conclusions: they simply shift the energy at which the coefficients A_ν^c vanish. This shift disappears if we take the boundary parameters B_c equal to S_c in the closed channels.

From Eqs. (II.16), (II.17), (II.19), (II.30), and (II.33), we find, after a straightforward calculation, the following value for Ψ_E^c in all space, in the one-channel, one-level case:

$$\Psi_E^c = 2iv_c^{-1/2}\Omega_c(1 - R_{cc}^0L_c^0)^{-1}(\overline{E}_\lambda - E + \tfrac{1}{2}i|\overline{\Gamma}_{\lambda c}|)^{-1} \times \left\{\text{Im}[\Omega_c^*(\overline{E}_\lambda - E + \tfrac{1}{2}i|\overline{\Gamma}_{\lambda c}|)(1 - R_{cc}^0L_c^0)I_c]_{\text{ext}}\psi_c - \tfrac{1}{2}v_c^{1/2}\hbar^{1/2}(2P_c)^{1/2} \times \left[\gamma_{\lambda c}X_\lambda + (E_\lambda - E)\sum_\nu \frac{\gamma_{\nu c}}{E_\nu - E}X_\nu\right]\right\}. \qquad \text{(II.34)}$$

This one-level expression has over Eq. (II.30) the merit of showing explicitly that Ψ_E^c is real, except for a phase factor.

III. RADIATIVE CAPTURE

We assume that the coupling between the nucleons and the electromagnetic field can be treated in first order perturbation theory. The transition amplitude for radiative capture from an entrance channel c into a final nuclear state Ψ_f is then given by

$$U_{cp} = \langle \Psi_f | EM | \Psi_E^c \rangle , \tag{III.1}$$

where p denotes the quantum numbers of the photon and EM is the photon emission operator, suitably normalized. If the contribution of the photon channels to the resonance widths and energy shifts is not negligible, one can still use for Ψ_E^c those expressions derived in Sec. II which involve the level matrix $\underline{A}$, provided that the quantities $\Delta_{\lambda\mu}$ and $\Gamma_{\lambda\mu}$ include contributions from the photon channels.[13] Since, however, $\Delta_{\lambda\mu p}$ cannot be factorized in a product of two quantities depending upon λ and μ, respectively, it is, in that case, not possible to introduce a channel matrix $\underline{R}$ without making a further approximation. Here we assume that the damping due to photon channels can be neglected, and give several forms for U_{cp}.

From Eqs. (II.3), (II.4), and (II.29), we obtain

$$U_{cp} = U_{cp}^{hs} + i\Omega_c \sum_{\omega,\eta} A_{\omega\eta} \Gamma_{\omega c}{}^{1/2} \Gamma_{\eta p}{}^{1/2} , \tag{III.2}$$

where

$$U_{cp}^{hs} = v_c{}^{-1/2} \Omega_c \langle \Psi_f | EM | (\Omega_c{}^{-1} I_c - \Omega_c O_c)_{\rm ext} \psi_c \rangle \tag{III.3}$$

is a purely imaginary quantity, while

$$\Gamma_{\eta p}{}^{1/2} = -\hbar^{1/2} \langle \Psi_f | EM | \tilde{X}_\eta \rangle . \tag{III.4}$$

Equation (III.2) is not useful since it involves the infinite matrix $\underline{A}$. A more practical expression is obtained from Eq. (II.30):

$$U_{cp} = U_{cp}^0 - i \frac{\overline{\Gamma}_{\lambda c}{}^{1/2} \overline{\Gamma}_{\lambda p}{}^{1/2}}{E - \overline{E}_\lambda + \frac{1}{2} i \overline{\Gamma}_\lambda} , \tag{III.5}$$

where

$$U_{cp}^0 = U_{cp}^{hs} - i\hbar^{1/2} \Omega_c \langle \Psi_f | EM | \tilde{\xi}_c^0 \rangle , \tag{III.6}$$

$$\overline{\Gamma}_{\lambda p}{}^{1/2} = -\hbar^{1/2} \Omega_c \langle \Psi_f | EM | \tilde{Y}_\lambda \rangle . \tag{III.7}$$

We note that the quantities U_{cp}^0, $\overline{\Gamma}_{\lambda c}$, and $\overline{\Gamma}_{\lambda p}$ are all complex. The shape of the resonance peak depends upon their relative phases and can in general not be predicted.

