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ON THE RADIAL PULSATION OF GASEOUS STARS

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ABSTRACT

An application of the virial theorem to the radial pulsation of gaseous stars leads to a simple derivation of the approximate formula

$$\tau = 2\pi \left(\frac{I_0}{-[3\Gamma_1 - 4]\,\Omega_0} \right)^{1/2} \tag{i}$$

for the period found earlier (*Ap. J.*, **94**, 124, 1941) by a variational method. In the foregoing formula I_0 denotes the moment of inertia and Ω_0 the potential energy of the star, and Γ_1 is the adiabatic exponent assumed constant through the star.

If the star is in steady uniform rotation, in the same approximation, the theorem gives

$$\tau = 2\pi \left(\frac{I_0}{-[3\Gamma_1 - 4]\Omega_0 + [5 - 3\Gamma_1]\omega_0 \mathfrak{M}} \right)^{1/2}, \tag{ii}$$

where ω_0 denotes the angular velocity and $\mathfrak{M}$ is the total angular momentum of the star. In the case of a homogeneous distribution of mass, equation (ii) reduces to a formula which can be derived directly from one of Poincaré's theorems. When $\frac{4}{3} < \Gamma_1 < \frac{5}{3}$ and under the circumstances of validity of the approximations used, the decrease in period due to the rotation is small.

1. *The application of the virial theorem to the steady pulsations of a gaseous star.*—For a cloud of particles which act on each other only by gravitational attraction or shocks we have the Lagrangian identity

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T + \Omega\,, \tag{1}$$

where I is the moment of inertia with respect to the origin, T the kinetic energy, and Ω is the gravitational potential energy of the star. Equation (1) is the form given by Jacobi to a theorem originally due to Lagrange for three bodies. For a spherical distribution of mass

$$I = \int_0^M r^2 dm(r) \quad \text{and} \quad \Omega = -G\int_0^M \frac{m(r)\,dm(r)}{r}\,, \tag{2}$$

where $m(r)$ denotes the mass interior to r and M is the total mass of the configuration.

In this paper we shall consider the application of equation (1) to the steady radial pulsations of a gaseous star. In studying this problem we shall adopt the Lagrangian mode of description, in which we follow each particle (or element of mass) during its motion. Let the distance r from the center of symmetry be used as such a Lagrangian coordinate. Further, let δr denote the displacement from the "equilibrium" position r_0. The conservation of mass clearly requires that

$$m(r_0+\delta r) = m(r_0) . \tag{3}$$

Let δI, δT, and $\delta\Omega$ denote the changes from their equilibrium values in the respective quantities at time t. Equation (1) gives

$$\frac{1}{2}\frac{d^2\delta I}{dt^2} = 2\delta T + \delta\Omega . \tag{4}$$

To a first order in δr we have (cf. eq. [2])

$$\delta I = 2\int_0^M r\,\delta r\,dm(r) \simeq 2\int_0^M \frac{\delta r}{r_0}\,dI_0 \tag{5}$$

and

$$\delta\Omega = G\int_0^M \frac{m(r)\,\delta(r)}{r^2}\,dm(r) \simeq -\int_0^M \frac{\delta r}{r_0}\,d\Omega_0 , \tag{5$'$}$$

where we have used the suffix zero to indicate that the quantities refer to the instant when $r = r_0$.

For small periodic oscillations we may write

$$\frac{\delta r}{r_0} = \xi = \xi_0 e^{i\sigma t} , \tag{6}$$

where $2\pi/\sigma$ denotes the period. Equations (5) and (5$'$) now become

$$\delta I = 2e^{i\sigma t}\int_0^M \xi_0 dI_0 \quad \text{and} \quad \delta\Omega = -e^{i\sigma t}\int_0^M \xi_0 d\Omega_0 . \tag{7}$$

Considering next the evaluation of δT, we may first observe that T consists of two parts: the kinetic energy due to thermal motions and the kinetic energy due to the vibrations. It is evident that the latter is of the second order in ξ and can therefore be ignored in a first-order theory. Accordingly, we need to consider only the variation in the total kinetic energy T, stored in the form of thermal motions. If the radiation pressure is negligible, we have the relation

$$2T_1 = 3\int_0^M \frac{p_g}{\rho}\,dm(r) , \tag{8}$$

where p_g denotes the gas pressure and ρ the density. On the other hand, if the radiation pressure is included, the usual argument which leads to equation (1) shows that equation (8) has to be replaced by (cf. eq. [66], below)

$$2T_1 = 3\int_0^M \frac{P}{\rho}\,dm(r) . \tag{9}$$

where P is the total pressure, including the gas and the radiation pressures. Thus, to the first order

$$2\delta T = 2\delta T_1 = 3\int_0^M \delta\left(\frac{P}{\rho}\right) dm(r_0) . \tag{10}$$

