THE ASTROPHYSICAL JOURNAL

AN INTERNATIONAL REVIEW OF SPECTROSCOPY AND ASTRONOMICAL PHYSICS

VOLUME 114 NOVEMBER 1951 NUMBER 3

THE NONRADIAL OSCILLATIONS OF GASEOUS STARS AND THE PROBLEM OF BETA CANIS MAJORIS*

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Received June 5, 1951

ABSTRACT

The general characteristics of the nonradial oscillations of a rotating star are summarized and compared with the observations of β Canis Majoris. It is shown that the existence of two periods very close to each other as well as a phase shift of a quarter-period between the broadening of the lines and the corresponding radial velocity can be accounted for if the oscillation corresponds to a spherical harmonic of degree 2 in a rotating star. However, for free oscillations, the sign of the phase shift is opposite to the one observed when the periods are in the correct ratio, and vice versa.

Forced oscillations are briefly discussed, and, although they offer a possibility of removing this discrepancy, a quantitative analysis presents difficulties due to the proximity of the hypothetical companion.

I. INTRODUCTION

In a recent paper, O. Struve[1] has discussed radial observations of β Canis Majoris extending over a period of more than forty years. His results confirm Meyer's interpretation[2] in terms of two harmonic radial-velocity-curves, V_1 and V_2, of very close periods τ_1 and τ_2, giving rise to a beat oscillation of period τ_3. According to Struve, the shortest period, $\tau_1 = 0.25002246$ day, and the corresponding amplitude, $A_1 = 5.8$ km/sec, have remained constant, while, as far as τ_2 and A_2 are concerned, the observations are best represented by adopting the values $\tau_2 = 0.2513015$ day and $A_2 = 4.2$ km/sec for the period 1909–1931 and $\tau_2' = 0.2513003$ day and $A_2' = 2.0$ km/sec for the period 1931–1948. The corresponding beat periods are $\tau_3 = 49.12$ and $\tau_3' = 49.17$ days.

Furthermore, the absorption lines exhibit a variation in width with the same period τ_2 as V_2, but with a phase lag of a quarter-period; that is, the maximum broadening occurs when V_2 goes through zero from positive to negative values. Struve's discussion suggests also that this broadening has changed with the amplitude A_2.

* This research was supported in part by funds of the Eugene Higgins Trust allocated to Princeton University.

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[1] *Ap. J.*, **112**, 520, 1950.

[2] *Pub. A.S.P.*, **46**, 202, 1934.

The explanation proposed by Struve requires the presence of a companion of very small dimensions and high density, one of the periods corresponding to the orbital motion and the other to an oscillation excited by resonance in the main component. This hypothesis raises difficulties which we will discuss later, but first we review the properties of the free oscillations of a rotating gaseous star.

II. THEORY OF FREE OSCILLATIONS

a) EQUATIONS OF MOTION, PERIODS, AND DISPLACEMENTS

It seems unlikely that purely radial oscillations are significant in this connection, as it would be difficult to explain both the very close periods and the observed broadening of the lines on such a theory. On the other hand, for nonradial oscillations, we know[3] that there will exist groups of oscillations of very close periods, provided that the star is in slow rotation.

In this case, if we suppose that the dependence with respect to the time t is of the form $\exp i\sigma t$, the Eulerian equation of motion, with respect to axis rotating with the angular velocity of the star $\vec{\Omega}$, can be written

$$-\sigma^2 \delta\mathbf{r} + 2i\sigma (\vec{\Omega} \times \delta\mathbf{r}) = -\frac{1}{\rho}\operatorname{grad} P' + \frac{\rho'}{\rho^2}\operatorname{grad} P + \operatorname{grad} U' , \tag{1}$$

where $\delta\mathbf{r}$ is the displacement and the other symbols have their usual meaning, the primes denoting the Eulerian variations of the corresponding variables. If Ω is small, so that we can neglect all its powers greater than the first, the components of this equation in polar co-ordinates are

$$-\sigma^2 \delta r - 2i\sigma\Omega r \sin^2\theta\, \delta\phi = -\frac{1}{\rho}\frac{\partial P'}{\partial r} + \frac{\rho'}{\rho^2}\frac{\partial P}{\partial r} + \frac{\partial U'}{\partial r}, \tag{2}$$

$$-\sigma^2 r\, \delta\theta - 2i\sigma\Omega \cos\theta\, r \sin\theta\, \delta\phi = -\frac{1}{\rho}\frac{\partial P'}{r\partial\theta} + \frac{\partial U'}{r\partial\theta}, \tag{3}$$

$$-\sigma^2 r \sin\theta\, \delta\phi + 2i\sigma\Omega \sin\theta\, \delta r + 2i\sigma\Omega \cos\theta\, r\, \delta\theta = -\frac{1}{\rho}\frac{\partial P'}{r\sin\theta\,\partial\phi} + \frac{\partial U'}{r\sin\theta\,\partial\phi}, \tag{4}$$

where the equilibrium values of the variables can be taken equal to their values in a purely spherical nonrotating configuration.