More specific results can be written in the case of only one particle channel. Equation (II.32) then yields

$$U_{cp} = 2i (v_c)^{-1/2} \Omega_c (1 - R_{cc} L_c^0)^{-1} (2P_c)^{1/2} \times \left(\sum_\omega \frac{\gamma_{\omega c} \gamma_{\omega p}}{E_\omega - E} + b_{cp} \right) , \tag{III.8}$$

where the quantities

$$\gamma_{\omega p} = -\tfrac{1}{2} v_c{}^{1/2} \hbar^{1/2} \langle \Psi_f | EM | X_\omega \rangle - (2P_c)^{-1/2} \langle \Psi_f | EM | \mathrm{Im}[\Omega_c^* L_c^0 I_c]_{\rm ext} \psi_c \rangle \gamma_{\omega c} , \tag{III.9}$$

$$b_{cp} = (2P_c)^{-1/2} \langle \Psi_f | EM | \mathrm{Im}[\Omega_c^* I_c]_{\rm ext} \psi_c \rangle \tag{III.10}$$

are *real*. Equation (III.8) shows that the capture cross section vanishes between two eigenvalues for which the products $\gamma_{\omega c}\gamma_{\omega p}$ have the same sign. This property is analogous to the Wigner theorem in the two-particle channel case.[14, 15] We note that the contribution of external capture is included. This contribution is, however, smoothly energy-dependent, but this should not, in practice, affect the validity of the result. If the contribution of the external region is neglected, one recovers the standard form of Wigner's theorem. If, on the contrary, the contribution of the internal region can be neglected, the capture cross section vanishes between two consecutive energies E_ω.

If the zero falls close enough to a resonance energy, it may yield asymmetric resonance peaks. It is thus of interest to study in more detail the one-level plus constant background approximation.

From Eqs. (II.34) and (III.1), we obtain

$$U_{cp} = 2i v_c{}^{-1/2} \Omega_c (1 - R_{cc}^0 L_c^0)^{-1} \rho_\lambda \frac{e_\lambda - E}{\overline{E}_\lambda - E + \frac{1}{2} i |\overline{\Gamma}_{\lambda c}|} , \tag{III.11}$$

where

$$\rho_\lambda = \langle \Psi_f | EM | \mathrm{Im}[\Omega_c^* (1 - R_{cc}^0 L_c^0) I_c]_{\rm ext} \psi_c \rangle - \tfrac{1}{2} v_c{}^{+1/2} \hbar^{1/2} (2P_c)^{1/2} \sum_\nu \frac{\gamma_{\nu c}}{E_\nu - E} \langle \Psi_f | EM | X_\nu \rangle , \tag{III.12}$$

$$e_\lambda = \rho_\lambda{}^{-1} \left\{ \langle \Psi_f | EM | \mathrm{Im}[(\overline{E}_\lambda + \tfrac{1}{2} i |\overline{\Gamma}_{\lambda c}|) \Omega_c^* (1 - R_{cc}^0 L_c^0) I_c]_{\rm ext} \psi_c \rangle - \tfrac{1}{2} v_c{}^{1/2} \hbar^{1/2} (2P_c)^{1/2} \left(\gamma_{\lambda c} \langle \Psi_f | EM | X_\lambda \rangle + E_\lambda \sum_\nu \frac{\gamma_{\nu c}}{E_\nu - E} \langle \Psi_f | EM | X_\nu \rangle \right) \right\} . \tag{III.13}$$