We shall now suppose that the pulsations are adiabatic, in which case

$$\delta\left(\frac{P}{\rho}\right) = \frac{P_0}{\rho_0}(\Gamma_1 - 1)\frac{\delta\rho}{\rho_0} , \tag{11}$$

where Γ_1 is the adiabatic exponent properly defined. Moreover, if $d\xi/dr_0$ is further assumed to be a small quantity of the first order everywhere, the equation of continuity leads to the relation

$$\frac{\delta\rho}{\rho_0} = -\left(3\,\xi_0 + r_0\,\frac{d\,\xi_0}{d\,r_0}\right) e^{i\sigma t}\,. \tag{12}$$

Combining equations (10), (11), and (12), we obtain

$$2\,\delta T = -\,9\,e^{i\sigma t}\int_0^M \frac{P_0}{\rho_0}(\Gamma_1 - 1)\;\xi_0 d\,m\,(r_0) - 3\,e^{i\sigma t}\int_0^{R_0} 4\pi P_0\,(\Gamma_1 - 1)\,\frac{d\,\xi_0}{d\,r_0}\;r_0^3 d\,r_0\,. \tag{13}$$

Integrating by parts the second of the two integrals on the right-hand side of equation (13), we find

$$\left.\begin{aligned}4\pi\int_0^{R_0} P_0\,(\Gamma_1 - 1)\;r_0^3\,\frac{d\,\xi_0}{d\,r_0}\,d\,r_0 &= 4\pi P_0\,(\Gamma_1 - 1)\;r_0^3\,\xi_0\Big|_0^{R_0} - 4\pi\int_0^{R_0} P_0\,\xi_0\,r_0^3\;d\Gamma_1\\ &\quad - 4\pi\int_0^{R_0}(\Gamma_1 - 1)\;\xi_0\,\frac{dP_0}{d\,r_0}\;r_0^3 d\,r_0 - 12\pi\int_0^{R_0} P_0\,(\Gamma_1 - 1)\;\xi_0 r_0^2 d\,r_0\,.\end{aligned}\right\} \tag{14}$$

The integrated part vanishes at both limits, and we are left with (cf. eq. [2])

$$\left.\begin{aligned}4\pi\int_0^{R_0} P_0\,(\Gamma_1 - 1)\;r_0^3\,\frac{d\,\xi_0}{d\,r_0}\,d\,r_0 &= -\int_0^M \frac{P_0}{\rho_0}\;\xi_0\,r_0\,\frac{d\Gamma_1}{d\,r_0}\,d\,m\,(r_0)\\ &\quad - 3\int_0^M \xi_0\,\frac{P_0}{\rho_0}(\Gamma_1 - 1)\,d\,m\,(r_0) - \int_0^M \xi_0\,(\Gamma_1 - 1)\,d\Omega_0\,,\end{aligned}\right\} \tag{15}$$

where in transforming the second of the three integrals on the right-hand side of equation (14) we have used the relation

$$4\pi\,r_0^3\,\frac{dP_0}{d\,r_0} = -\,4\pi\,r_0^3\,\frac{Gm\,(r_0)}{r_0^2}\;\rho_0 = -\,\frac{Gm\,(r_0)\;d\,m\,(r_0)}{r_0 d\,r_0} = \frac{d\Omega_0}{d\,r_0}\,. \tag{16}$$

Equations (13) and (15) now give

$$2\,\delta T = 3\,e^{i\sigma t}\int_0^M \frac{P_0}{\rho_0}\;\xi_0\,r_0\,\frac{d\Gamma_1}{d\,r_0}\,d\,m\,(r_0) + 3\,e^{i\sigma t}\int_0^M \xi_0\,(\Gamma_1 - 1)\;d\Omega_0\,. \tag{17}$$