We must take into account the equation of continuity,

$$\rho' = -\operatorname{div} \rho\, \delta\mathbf{r} , \tag{5}$$

and the adiabatic relation,

$$P' = -\frac{\gamma P}{\rho}\operatorname{div}\delta\mathbf{r} - \delta\mathbf{r}\cdot\operatorname{grad} P = \frac{\gamma P}{\rho}\rho' - \delta\mathbf{r}\left(\operatorname{grad} P - \frac{\gamma P}{\rho}\operatorname{grad}\rho\right). \tag{6}$$

We can suppose that δr, ρ', P', and U' are expanded in series of spherical harmonics and consider only the general term,

$$F(r,\theta,\phi) = f(r) P_s^m(\cos\theta)\, e^{\pm im\phi} . \tag{7}$$

These solutions must satisfy the boundary conditions

$$\delta r = 0 \quad \text{at} \quad r = 0 ,$$

[3] T. G. Cowling and R. A. Newing, *Ap. J.*, **109**, 149, 1949; also P. Ledoux, *Mém. Soc. R. Sci. Liège*, 4th ser., **9**, 263, 1949.

$$P' + \delta r \cdot \operatorname{grad} P = 0 \;, \qquad \frac{\partial U'}{\partial r} + \frac{s+1}{r} U' = 0 \;, \qquad \text{at} \qquad r = R \,. \tag{8}$$

If we denote, by a suffix zero, the solutions for $\Omega = 0$, we can easily verify that the corresponding displacements $\delta r_{k,\,0}$ are orthogonal:

$$\int_0^M \delta r_{k,\,0} \cdot \delta r^*_{l,\,0} dm = \int_0^M (\delta r_{k,\,0} \delta r^*_{l,\,0} + r^2 \delta \theta_{k,\,0} \delta \theta^*_{l,\,0} + r^2 \sin^2 \theta \, \delta \phi_{k,\,0} \delta \phi^*_{l,\,0}) dm = 0 \,, \quad (k \neq l) \,. \tag{9}$$

If we retain the terms in Ω, $\sigma_{k,\,0}$ is increased by a small amount σ'_k and $\delta r_{k,\,0}$ by a small vector $\delta r'_k$, which can be represented by a series in terms of $\delta r_{i,\,0}$,

$$\delta r_k = \delta r_{k,\,0} + \sum_i a_{k,\,i} \delta r_{i,\,0} \,.$$

Then equations (5) and (6) and Poisson's equation give

$$\rho'_k = \rho'_{k,\,0} + \sum_i a_{k,\,i} \rho'_{i,\,0} \,, \qquad P'_k = P'_{k,\,0} + \sum_i a_{k,\,i} P'_{i,\,0} \,, \qquad U'_k = U'_{k,\,0} + \sum_i a_{k,\,i} U'_{i,\,0} \,.$$

Introducing these expressions into equation (1) and keeping first-order terms only, we obtain

$$2 \sigma_{k,\,0} \sigma'_k \delta r_{k,\,0} = 2 i \sigma_{k,\,0} (\vec{\Omega} \times \delta r_{k,\,0}) - \sum_i a_{k,\,i} (\sigma^2_{k,\,0} - \sigma^2_{i,\,0}) \, \delta r_{i,\,0} \,.$$

If we multiply the components of this equation, respectively, by $\delta r^*_{k,\,0}$, $r \delta \theta^*_{k,\,0}$, $r \sin \theta \, \delta \phi^*_{k,\,0}$, integrate the sum over the whole mass, and use relation (9), we find

$$\sigma'_k = \frac{i \int_0^M (\vec{\Omega} \times \delta r_{k,\,0}) \cdot \delta r^*_{k,\,0} dm}{\int_0^M (\delta r_{k,\,0} \cdot \delta r^*_{k,\,0}) dm} \,, \tag{10}$$

which is identical to the formula obtained by T. G. Cowling and R. A. Newing[3] from the application of Rayleigh's principle.

The displacements $\delta r_{k,\,0}$ can be deduced from equations (2), (3), and (4) with $\Omega = 0$. Furthermore, T. G. Cowling's discussion of the polytropic case[4] has shown that, even for harmonics of small degree ($s = 2, 3, \ldots$) a reasonable approximation can be obtained by neglecting U'. In that case, using equations (2) and (6) and the following definitions,

$$y = \frac{P'}{\rho} \,, \qquad g = -\frac{1}{\rho} \frac{\partial P}{\partial r} \,, \qquad A = \frac{1}{\gamma P} \frac{\partial P}{\partial r} - \frac{1}{\rho} \frac{\partial \rho}{\partial r} \,, \tag{11}$$

we have

$$\delta r_{k,\,0} = \frac{1}{\sigma^2_{k,\,0} - g A} \left(\frac{\partial y}{\partial r} - y A \right) P^m_s e^{\pm i m \phi} = a(r) P^m_s e^{\pm i m \phi} \,, \tag{2'}$$

$$r \delta \theta_{k,\,0} = \frac{y}{\sigma^2_{k,\,0} r} \frac{\partial P^m_s}{\partial \theta} e^{\pm i m \phi} = b(r) \frac{\partial P^m_s}{\partial \theta} e^{\pm i m \phi} \,, \tag{3'}$$