The zero at e_λ is meaningful only if it falls in the region of validity of the one-level plus constant background approximation. In practice, it will be observable only if it lies within or close to the energy range $(\bar{E}_\lambda - |\bar{\Gamma}_{\lambda c}|, \bar{E}_\lambda + |\bar{\Gamma}_{\lambda c}|)$. If the contribution of the external region can be neglected, and if $\langle \Psi_f | EM | X_\lambda \rangle = 0$, we find $e_\lambda = E_\lambda$ which is a consequence of the fact that $\Psi^c_{E_\lambda}$ is proportional to X_λ in the internal region. This for instance applies to $T=0$ resonance in light even-even nuclei,[22] and to the asymmetry of isobaric analog resonances as observed in (p,n) reactions.[11] These examples show that the type of asymmetry which is observed, i.e., the phase of interference between the resonance and the background, may sometimes be given a physical interpretation or, alternatively, yield detailed information on the scattering wave function. Another case concerns the asymmetry of the (p,γ) resonance peak at the isobaric analog resonances. A dip is often observed between the resonance energy and the giant dipole resonance.[9] This is probably related to the extension of Wigner's theorem stated above, since it appears that in these cases the products $\gamma_{\omega c}\gamma_{\omega p}$ have the same sign for both resonances.[23,24] In a more detailed analysis, however, the existence of other open channels and of a spreading of the two states (intermediate structure) should be taken into account.

In order to acquire a better physical understanding of the physical origin of the zero in the capture cross section, it is useful to discuss in more detail two extreme examples. We assume for simplicity that external capture can be neglected. Equations (III.12) and (III.13) yield then

$$e_\lambda = E_\lambda + \gamma_{\lambda c} \frac{\langle \Psi_f | EM | X_\lambda \rangle}{\langle \Psi_f | EM | Z_0 \rangle}, \qquad \text{(III.14)}$$

where

$$Z_0 = \sum_{\nu \neq \lambda} \frac{\gamma_{\nu c}}{E_\nu - E} X_\nu . \qquad \text{(III.15)}$$

The state $\gamma_{\lambda c} Z_0$ plays the role of a "background state." The internal wave function reads

$$\Psi^c_E = -i\hbar^{1/2}\Omega_c (E_\lambda - E - \beta_{\lambda c}\gamma_{\lambda c})^{-1}(1 - R^0_{cc}L^0_c)^{-1} \times [\gamma_{\lambda c} X_\lambda + (E_\lambda - E) Z_0] . \qquad \text{(III.16)}$$

At the energy e_λ, the contributions of X_λ and of Z_0 to the radiative capture amplitude cancel each other. The zero may be observable if

$$|e_\lambda - E_\lambda| < \frac{|L^0_c| \gamma_{\lambda c}^2}{|1 - R^0_{cc} L^0_c|}, \qquad \text{(III.17)}$$

which amounts to

$$\langle \Psi_f | EM | X_\lambda \rangle < \frac{|L^0_c|}{|1 - R^0_{cc} L^0_c|} \langle \Psi_f | EM | Z_0 \rangle \gamma_{\lambda c} . \qquad \text{(III.18)}$$

Inequality (III.18) means that the radiative width amplitude of X_λ must be small compared to that of the background state. Three limiting cases are of illustrative interest. (i) $\langle \Psi_f | EM | X_\lambda \rangle = 0$. Then $e_\lambda = E_\lambda$ and a dip appears in the radiative capture cross section if B_c is chosen equal to S_c. This may probably apply to the data reported in Ref. 6. (ii) $\langle \Psi_f | EM | Z_0 \rangle = 0$. Then the zero is rejected to infinity, and a symmetric resonance peak is observed. (iii) $\gamma_{\lambda c} = 0$. Then the zero apparently falls at E_λ. However, the damping effect of the photon channels can no longer be neglected, and Eq. (III.1) may not be used.

IV. STABILITY OF THE RESONANCE PARAMETERS

The use of R-matrix theory in the analysis of experimental data has been criticized on the basis that it involves the arbitrary boundary parameters a_c and B_c, upon which the collision matrix should not depend. A related problem concerns the determination of those values of a_c and B_c for which the one-level approximation is valid. In the present section, we show that the choice of a_c and B_c is largely arbitrary, in the sense that the analysis of an isolated resonance usually yields correct values for the reduced widths $\gamma_{\lambda c}^2(B_c)$, for a wide range of values for B_c. In other words, the resonance parameters are stable under a change of B_c, although the quantities E_λ and $\gamma_{\lambda c}^2$ depend upon B_c.