Finally, introducing equations (7) and (17) into equation (4), we obtain

$$-\,\sigma^2\int_0^M \xi_0 dI_0 = 3\int_0^M \frac{P_0}{\rho_0}\;\xi_0\,r_0\,\frac{d\Gamma_1}{d\,r_0}\,d\,m\,(r_0) + 3\int_0^M \xi_0\,(\Gamma_1 - 1)\;d\Omega_0 - \int_0^M \xi_0 d\Omega_0\,, \tag{18}$$

or

$$\sigma^2 = -\,\frac{\displaystyle\int_0^M (3\Gamma_1 - 4)\;\xi_0 d\Omega_0 + 3\int_0^M \frac{P_0}{\rho_0}\;\xi_0\,r_0\,\frac{d\Gamma_1}{d\,r_0}\,d\,m\,(r_0)}{\displaystyle\int_0^M \xi_0 d\,I_0}\,. \tag{19}$$

Equation (19) for the period is of the same general form as the one obtained in an earlier paper.[1] However, in that paper σ^2 was defined, in accordance with the Ritz principle, as the minimum of a similar expression which included terms of the second order in ξ_0.

[1] P. Ledoux and C. L. Pekeris, *Ap. J.*, **94**, 124, 1941. This paper will be referred to as "I."

If Γ_1 is assumed to be independent of r_0, equation (19) reduces to

$$\sigma^2 = -\frac{(3\Gamma_1 - 4)\int_0^M \xi_0 d\Omega_0}{\int_0^M \xi_0 dI_0} . \qquad (20)$$

If we further suppose that ξ_0 is a constant, we recover the formula

$$\sigma^2 = -\frac{(3\Gamma_1 - 4)\,\Omega_0}{I_0} , \qquad (21)$$

obtained in paper I. As has been shown in paper I, equation (21) already provides a fair approximation to the true periods if the central condensation of the star is not too high.

The extension of the foregoing analysis to include second-order terms is feasible and may give indications as to the manner in which σ^2 may be expected to change when the amplitudes of the pulsation become large. However, in such cases the assumption that the pulsations are adiabatic will require reconsideration.

2. *Application of one of Poincaré's theorems*[2] *to the pulsation of a rotating configuration.*—The problem of a rotating gaseous configuration in pulsation is a difficult one. But we may expect that a general theorem such as the one we have referred to will give us some idea as to the effect of rotation on the period.

Consider, then, the case of a uniformly rotating gaseous configuration in which at any given instant the angular velocity is the same throughout the mass. Since we have in view the pulsation of such a configuration, we must allow $\boldsymbol{\omega}$ to be a function of time. Under these conditions the equation of motion in a frame of reference rotating with an angular velocity $\boldsymbol{\omega}$ is

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{\rho}\,\text{grad}\,P + \text{grad}\,\mathfrak{B} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) - 2(\boldsymbol{\omega}\times\boldsymbol{v}) - \left(\frac{d\boldsymbol{\omega}}{dt}\times\boldsymbol{r}\right), \qquad (22)$$

where P denotes the total pressure, ρ the density, and $\mathfrak{B}$ the gravitational potential governed by Poisson's equation,

$$\nabla^2\mathfrak{B} = -4\pi G\rho . \qquad (23)$$

Under the circumstances envisaged,

$$\boldsymbol{v} = \frac{d\,\delta\boldsymbol{r}}{dt} , \qquad (24)$$

where $\delta\boldsymbol{r}$ denotes the displacement, at time t, of an element of gas whose equilibrium position is $\boldsymbol{r}_0$ (say). Since the problem admits of an axial symmetry, it would appear that $\delta\boldsymbol{r}$ will lie in the meridian plane. If we assume that this is the case, $\boldsymbol{v}$ will also lie in this plane, and the two last terms on the right-hand side of equation (22) are the only vectors in this equation which are normal to the meridian plane. Accordingly, we must require that

$$2(\boldsymbol{\omega}\times\boldsymbol{v}) + \frac{d\boldsymbol{\omega}}{dt}\times\boldsymbol{r} \equiv 0 . \qquad (25)$$

In spherical polar co-ordinates (with the z-axis along the axis of rotation) equation (25) takes the form

$$2\left(\omega r \cos\vartheta\,\frac{d\vartheta}{dt} + \omega\sin\vartheta\,\frac{dr}{dt}\right) + r\sin\vartheta\,\frac{d\omega}{dt} = 0 , \qquad (26)$$

[2] *Leçons sur les hypothèses cosmogoniques*, p. 22, Paris, 1911.