[4] *M.N.*, **101**, 367, 1941; cf. also Z. Kopal, *Ap. J.*, **109**, 509, 1949; and E. Sauvenier-Goffin, *Bull. Soc. R. Sci. Liége*, **20**, 20, 1951.

$$r \sin\theta\, \delta\phi_{k,0} = \frac{\pm imy}{\sigma^2_{k,0} r} \frac{P^m_s}{\sin\theta} e^{\pm im\phi} = \pm im\, b(r) \frac{P^m_s}{\sin\theta} e^{\pm im\phi} . \tag{4'}$$

Introducing these expressions into equation (10) and making use of well-known properties of the spherical harmonics, we obtain

$$\sigma'_k = \pm m\Omega \frac{\int_0^R \rho (2ab + b^2) r^2 dr}{\int_0^R \rho [a^2 + s(s+1) b^2] r^2 dr} = \pm m\Omega C_k , \tag{10'}$$

where C_k is a constant for a given mode. For the homogeneous compressible model, this expression reduces to the known value,[5]

$$\sigma'_k = \pm m\Omega \frac{2\alpha_k + 1}{\alpha_k^2 + s(s+1)} , \qquad \text{where} \qquad \alpha_k = \frac{3\sigma_k^2}{4\pi G\rho} . \tag{12}$$

Since, for a given value of the degree s, the rank m can take the values $0, 1, 2, \ldots, s$, we now have $(2s+1)$ frequencies $\sigma_{k,m}$ associated with each frequency $\sigma_{k,0}$ of the nonrotating star,

$$\sigma_{k,m} = \sigma_{k,0} \pm m\Omega C_k , \qquad (m = 1, 2, \ldots, s) .$$

We can limit our discussion here to the case $s = 2$, which is the most interesting one, and, if we neglect the nonsymmetrical oscillations corresponding to $m = 1$, we are left with three frequencies σ and $\sigma \pm 2\Omega C$ for each mode. The first one is the frequency of a stationary oscillation symmetrical with respect to the axis of rotation, while the other two correspond, respectively, to waves traveling in the opposite or the same direction as the rotation of the star with relative angular velocities $\mp(\sigma/2 \pm \Omega C)$. But, since our axes are themselves rotating with velocity Ω, the absolute angular velocities are, respectively, $\mp[\sigma/2 \mp \Omega(1 - C)]$ and the corresponding frequencies for observers at rest are

$$\sigma, \qquad \sigma - 2\beta\Omega , \qquad \sigma + 2\beta\Omega , \qquad \text{where} \qquad \beta = 1 - C . \tag{13}$$

b) DISCUSSION OF THE VALUES OF C

In the case of an incompressible fluid, C is just equal to $\frac{1}{2}$, but it is already appreciably smaller for the homogeneous compressible model, in which case equation (12) gives the maximum value $C \simeq \frac{1}{4}$ for the lowest stable mode. In more general cases the value of C will depend on the behavior of a and b through the star. To discuss this dependence, let us consider equation (2′) and the equation obtained in developing the last two members of equation (6),

$$\frac{dy}{dr} - yA = (\sigma^2 - gA)\, a , \tag{14}$$

$$\frac{d}{dr}(r^2 a) + \frac{1}{\gamma P} \frac{\partial P}{\partial r} r^2 a = \left[\frac{s(s+1)}{\sigma^2} - \frac{\rho r^2}{\gamma P}\right] . \tag{15}$$

With the help of the following definitions:

$$v = r^2 a P^{1/\gamma} , \qquad w = \rho y P^{-1/\gamma} ,$$

these equations become

$$r^2 \frac{dw}{dr} = (\sigma^2 - gA)\, \rho P^{-2/\gamma} v , \tag{16}$$

[5] Ledoux, *op. cit.*, p. 283.

$$\frac{dv}{dr}=\left[\frac{s(s+1)}{\sigma^2}-\frac{\rho r^2}{\gamma P}\right]\rho^{-1}P^{2/\gamma}w\,. \tag{17}$$

A general discussion of the equations with respect to the stability cannot be attempted here, but equations (16) and (17) show that if A is positive everywhere—that is, if the star is convectively stable in all its parts—there is no solution with σ^2 negative (instability) which satisfies the boundary conditions.

We will limit ourselves to this case, because regions of small extent, with slightly superadiabatic gradients, will not affect our conclusions for the modes in which we are interested.

These equations have regular singularities at $r = 0$ and $r = R$; and, taking the boundary conditions (8) into account, one finds that near the center, where A varies as r, one has the asymptotic relations

$$y\to\sigma^2 r a\to r^s \quad \text{or} \quad b\to a\to r^{s-1}\,; \qquad (r\to 0)\,, \tag{18}$$

while near the surface, where A becomes infinite, one finds immediately

$$y\to g a \quad \text{or} \quad b\to\frac{a}{\alpha} \qquad (r\to R)\,, \tag{19}$$

if α is defined by

$$\sigma^2=\frac{4\pi G\bar{\rho}}{3}\,\alpha\,. \tag{20}$$

As $\rho r^2/\gamma P$ varies in a monotonic fashion from zero at the center to $+\infty$ at the surface, the bracket in equation (17) vanishes once in the interval $0 < r < R$, let us say at r_v. On the other hand, gA varies also from 0 to ∞ but is not necessarily monotonic, so that there are either one or an odd number of points r_w where the bracket in equation (16) vanishes. At these points, r_v and r_w, the equations have regular singularities where v and w have true maxima or minima.