The problem of the "best" choice of B_c has been discussed by several authors, in the case of the one-level approximation without background. Most of them[1,13,16,19] advocate the value

$$B_c = S_c(E_\lambda), \qquad \text{(IV.1)}$$

for which the level shift $\Delta_{\lambda\lambda c}(E_\lambda)$ vanishes. It was pointed out in Refs. 17 and 18 that a deviation from the choice (IV.1) in the *open* channels usually leads to only a small shift of the resonance energy from E_λ, and that, more importantly, the choice (IV.1) is usually not the one for which the one-level approximation without background is justified. In fact, it appears that the value of a_c is the most critical one. In the numerical study of a many-channel model, it was found in Ref. 23 that, in contrast, the one-level plus constant background approximation appears to be justified for a wide variety of values for B_c, in the vicinity of the choice (IV.1), for a given arbitrary a_c. We shall now explain and extend these numerical observations.

In a practical analysis of an isolated resonance, the channel radii a_c in the open channels are fixed *a priori*, and chosen close to the sum of the radii of the fragments in channel c. The problem then amounts to the study of the change of the resonance parameters when B_c is modified. The following differential equations hold[19]:

$$\frac{\partial E_\lambda}{\partial B_c} = -\gamma_{\lambda c}{}^2 , \tag{IV.2}$$

$$\frac{\partial(\gamma_{\lambda s}\gamma_{\lambda t})}{\partial B_c} = \gamma_{\lambda s}\gamma_{\lambda c} R^0_{ct}(E_\lambda) + \gamma_{\lambda t}\gamma_{\lambda c} R^0_{cs}(E_\lambda) . \tag{IV.3}$$

The one-level approximation with constant background is given by Eqs. (II.22)–(II.25). We limit the discussion to the case when the background is diagonal. In practice, one chooses the boundary parameters a_c and B_c, and one fits the background cross section with Eq. (II.25). Since the experimental background is given, and since the channel radii a_c are fixed *a priori*, the variation with B_c of the value of R^0_{cc} obtained from the fit is determined by the equation

$$\frac{\partial U^0_{cc}}{\partial B_c} = 0 , \tag{IV.4}$$

which yields

$$\frac{\partial R^0_{cc}}{\partial B_c} = (R^0_{cc})^2 . \tag{IV.5}$$

It is then straightforward to obtain, from Eqs. (II.19), (II.29), (II.24), and (IV.5), the stationarity relations

$$\frac{\partial \overline{E}_\lambda}{\partial B_c} = \frac{\partial \overline{\Gamma}_{\lambda c}}{\partial B_c} = 0 . \tag{IV.6}$$

Hence the resonance parameters are independent of the boundary parameters B_c, as announced above. We conclude that the choice of B_c is largely arbitrary, with, however, a limit being set by the fact that the level shift $\Delta_{\lambda\lambda c}$ should be smaller than the spacing between eigenvalues, so that $\underline{R}^0$ can indeed be approximated by a constant. As discussed in Ref. 1, the latter limitation is not a stringent one. In other words, one may analyze the experimental data with several values of B_c, and be confident that the values $E_\lambda(B_c)$ and $\gamma_{\lambda c}{}^2(B_c)$ have a physical meaning. As shown in Ref. 25, it can happen that two R-matrix eigenvalues fall within the observed resonance, but this event is very exceptional and unpredictable.

The result obtained above reconciles the facts that, on the one hand, the assumption that R^0 is constant should not be critically dependent upon the choice of B_c and that, on the other hand, the reduced widths $\gamma_{\lambda c}{}^2$ and the eigenvalue E_λ vary with B_c. Another way to interpret the result is the following. The collision matrix $\underline{U}$ is independent of B_c; since the background part $\underline{U}^0$ is automatically independent of B_c, the same must be true for the resonance part, and hence for the resonance parameters. This reasoning makes one expect that a similar property holds for the dependence upon a_c, but we have not been able to prove it. It also appears likely that Eqs. (IV.2) and (IV.3), and similar ones for the dependence upon a_c, could be derived directly from the property that the collision matrix is independent of the boundary parameters.