or, somewhat differently,

$$\frac{1}{\omega}\frac{d\omega}{dt} = -\frac{2}{r}\frac{dr}{dt} - 2\cot\vartheta\,\frac{d\vartheta}{dt}. \tag{27}$$

Equation (27) admits of immediate integration, and we have

$$\omega r^2 \sin^2\vartheta = \text{constant}. \tag{28}$$

The foregoing equation (28) expresses merely the conservation of angular momentum. We must, of course, expect this to be an integral of our problem. Thus, for pulsations in which the angular momentum is identically conserved, the displacements $\delta \boldsymbol{r}$ lie in the meridian planes, and the equation of motion (22) simplifies to

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{\rho}\operatorname{grad} P + \operatorname{grad}\mathfrak{B} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}). \tag{29}$$

On the other hand,

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = -\tfrac{1}{2}\omega^2\operatorname{grad}(x^2+y^2) = -\tfrac{1}{2}\omega^2\operatorname{grad}(r^2\sin^2\vartheta). \tag{30}$$

We can, accordingly, re-write equation (29) in the standard form

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{\rho}\operatorname{grad} P + \operatorname{grad}\phi, \tag{31}$$

where

$$\phi = \mathfrak{B} + \tfrac{1}{2}\omega^2(x^2+y^2). \tag{32}$$

At the equilibrium position, $\boldsymbol{v} = 0$; and equation (31) reduces to

$$\frac{1}{\rho_0}\operatorname{grad}_0 P_0 = \operatorname{grad}_0\phi_0, \tag{33}$$

where, as in § 1, the suffix zero is used to indicate that the quantities are assigned their equilibrium values.

Proceeding as in Poincaré's analysis, we take the divergence of equation (31) and integrate over the entire volume occupied by the configuration. We obtain

$$\int_V \frac{d}{dt}(\operatorname{div}\boldsymbol{v})\,dV = \int_V (2\omega^2 - 4\pi G\rho)\,dV - \int_V \operatorname{div}\left(\frac{1}{\rho}\operatorname{grad} P\right)dV, \tag{34}$$

use having been made of the relation (cf. eq. [23])

$$\left.\begin{aligned}\operatorname{div}\operatorname{grad}\phi &= \nabla^2\mathfrak{B} + \frac{\omega^2}{2}\nabla^2(x^2+y^2)\\ &= -4\pi G\rho + 2\omega^2.\end{aligned}\right\} \tag{35}$$

The first of the two integrals on the right-hand side of equation (34) is readily evaluated; and, transforming the second into a surface integral by Gauss's theorem, we find

$$\int_V \frac{d}{dt}(\operatorname{div}\boldsymbol{v})\,dV = 2\omega^2 V - 4\pi GM - \int_S \frac{1}{\rho}\operatorname{grad} P\cdot \boldsymbol{1}_{dS}\,dS, \tag{36}$$

where the surface integral is extended over the entire bounding surface S of the configuration and $\boldsymbol{1}_{dS}$ is a unit vector normal to element of surface dS of S.

Now, in virtue of equation (24), we may regard $\boldsymbol{v}$ as a quantity of the first order of smallness. Hence, in an approximation in which we ignore all quantities of the second order of smallness we may interchange the differentiation with respect to t and an integration over the spatial co-ordinates. Thus, in this approximation equation (36) becomes

$$\frac{d^2 V}{dt^2} = 2\omega^2 V - 4\pi G M - \int_S \frac{1}{\rho} \operatorname{grad} P \cdot \mathbf{1}_{dS} dS . \tag{37}$$

Writing $V = V_0 + \delta V$, etc., in equation (37), we obtain

$$\frac{d^2 \delta V}{dt^2} = 2\delta(\omega^2 V) - \delta \int_S \frac{1}{\rho} \operatorname{grad} P \cdot \mathbf{1}_{dS} dS . \tag{38}$$

We shall now suppose that δr is along r and that $\delta r / r_0$ can again be represented as in equation (6). Under these circumstances equation (28) allows us to conclude that

$$\frac{\delta \omega}{\omega_0} = -2 \frac{dr}{r_0} = -2\xi_0 e^{i\sigma t} . \tag{39}$$

Moreover, it is clear that, to be consistent with our assumption of uniform rotation (ω independent of spatial co-ordinates), we should also suppose that

$$\xi_0 = \text{constant} \tag{40}$$

throughout the entire configuration. As we have seen in § 1, this assumption leads, in most cases, to a fair approximation for the period and is therefore to be regarded as not too restrictive. On the assumption (40)

$$\frac{\delta \rho}{\rho_0} = -3\xi_0 e^{i\sigma t} \quad \text{and} \quad \frac{\delta V}{V_0} = 3\xi_0 e^{i\sigma t} . \tag{41}$$