For a given value of s, the solutions can be divided into two groups, according to the relative positions of the points r_v and r_w nearest the center; and Cowling's nomenclature for the polytropes can be adopted here, too. If the point r_w occurs first—and this is certainly the case for σ^2 small enough—the first node of w (or P') is nearer the center than the first node of v (or δr). This class of solutions comprises the g oscillations, for which σ^2 tends toward zero as the number of nodes increases indefinitely. On the other hand, when a point r_v occurs first—and this is certainly the case for σ^2 large enough—we have the p oscillations, for which σ^2 tends toward infinity as the number of nodes increases. There is one oscillation called f by Cowling, which has no node in $0 < r < R$ and separates these two classes.

For the standard model, which can be considered typical in this respect, the value of α_f is of the order of 10, and we can conclude from equations (18) and (19) that, except for the high g modes, b will always be smaller than a, especially near the surface. In that case the definition (10′) shows that C will be of the order of $\overline{(2b/a)}$, or 0.2 for the standard model. When b tends to become of the same order of magnitude as a or larger (g oscillations), the product (ab) changes sign along the radius, and its integral in equation (10′) tends to vanish, so that even in this case C can never become much larger than $\frac{1}{7}$. For instance, the numerical integrations[4] carried out for the standard model with $s = 2$ and $\gamma = \frac{5}{3}$ give

$$\alpha_f=8.69\,,\qquad \alpha_{g_1}=4.72\,,\qquad \alpha_{g_2}=2.92\,; \tag{21}$$

and, using the values of the corresponding amplitudes in equation (10′), one finds

$$C_f=0.179\,,\qquad C_{g_1}=0.146\,,\qquad C_{g_2}=0.137\,. \tag{22}$$

Unfortunately, we have no numerical information on the p oscillations except for the homogeneous compressible model; but we know that the corresponding α_p increase, α_{p_1} being probably of the order of 15, while the C_p will slowly decrease.

III. COMPARISON WITH THE OBSERVATIONS

a) PERIODS

According to Struve, β Canis Majoris, which is of spectral type B, has some giant characteristics, and we will adopt for its mean density, $\bar{\rho}$, a value of the order of 0.02, which is about one-tenth the density of a main-sequence star of the same type.[6] With an observed period of 0.25 day, this gives, according to definition (20), a value for α of the order of 15.

A comparison with the theoretical values (21) points toward the p_1 oscillation rather than the f or one of the g oscillations. However, some of the stars belonging to this class of variables are members of the main sequence and have only slightly shorter periods, so that the corresponding α's would be of the order of 2–2.5, which would imply g oscillations in these cases. This is not very satisfactory, but the giant characteristics of β Canis Majoris might be due to a distended photosphere caused by some surface agency,[7] which would leave its mean density closer to that of the main-sequence stars.

Perhaps at this preliminary stage the only significant feature as far as the periods are concerned is that they fall in the range of theoretical values corresponding to simple modes, which do not require a too complicated mechanism for their excitation or maintenance.

As the two traveling waves will have the same general properties, we can already suspect that we will have to identify the two observed oscillations with the stationary pulsation and one or the other of the traveling waves. According to the set of values (13), the corresponding frequencies would differ in either case by an amount of the order of 2Ω, which should be equal to the observed beat frequency. For instance, for the f mode of the standard model, the difference would be 1.642Ω. If we put this equal to 49 days, we find a period of rotation for β Canis Majoris of the order of 80 days, corresponding to linear velocities at the equator of the order of 8–10 km/sec.

b) MEAN VELOCITY COMPONENT ALONG THE LINE OF SIGHT AND BROADENING

Let us consider the combined effects of the rotation and the oscillation on the absorption lines. Apart from terms wholly negligible—of the order of Ω/σ—we can obtain the velocities at the surface by multiplying equations (2′), (3′), and (4′) by $i\sigma e^{i\sigma t}$, putting $r = R$, $s = 2$, and $m = 0$ or 2. The two corresponding real solutions differ only by an angle $\pi/2$ in the argument, and it is sufficient to discuss one of them for each of the two values of m. We obtain

$$\begin{aligned} v_{r,0} &= \tfrac{1}{2} A_0 (3\cos^2\theta - 1)\cos\sigma t\,, \\ v_{\theta,0} &= -3B_0 \cos\theta \sin\theta \cos\sigma t\,, \qquad (m = 0) \qquad (23) \\ v_{\psi,0} &\simeq 0\,. \end{aligned}$$

With $A_0 = \sigma a_0(R)$, $B_0 = \sigma b_0(R)$,

$$\begin{aligned} v_{r,2} &= 3A_2 \sin^2\theta \cos[(\sigma \mp 2\beta\Omega)t \pm 2\psi]\,, \\ v_{\theta,2} &= 6B_2 \sin\theta \cos\theta \cos[(\sigma \mp 2\beta\Omega)t \pm 2\psi]\,, \qquad (m = 2) \qquad (24) \\ v_{\psi,2} &= \mp 6B_2 \sin\theta \sin[(\sigma \mp 2\beta\Omega)t \pm 2\psi]\,. \end{aligned}$$

[6] G. P. Kuiper, *Ap. J.*, **88**, 429 and 472, 1938; according to Dr. Kuiper, the temperatures around B, as given there, are too low and should be raised by one subclass.