Finally, several comments about the "natural" choice (IV.1) are in order. This choice is in general not the one for which $\underline{R}^0$ is minimum, and should therefore not be expected to favor the one-level approximation without background. In fact, Eq. (II.25) shows that the assumption $\underline{R}^0 = 0$ can only be satisfied for suitable values of the channel radii, and that, for these values of a_c, the choice of B_c is largely arbitrary, in keeping with Eqs. (IV.2) and (IV.3). The value (IV.1) was recommended in Refs. 16 and 19 on the basis that $\overline{\Gamma}_{\lambda c}$ would then be stable under a variation of a_c, i.e.,

$$\frac{\partial \overline{\Gamma}_{\lambda c}}{\partial a_c} = 0 . \tag{IV.7}$$

However, the derivation of Eq. (IV.7) contains a slight flaw, because it implies the assumption that the one-level approximation without background remains valid when a_c is varied or equivalently that $\partial \underline{R}^0/\partial a_c = 0$, which is not true. Although not critical, the choice (IV.1) is nevertheless favorable, for the following reasons. Firstly, it ensures that the eigenvalue E_λ falls close to the resonance energy. Secondly, it is the one for which $\alpha_{\lambda c} \simeq \gamma_{\lambda c}$ for $|P_c| \ll 1$, so that the value of $\underline{R}^0$ is not needed at low energy. This is useful, since $\underline{R}^0$ is usually only poorly determined by the background cross section. We recall, however, that the choice (IV.1) does not imply that $\underline{R}^0$ is small.

V. SUMMARY AND CONCLUSIONS

The main merit of R-matrix theory is probably that any approximation on the R matrix yields a unitary expression for the collision matrix. Moreover, the R matrix has a simple analytic structure, and few-level approximations can be introduced in an easy and natural way. However, the physical interpretation of any approximation in terms of nuclear models normally rests upon the use of a truncated basis of wave functions. It is therefore of interest to derive the expressions of the scattering wave function Ψ^c_E which correspond to the few-level approximations of the R matrix. The index c refers to the entrance channel and E to the energy. In Sec. II, we derive several forms for Ψ^c_E, in terms of the full level matrix $\underline{A}$, of a truncated level matrix $\overline{\underline{A}}$ which pertains to a few selected levels, of the full R matrix, etc. We discuss in particular the one-level and the one-channel cases. We also include the external region in our study. Some of these results had been previously obtained by Lane and Lynn[21,26] in the special case of isolated resonances. In Sec. III we apply our results to the study of radiative capture at low energy, with particular emphasis on the existence and on the physical interpretation of asymmetric resonance peaks. Finally, we show in Sec. IV that the one-level plus constant background approximation is usually justified for a wide range of values of the boundary parameters B_c and a_c.

Many of the problems studied in this paper were brought to our attention by Dr. A. M. Lane, to whom we are very grateful for a stimulating correspondence.

APPENDIX

Here we show, in the frame of general scattering theory, that Ψ_E^c is a real function of the nucleon coordinates, in the one-channel case. The Lippmann-Schwinger equation[27] reads, with previous notation,

$$\Psi_E^c = \chi_E^c + (E^+ - H_0)^{-1} V \Psi_E^c \,, \qquad \text{(A1)}$$

where χ_E^c is an eigenstate of the free Hamiltonian H_0. The transition matrix is given by

$$V\Psi_E^c = \sum_{c'} T_{cc'} \chi_E^{c'} \,. \qquad \text{(A2)}$$

We also introduce the standing wave eigenstate, which is a solution of

$$\overline{\Psi}_E^c = \chi_E^c + P(E - H_0)^{-1} V \overline{\Psi}_E^c \,, \qquad \text{(A3)}$$

where P indicates that the principal value integral should be taken. Clearly, $\overline{\Psi}_E^c$ is a real function. From Eqs. (A1)–(A3), we find

$$\Psi_E^c - \overline{\Psi}_E^c = P(E - H_0)^{-1} V[\Psi_E^c - \overline{\Psi}_E^c] - i\pi \sum_{c'} T_{cc'} \chi_E^{c'} \,. \qquad \text{(A4)}$$

In the one-channel case, the comparison between Eqs. (A4) and (A3) yields

$$\Psi_E^c = \overline{\Psi}_E^c (1 - i\pi T_{cc}) \,, \qquad \text{(A5)}$$

which shows that Ψ_E^c is, like $\overline{\Psi}_E^c$, a real function of the nucleon coordinates.

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