Further, according to equations (39) and (41),

$$2\delta(\omega^2 V) = 2\omega_0^2 V_0 \left(2\frac{\delta\omega}{\omega_0} + \frac{\delta V}{V_0}\right) = -2\omega_0^2 V_0 \xi_0 e^{i\sigma t} . \tag{42}$$

Again, to a first approximation, we have

$$\delta \int_S \frac{1}{\rho} \operatorname{grad} P \cdot \mathbf{1}_{dS} dS = \int_{S_0} \delta\left(\frac{1}{\rho} \operatorname{grad} P\right) \cdot \mathbf{1}_{dS_0} dS_0 + \int \frac{1}{\rho_0} \operatorname{grad}_0 P_0 \cdot \delta(\mathbf{1}_{dS} dS) . \tag{43}$$

As in § 1, we shall assume that during the pulsations the changes of state take plac adiabatically. Then

$$\delta\left(\frac{1}{\rho} \operatorname{grad} P\right) = -\frac{\delta\rho}{\rho_0}\frac{1}{\rho_0} \operatorname{grad}_0 P_0 + \frac{1}{\rho_0} \operatorname{grad}_0 \left(\Gamma_1 P_0 \frac{\delta \rho}{\rho_0}\right) + \frac{1}{\rho_0} (\delta \operatorname{grad}) P_0 . \quad (44$$

Now the components proportional to $\partial/\partial r$ and $\partial/r\partial\vartheta$ of the gradient operator at tim have the values

$$\frac{1}{1+\xi}\frac{\partial}{\partial r_0} \quad \text{and} \quad \frac{1}{1+\xi}\frac{1}{r_0}\frac{\partial}{\partial \vartheta} . \quad (4$$

Accordingly,

$$\delta(\operatorname{grad}) = -\xi_0 e^{i\sigma t} \operatorname{grad}_0 . \quad (4$$

Restricting ourselves further to the case where Γ_1 is a constant, we have

$$\frac{1}{\rho_0} \operatorname{grad}_0 \left(\Gamma_1 P_0 \frac{\delta\rho}{\rho_0}\right) = \frac{\Gamma_1 P_0}{\rho_0} \operatorname{grad} \frac{\delta\rho}{\rho_0} + \frac{\Gamma_1}{\rho_0}\frac{\delta\rho}{\rho_0} \operatorname{grad}_0 P_0 , \quad (4$$

or, since $\delta\rho/\rho_0$ is independent of spatial co-ordinates (cf. eqs. [40] and [41]),

$$\frac{1}{\rho_0}\,\text{grad}_0\left(\Gamma_1 P_0\,\frac{\delta\rho}{\rho_0}\right) = -3\,\xi_0 e^{i\sigma t}\,\frac{\Gamma_1}{\rho_0}\,\text{grad}_0 P_0\,. \tag{48}$$

Combining equations (44), (46), and (48) and after some further minor reductions, we find

$$\delta\left(\frac{1}{\rho}\,\text{grad}\,P\right) = \xi_0 e^{i\sigma t}\,(2-3\Gamma_1)\,\frac{1}{\rho_0}\,\text{grad}_0 P_0\,. \tag{49}$$

Now, since the bounding surface S of the configuration must be an equipotential surface,

$$\mathbf{1}_{dS} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}\,. \tag{50}$$

Also, we may write

$$dS = 2\pi r^2 \sin\vartheta\, d\vartheta\,. \tag{50'}$$

Further, under the assumptions made concerning $\delta\mathbf{r}$, it is evident that the direction of the normal is left unchanged during the pulsation. Hence,

$$\delta\,(\mathbf{1}_{dS} dS) = 2\xi_0 e^{i\sigma t}\,(\mathbf{1}_{dS_0} dS_0)\,. \tag{51}$$

Introducing equations (49) and (51) into equation (43), we have

$$\delta\int_S \frac{1}{\rho}\,\text{grad}\,P\cdot\mathbf{1}_{dS} dS = (4-3\Gamma_1)\,\xi_0 e^{i\sigma t}\int_{S_0}\frac{1}{\rho_0}\,\text{grad}_0 P\cdot\mathbf{1}_{dS_0} dS_0\,. \tag{52}$$