[7] Cf. O. Struve and P. Swings, *Ap. J.*, **94**, 99, 1941.

With $A_2 = \sigma a_2(R)$, $B_2 = \sigma b_2(R)$. In these expressions, ψ is now the longitude measured from a fixed direction coinciding, at $t = 0$, with the origin of ϕ.

Let us suppose that the line of sight makes an angle θ_0 with the axis of rotation, and we define a new right-handed system of fixed co-ordinates (x, y, z), x being along the origin of ψ, and z along the axis of rotation. If the velocity component along the line of sight V_i is counted positively away from us, as is usual in the observations, we find, for the stationary oscillation ($m = 0$),

$$V_{i,0} = \left\{ \left[\frac{A_0}{2} (1 - 3z^2)\, x + 3B_0 z^2 x \right] \sin\theta_0 + \left[\frac{A_0}{2} (1 - 3z^2)\, z + 3B_0 z\, (z^2 - 1) \right] \cos\theta_0 \right\} \cos\sigma t\,, \tag{25}$$

and, for the traveling waves ($m = 2$),

$$\begin{aligned} V_{i,2} = {} & [3A_2 (z^2 - 1)\, x - 6B_2 z^2 x] \cos[(\sigma \mp 2\beta\Omega)\, t \pm 2\psi] \sin\theta_0 \\ & \mp 6B_2 y \sin[(\sigma \mp 2\beta\Omega)\, t \pm 2\psi] \sin\theta_0 \\ & - (3A_2 - 6B_2)(1 - z^2) \cos[(\sigma \mp 2\beta\Omega)\, t \pm 2\psi] \cos\theta_0 \\ & + y\Omega \sin\theta_0\,, \end{aligned} \tag{26}$$

where we have included the velocity of rotation.

In turn, these co-ordinates can be expressed in terms of polar co-ordinates d and Θ in the plane normal to the line of sight, as this facilitates the integrations on the visible disk,

$$\begin{aligned} x &= d \cos\Theta \cos\theta_0 + \sqrt{(1 - d^2)} \sin\theta_0\,, \\ y &= d \sin\Theta\,, \\ z &= \sqrt{(1 - d^2)} \cos\theta_0 - d \cos\Theta \sin\theta_0\,, \end{aligned} \tag{27}$$

Θ being measured from the projection of the origin of ψ.

1. *The stationary oscillation ($m = 0$).*—In this case, according to equation (25), all the terms oscillate in phase, and, using the transformation (27), one has, for the mean radial velocity over the whole disk,

$$\bar{V}_{i,0} = -\left(\tfrac{2}{3} - \sin^2\theta_0\right) \left[\tfrac{1}{5}(A_0 + 3B_0) + \frac{\delta}{16}(3A_0 + 6B_0)\right] \left(\tfrac{1}{2} + \tfrac{1}{3}\delta\right)^{-1} \cos\sigma t\,, \tag{28}$$

where δ is the coefficient of limb darkening, so that the intensity at each point of the disk is

$$I = I_0 [1 + \delta \sqrt{(1 - d^2)}]\,. \tag{29}$$

Of course, there are also brightness variations along the surface due to the oscillation itself; but at this stage the theory could not give any useful information on this point, and we know from the observations that these variations are very small in any case and not likely to offset our qualitative conclusions.

From equation (28) we see that $\bar{V}_{i,0}$ vanishes for a value of θ_0 close to 54°, the whole effect then reducing to a broadening of the lines with a period π/σ equal to half the period of the velocity variation and in phase with $|\cos\sigma t|$. However, if we consider a line of sight near the pole or the equator and assume that B_0 is approximately equal to $A_0/8$, as for the f oscillation, we find that the amplitude of $\bar{V}_{i,0}$ is of the order of $A_0/3$ or $A_0/6$ for θ_0 equal to 0 and $\pi/2$, respectively, and its period in both cases is $2\pi/\sigma$. A rough estimate made on the basis of the observed amplitudes of β Canis Majoris shows that in these cases, assuming δ to be of the order of $\frac{3}{2}$, the broadening of the lines due to the oscillation is very small.

2. *The traveling waves* ($m = 2$).—As far as $V_{i,\,2}$ is concerned, the algebra becomes rather complicated if θ_0 is kept as a parameter, and we shall limit ourselves here to the cases corresponding to $\theta_0 = 0$ and $\theta_0 = \pi/2$. This is sufficient, as the solution changes progressively from one form to the other as θ_0 varies from 0 to $\pi/2$. If the line of sight is along the axis of rotation ($\theta_0 = 0$), $V_{i,\,2}$ becomes

$$V_{i,\,2} = -(3A_2 - 6B_2)\sqrt{(1-d^2)}\,d^2 \cos[(\sigma \mp 2\beta\Omega)t \pm 2\Theta]. \tag{30}$$

The mean value $\bar{V}_{i,\,2}$ vanishes because of the integration with respect to Θ from 0 to 2π, and the lines will simply be broadened and their mean wave lengths will not vary in the course of time. If we combine this result with the corresponding case for $V_{i,\,0}$, the lines would appear more or less broad, with a periodic asymmetry of period $2\pi/\sigma$ more or less marked according to the relative amplitudes of the two oscillations.

If the line of sight is in the equatorial plane, equation (26) in terms of d and Θ becomes, if we expand the factors containing the time,

$$\begin{aligned} V_{i,\,2} = &-\{(3A_2 - 6B_2)[1 - d^2(1+\sin^2\Theta)] + 6B_2\}(1-d^2)^{1/2} \\ &\times \cos[(\sigma \mp 2\beta\Omega)t - \pi] \pm [2(3A_2 - 6B_2)(1-d^2) + 6B_2]\,d \sin\Theta \sin \\ &\times [(\sigma \mp 2\beta\Omega)t] + \Omega R d \sin\Theta. \end{aligned} \tag{31}$$

The interesting point here is that the component of velocity along the line of sight can be divided into two parts, which are 90° out of phase. The first part, when integrated over all the disk, gives a mean component

$$\bar{V}_{i,\,2} = [\tfrac{2}{5}(A_2 + 3B_2) + \tfrac{3}{8}\delta(A_2 + 2B_2)](\tfrac{1}{2} + \tfrac{1}{3}\delta)^{-1}\cos[(\sigma \mp 2\beta\Omega)t - \pi], \tag{32}$$

which in the observations will appear as a variable radial velocity of period $(2\pi/\sigma)(1 \pm 2\beta\Omega/\sigma)$ and amplitude approximately equal to A_2 if we suppose again that $B_2 \simeq A_2/8$ and $\delta = \frac{3}{2}$. Since this should correspond to the observed radial velocity V_2, A_2 should be of the order of 2–4 km/sec.

The second term of $V_{i,\,2}$, being proportional to $\sin\Theta$, vanishes when integrated over the whole disk and thus corresponds only to a periodic broadening of the lines 90° out of phase with the corresponding radial velocity (32), and it will combine with the broadening due to the rotation represented by the third term.

If the equatorial linear velocity (ΩR) is small compared to the amplitude of the oscillation, the broadening will vary as $|\sin(\sigma \mp 2\beta\Omega)t|$, and its period will be only half that of the radial-velocity-curve. But if (ΩR) is of the same order of magnitude as the amplitude or larger, the broadening will have the same period as the radial-velocity-curve. Essentially what happens is that the rotation appears reinforced at a given phase and then decreased half a period later.[8] According to our previous estimates, (ΩR) here would be of the order of $3A_2$, and, combining the two last terms of equation (31) with $B_2 \simeq A_2/8$, we find for the component $V'_{i,\,2}$, giving rise to the broadening,

$$V'_{i,\,2} = A_2\left\{3 + (5.25 - 4.5d^2)\cos\left[(\sigma \mp 2\beta\Omega)t \mp \frac{\pi}{2}\right]\right\} d \sin\Theta. \tag{33}$$

If we consider, on the visible disk, sectors of 45°, each divided into ten parts which contribute equally to the formation of the lines, assuming the same law of darkening as before, the values taken by $V'_{i,\,2}$ at the center of each of these subdivisions are given in Table 1.

From this table we see that the total effective range of velocities at the phases of

[8] Cf. Struve and Swings, *op. cit.*, p. 103, end of second paragraph.

maximum and minimum broadening are, respectively, of the order of +10 to −10 km/sec and +2 to −2 km/sec. Although these figures are rather small, the corresponding variation might be sufficient to give rise to a differential effect such as the one observed. Furthermore, it is satisfactory that, on this picture, the broadening and the amplitude of the corresponding radial velocity are strongly correlated, as suggested by the observations.

However, there is a major difficulty as to the sign of the phase shift Φ_1 of the broadening with respect to the radial velocity. Since the broadening is associated with the longest period, we see from Table 2, where the results are summarized, that, to represent

TABLE 1

VALUES OF $V_{i,2}'$ AT PHASES $\Phi \equiv [(\sigma \mp 2\beta\Omega)t \mp \pi/2] = 0$ or π