The foregoing equation can be further simplified by the use of equations (32) and (33). Thus,

$$\left.\begin{aligned}\delta\int_S \frac{1}{\rho}\,\text{grad}\,P\cdot\mathbf{1}_{dS} dS &= (4-3\Gamma_1)\,\xi_0 e^{i\sigma t}\int_V \text{div}_0\,\text{grad}_0\,\phi_0 dV_0\\ &= (4-3\Gamma_1)\,\xi_0 e^{i\sigma t}\,(2\omega_0^2 V_0 - 4\pi G M)\,.\end{aligned}\right\} \tag{53}$$

Equation (38) now reduces to (cf. eqs. [41], [42], and [53])

$$\sigma^2 = \tfrac{4}{3}\pi G\,(3\Gamma_1-4)\,\bar{\rho} + \tfrac{2}{3}\omega_0^2\,(5-3\Gamma_1)\,, \tag{54}$$

where $\bar{\rho}$ denotes the mean density. For a mass devoid of rotation, equation (54) gives

$$\sigma^2 = \tfrac{4}{3}\pi G\,(3\Gamma_1-4)\,\bar{\rho}\,. \tag{55}$$

In this case of no rotation a better approximation, taking into account the variation of ξ_0 with r_0, can readily be found. The final result can be expressed in the form

$$\sigma^2 = \frac{4}{3}\,\pi G\bar{\rho}\left[\,(3\Gamma_1-4) + \Gamma_1 R\left(\frac{1}{\xi_0}\frac{d\xi_0}{dr_0}\right)_R\right], \tag{56}$$

where R is the radius of the spherical star considered. However, equation (56) has the disadvantage that it makes σ^2 spuriously sensitive to the boundary values.

According to equation (54) for $\frac{4}{3} < \Gamma_1 < \frac{5}{3}$, the effect of rotation is to increase σ^2, and this implies a decrease in the period. But this can only be regarded as a qualitative indication. For equation (55) predicts for the standard model, for example, a value of σ which is smaller than the true value by a factor 3.6. On the other hand, equation (55) predicts a value which approaches the true one in the limit of a uniform distribution. As our discussion in paper I has shown, the effect of the central condensation is to multiply the value of σ^2 given by equation (54) by a factor of the order of

$$-\frac{3\Omega_0}{4\pi G\bar{\rho}I_0}. \tag{57}$$

We may assume that this is true of the first term of equation (54). But to find the corresponding factor for the second term we need a more detailed discussion, which is undertaken in the following section.

3. *The application of the virial theorem to the pulsation of a rotating star.*—The statement of the virial theorem for a rotating configuration can be deduced from (1) if proper allowance is made for the fact that in this equation T refers to the kinetic energy in a fixed frame of reference. However, we shall derive the necessary formulation of the theorem more directly from the equation of motion (31). Re-writing this equation in the form

$$\frac{d\boldsymbol{v}}{dt}\,dm = -\,(\text{grad}\,P)\,dV + (\text{grad}\,\mathfrak{V})\,dm + [\tfrac{1}{2}\omega^2\,\text{grad}\,(x^2+y^2)\,]\,dm\,, \tag{58}$$

we resolve it into its Cartesian components. We have

$$\left.\begin{aligned}\frac{d^2x}{dt^2}\,dm &= -\frac{\partial P}{\partial x}\,dV + \frac{\partial\mathfrak{V}}{dx}\,dm + \omega^2 x\,dm\,,\\ \frac{d^2y}{dt^2}\,dm &= -\frac{\partial P}{\partial y}\,dV + \frac{\partial\mathfrak{V}}{\partial y}\,dm + \omega^2 y\,dm\,,\\ \frac{d^2z}{dt^2}\,dm &= -\frac{\partial P}{\partial z}\,dV + \frac{\partial\mathfrak{V}}{\partial z}\,dm\,.\end{aligned}\right\} \tag{59}$$