Θ		d									
		0.2	0.346	0.456	0.542	0.621	0.692	0.759	0.822	0.889	0.959
0	$\Phi=0$...	0	0	0	0	0	0	0	0	0	0
	$\Phi=\pi$...	0	0	0	0	0	0	0	0	0	0
45°	$\Phi=0$...	+3.42	+5.66	+ 7.07	+ 7.96	+ 8.58	+ 8.95	+ 9.11	+ 9.08	+ 8.86	+ 8.36
	$\Phi=\pi$...	−0.88	−1.26	− 1.27	− 1.06	− 0.68	− 0.14	+ 0.55	+ 1.38	+ 2.46	+ 3.84
90°	$\Phi=0$...	+4.84	+8.00	+10.00	+11.27	+12.14	+13.55	+12.88	+12.84	+12.53	+11.82
	$\Phi=\pi$...	−1.24	−1.78	− 1.80	− 1.50	− 0.96	− 0.19	+ 0.48	+ 1.95	+ 3.18	+ 5.44
135°	$\Phi=0$...	+3.42	+5.66	+ 7.07	+ 7.96	+ 8.58	+ 8.95	+ 9.11	+ 9.08	+ 8.86	+ 8.36
	$\Phi=\pi$...	−0.88	−1.26	− 1.27	− 1.06	− 0.68	− 0.14	+ 0.55	+ 1.38	+ 2.46	+ 3.84
180°	$\Phi=0$...	0	0	0	0	0	0	0	0	0	0
	$\Phi=\pi$...	0	0	0	0	0	0	0	0	0	0
225°	$\Phi=0$...	−3.42	−5.66	− 7.07	− 7.96	− 8.58	− 8.95	− 9.11	− 9.08	− 8.86	− 8.36
	$\Phi=\pi$...	+0.88	+1.26	+ 1.27	+ 1.06	+ 0.68	+ 0.14	− 0.55	− 1.38	− 2.46	− 3.84
270°	$\Phi=0$...	−4.84	−8.00	−10.00	−11.27	−12.14	−13.55	−12.88	−12.84	−12.53	−11.82
	$\Phi=\pi$...	+1.24	+1.78	+ 1.80	+ 1.50	+ 0.96	+ 0.19	− 0.48	− 1.95	− 3.18	− 5.44
315°	$\Phi=0$...	−3.42	−5.66	− 7.07	− 7.96	− 8.58	− 8.95	− 9.11	− 9.08	− 8.86	− 8.36
	$\Phi=\pi$...	+0.88	+1.26	+ 1.27	+ 1.06	+ 0.68	+ 0.14	− 0.55	− 1.38	− 2.46	− 3.84

TABLE 2

THEORETICAL PREDICTIONS ON THE THEORY OF NONRADIAL OSCILLATIONS

m	Type of Oscillation	Periods	θ_0: Direction of Line of Sight	$\bar{V}_i$	Broadening	Φ_1
0	Stationary (P_2^0)	τ_0	0° (polar axis) 54° 90° (equator)	$A_0/3$ 0 $A_0/6$	Very small Large, variable with period $=\tau_0/2$ Very small	0 — 0
2	Wave traveling in the same direction as the rotation ($P_2^2 e^{-2i\phi}$)	$\tau_0-2\beta\Omega$	0° (polar axis) 90° (equator)	0 A_2	Large, constant Large, variable with period $=\tau_0-2\beta\Omega$	— $-\pi/2$
	Wave traveling in the opposite direction ($P_2^2 e^{+2i\phi}$)	$\tau_0+2\beta\Omega$	0° (polar axis) 90° (equator)	0 A_2	Large, constant Large, variable with period $=\tau_0+2\beta\Omega$	— $+\pi/2$

the observations, we should take the stationary oscillation and the wave traveling in the opposite direction to the rotation, assuming the line of sight to be close to the equatorial plane. But then the broadening is advanced by one-quarter period with respect to the radial velocity. In other words, the maximum broadening would occur when the radial velocity V_2 goes through zero from negative to positive values, which is contrary to the observations. Of course, the wave traveling in the same direction as the rotation would give us the right phase shift, but then the periods stand in the wrong ratio. Thus, although free oscillations, corresponding to P_2^0 and P_2^2, give us periods very close to one another and a possibility of accounting for the periodic broadening, the last discrepancy forces us to reject them as an explanation. Furthermore, even if one of the set of solutions of Table 2 was in complete agreement with observations, it would still be very difficult, for purely free oscillations, to explain why one of the traveling waves would be excited and not the other.

IV. FORCED OSCILLATIONS

Let us suppose that, as suggested by Struve, the main star of mass M, radius R, and mean density $\bar{\rho}$ has a very small companion of mass m describing a circular orbit of radius a with the angular velocity ω, then

$$\omega^2 = \frac{4\pi G \bar{\rho}}{3}\left(\frac{R}{a}\right)^3 . \tag{34}$$

The gravitational potential of m at a point r, θ, or ψ with respect to the absolute axis defined previously, can be written, provided that r is less than a, in the form

$$U_2' = \frac{Gm}{a}\sum_{s=1}^{\infty}\left(\frac{r}{a}\right)^s \left\{ P_s(\mu) P_s(0) + 2\sum_{m=1}^{s} \frac{(s-m)!}{(s+m)!} P_s^m(\mu) P_s^m(0) \cos m(\omega t - \psi) \right\} \tag{35}$$

if the mass m coincides at $t = 0$ with the origin of ψ. Since $P_s(0) = 0$, if s is odd and $P_s^m(0) = 0$ if $(s - m)$ is odd, U_2' will contain only terms which are symmetrical with respect to the equator.

The first nonzero term in equation (35) is proportional to P_1^1 and corresponds to the displacement of the center of gravity of M around m. In Struve's hypothesis, this is supposed to account for the velocity V_1 of β Canis Majoris.[9] But the only other term capable of giving rise to an oscillation of very close period is one in P_s^1, P_3^1 being the most likely. However, as it would be a forced oscillation, it is difficult to see why the two periods would not be exactly equal. On the other hand, the resulting motion would not differ very much from the corresponding free oscillation which is in resonance. But then, even in the most favorable case, when the line of sight is in the equatorial plane, the corresponding velocity component would be

$$\begin{aligned} V_{i,3} = &- \{\tfrac{3}{2}(5d^2\cos^2\Theta - 1)[A_3 + (B_3 - A_3)d^2] + 15B_3 d^2(1-d^2)\cos^2\Theta\}\cos\omega t \\ &+ \{\tfrac{3}{2}(5d^2\cos^2\Theta - 1)(A_3 - B_3) - 6B_3\} d(1-d^2)^{1/2}\sin\Theta \sin\omega t \\ &+ \Omega R d \sin\Theta , \end{aligned} \tag{36}$$

if we include the component due to the rotation of M. Since B_3 is appreciably smaller than A_3, the first two terms will change sign for approximately $d^2 = \frac{1}{5}\cos^2\Theta$. As a result, the average radial velocity over the whole disk will be very small, and the resulting broad-

[9] In a paper on 12 Lacertae, which the author had the privilege of reading before its publication, O. Struve suggests that the binary motion might account for the velocity V_2 rather than for V_1, the associated broadening being due to some kind of turbulence on the hemisphere facing m. However, in the case of β Canis Majoris, at any rate, this would not facilitate the explanation.

ening, which will be important, will vary with the period π/ω, equal to half the period of the corresponding radial velocity, whatever be the value of Ω. It is only in the case of the oscillation corresponding to P_2^2 that, as we have seen, its combination with the rotation can give a period for the broadening which is equal to the period of the corresponding radial velocity.

Furthermore, if we use the same data as before for β Canis Majoris, we find from formula (34) that the agreement between observed and theoretical periods would require a value of a/R of the order of 0.4 or, if we put $(a/R) = 1$, a value of $\bar{\rho}$ fifteen times greater than the value previously adopted. This seems unreasonable in both cases. An analysis of the light-variation[10] precise enough to decide whether these two periods are also present in the light-curve would be particularly useful in this respect, as one would not expect any light-variation associated with the orbital motion, since any eclipsing effect would probably be negligible.

On the other hand, if we assume that the ratio m/M is so small that the motion of the center of gravity of M is negligible, then the first term in equation (35) which could give rise to resonance is the term in P_2^2. First, the observed frequency in this case would be 2ω, and, to bring its value as given by equation (34) into agreement with the observations, a/R would have to be of the order of 0.625. This already seems more reasonable, since, if we suppose, for instance, that the main component is built on the standard model, then 97 per cent of its mass would already lie inside the orbit of the companion. On the other hand, an increase in $\bar{\rho}$ by a factor of 3.7, which is equivalent to a decrease in radius by a factor of 1.6, would make $a/R \simeq 1$. Changes of this magnitude are perhaps not ruled out entirely. The only other oscillation of very close period which could represent V_1 is the one in P_2. But the corresponding term in equation (35) is not periodic and would result only in a permanent deformation. However, as the amplitude of P_2^2 increases, the linear approximation breaks down, and one might expect some transfer of energy from P_2^2 to P_2 due to their nearly equal periods. Nevertheless, it seems somewhat unlikely that this could amplify P_2 to the extent observed, unless there is an incipient instability toward this mode of the same type as the vibrational instability, which in cepheids leads to radial pulsations. In any case, the corresponding period, τ_1, would be practically equal to that of the free oscillation.

The forced oscillation P_2^2 corresponds here to a wave traveling in the same direction as the revolution of m, and it is probable that the rotation of M, although much slower, will also be in the same sense. In that case, we know from our previous discussion that the phase shift between broadening and radial velocity will have the right sign. As to the corresponding period τ_2', it must be equal to π/ω, since P_2^2 is a forced oscillation. Although τ_2' must be very close to the free period τ_2 of P_2^2, it is probable that the small difference needed to have $\tau_2' > \tau_1$, as required by the observations, is not excluded. Of course, if this is the case, the fact that the broadening has the largest period in β Canis Majoris is purely accidental, and one would expect the reverse to occur in other stars.

The variations in pressure and temperature in the reversing layer could also affect the lines, although one might be tempted to think that their amplitude would be small, as are the changes in luminosity. However, in order to decide on this point, a detailed study of the conditions in the external layers would be necessary, and such a study is beyond the scope of this paper.

On the other hand, the investigation of A. B. Underhill[11] gives the impression that motions must be mainly responsible for the line profiles. In this respect, one must expect that the general nonuniform velocity field corresponding to P_2^2 would also be accompanied by an appreciable turbulence.

The fact that the companion has to be so very close to the main star if, in fact, it does

[10] E. A. Fath, *Lick Obs. Bull.*, **17**, 116, 1935.

[11] *Ap. J.*, **104**, 388, 1946.

not share a common photosphere with it[12] makes a precise discussion rather difficult. On the other hand, the observed changes in the amplitude and period of V_2 may be connected with this.

Of course, we need much more observational data, especially on other stars of the same type, before we can reach any definite conclusion, but it is hoped that this analysis in terms of nonradial oscillations will help in the search for a final explanation.

It is a pleasure to record here the many interesting discussions which I had with Dr. M. Schwarzschild in the course of this work.

[12] Cf., in this respect, Hoyle and Lyttleton's theory of the cepheids, *M.N.*, **103**, 21, 1943.