Multiplying the foregoing equations by x, y, and z, respectively, adding them together, and integrating the result over the entire configuration, we obtain

$$\left.\begin{aligned}\frac{1}{2}\int_0^M \frac{d^2}{dt^2}(x^2+y^2+z^2)\,dm &= \int_0^M\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]dm\\ &+\int_0^M\left(x\frac{\partial\mathfrak{V}}{\partial x}+y\frac{\partial\mathfrak{V}}{\partial y}+z\frac{\partial\mathfrak{V}}{\partial z}\right)dm+\int_0^M\omega^2(x^2+y^2)\,dm\\ &-\int_0^V\left(x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}+z\frac{\partial P}{\partial z}\right)dm\,,\end{aligned}\right\} \tag{60}$$

where use has been made of the relations

$$x\frac{d^2x}{dt^2}=\frac{1}{2}\frac{d^2x^2}{dt^2}-\left(\frac{dx}{dt}\right)^2,\quad\text{etc.} \tag{61}$$

Equation (60) can be written alternatively in the form

$$\frac{1}{2}\frac{d^2I}{dt^2}=2T_2+\Omega-\int_V \boldsymbol{r}\cdot\text{grad}\,P\,dV+\int_0^{\mathfrak{M}}\omega\,d\mathfrak{M}\,, \tag{62}$$

where I and Ω have the same meanings as in § 1, T_2 is the kinetic energy with respect to the chosen axis, and $\mathfrak{M}$ is the total angular momentum ($d\mathfrak{M} = \omega^2[x^2 + y^2]dm$). Since

$$\text{div}\,(\boldsymbol{r}P) = \boldsymbol{r}\cdot\text{grad}\,P + 3P\,, \tag{63}$$

we have

$$\left.\begin{aligned}\int_V \boldsymbol{r}\cdot\text{grad}\,P\,dV &= \int_V \text{div}\,(\boldsymbol{r}P)\,dV - 3\int_V P\,dV\\ &= \int_S (\boldsymbol{r}P)\cdot \mathbf{1}_{dS}\,dS - 3\int_V P\,dV\,.\end{aligned}\right\} \tag{64}$$

But as P vanishes on the bounding surface, we have

$$\int_V \mathbf{r}\cdot\operatorname{grad} P\,dV = -3\int_V P\,dV\,. \qquad (65)$$

Equation (62) therefore becomes

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2T_2 + \Omega + 3\int_V P\,dV + \int_0^{\mathfrak{M}} \omega\, d\mathfrak{M}\,. \qquad (66)$$

In considering the variation of this equation we shall (as in § 2) consider displacements $\delta \mathbf{r}$, which lie in the meridian planes and which can further be represented as

$$\frac{\delta r}{r_0} = \xi_0 e^{i\sigma t} \qquad (\xi_0 = \text{constant})\,. \qquad (67)$$

Also, ξ_0 is assumed to be a quantity of the first order of smallness. For such variations T_2 is of the second order of smallness and can be ignored. We can therefore write

$$\frac{1}{2}\frac{d^2 \delta I}{dt^2} = \delta\Omega + 3\int_0^M \delta\left(\frac{P}{\rho}\right) dm(r) + \delta\int_0^{\mathfrak{M}} \omega\, d\mathfrak{M}\,. \qquad (68)$$

Remembering that ξ_0 is assumed to be a constant, we have

$$\delta I = 2\xi_0 e^{i\sigma t} I_0 \qquad \text{and} \qquad d\Omega = -\xi_0 e^{i\sigma t}\Omega_0\,. \qquad (69)^{3}$$

Moreover, for adiabatic pulsations under our present conditions we have (cf. eqs. [11] and [12])

$$\delta\left(\frac{P}{\rho}\right) = \frac{P_0}{\rho_0}(\Gamma_1 - 1)\frac{\delta\rho}{\rho_0} = -3\xi_0 e^{i\sigma t}(\Gamma_1 - 1)\frac{P_0}{\rho_0}\,. \qquad (70)$$

Assuming, further, that Γ_1 is a constant, we have

$$\int_0^M \delta\left(\frac{P}{\rho}\right) dm(r) = -3(\Gamma_1 - 1)\,\xi_0 e^{i\sigma t}\int_{V_0} P_0\,dV_0\,, \qquad (71)$$

and equation (68) becomes

$$-\sigma^2 \xi_0 e^{i\sigma t} I_0 = -\xi_0 e^{i\sigma t}\Omega_0 - 3(\Gamma_1 - 1)\,\xi_0 e^{i\sigma t}\int_{V_0} 3P_0\,dV_0 + \delta\int_0^{\mathfrak{M}} \omega\, d\mathfrak{M}\,. \qquad (72)$$

On the other hand, for the equilibrium position equation (66) gives

$$\Omega_0 + 3\int_{V_0} P_0\,dV_0 + \int_0^{\mathfrak{M}} \omega_0\, d\mathfrak{M} = 0\,. \qquad (73)$$

Combining equations (72) and (73), we have

$$-\sigma^2 \xi_0 e^{i\sigma t} I_0 = (3\Gamma_1 - 4)\,\xi_0 e^{i\sigma t}\Omega_0 + 3(\Gamma_1 - 1)\,\omega_0 \mathfrak{M}\,\xi_0 e^{i\sigma t} + \delta\int_0^{\mathfrak{M}} \omega\, d\mathfrak{M}\,. \qquad (74)$$

Now the conservation of angular momentum insures that $\mathfrak{M}$ and $d\mathfrak{M}$ remain constant during the pulsation. Hence,

$$\delta\int_0^{\mathfrak{M}} \omega\, d\mathfrak{M} = \int_0^{\mathfrak{M}} \frac{\delta\omega}{\omega_0}\,\omega_0\, d\mathfrak{M}\,; \qquad (75)$$

or, using equation (39),

$$\delta\int_0^{\mathfrak{M}} \omega\, d\mathfrak{M} = -2\xi_0 e^{i\sigma t}\int_0^{\mathfrak{M}} \omega_0\, d\mathfrak{M} = -2\xi_0 e^{i\sigma t}\omega_0 \mathfrak{M}\,. \qquad (76)$$

[3] The relation for $\delta\Omega$ follows from the fact that Ω is of dimension -1 in the relative distances.

Substituting this in equation (74), we finally obtain

$$\sigma^2 = -\frac{(3\Gamma_1 - 4)\,\Omega_0}{I_0} + \frac{(5 - 3\Gamma_1)\,\omega_0 \mathfrak{M}}{I_0}. \tag{77}$$

Just as equation (21) for the period is superior to equation (55), we may expect that equation (77) is a correspondingly better approximation for the period when there is rotation than equation (54). On the other hand, equations (77) and (54) become identical in the limit of a homogeneous distribution.

Now the factor introduced by our present method in the second term of (77) is never very different from $2\omega_0/3$, while Ω_0/I_0 can be much larger than $4\pi G\bar{\rho}/3$. For instance, in the case of the standard model, $\Omega_0/I_0 = 13.26 \times 4\pi G\bar{\rho}/3$. If we consider a star having the same distribution of density as the standard model but rotating with an angular velocity ω and neglect its deviations from spherical shape, we have

$$\sigma^2 = 13.26\,(3\Gamma_1 - 4)\frac{g}{R} + \frac{2}{3}(5 - 3\Gamma_1)\frac{f}{R}, \tag{78}$$

where we have used g to denote the gravitational attraction at the surface and f the centrifugal force at the equator. If we take its spheroidal shape into consideration, Ω_0/I_0 will decrease and $\mathfrak{M}/I_0$ will increase. However, the maximum value of $\mathfrak{M}/I_0$ is 1; and Ω_0/I_0 is not likely to become much smaller for stable configurations than its spherical value, since the central condensation will persist; and $\Omega_0/I_0 = 10 \times 4\pi G\bar{\rho}/3$ is likely to be a minimum value. With these values (78) becomes

$$\sigma^2 \simeq 10\,(3\Gamma_1 - 4)\frac{g}{R} + (5 - 3\Gamma_1)\frac{f}{R}. \tag{79}$$

Now let us consider more closely the implications of our assumptions concerning $\delta \boldsymbol{r}$. For displacements of the kind we have considered, isosteric as well as isobaric, surfaces will remain as such, since $\delta\rho/\rho_0 =$ constant and $\delta P/P_0 = \Gamma_1 \delta\rho/\rho_0 =$ constant. But on such a surface ϕ will not remain a constant, and grad ϕ and grad P will not continue to be parallel and there will be a tendency to create a current causing circulation. To the first order of approximation the restoring force along $\boldsymbol{r}$ is

$$F_r = \xi\,(3\Gamma_1 - 4)\frac{\partial \mathfrak{V}_0}{\partial r_0} - \xi\,(5 - 3\Gamma_1)\,\omega_0^2 r_0 \sin^2\vartheta. \tag{80}$$

But there is a component along ϑ as well, and this has the value

$$F_\vartheta = \xi\,(3\Gamma_1 - 4)\frac{1}{r_0}\frac{\partial \mathfrak{V}}{\partial \vartheta} - \xi\,(5 - 3\Gamma_1)\,\omega_0^2 r \sin\vartheta \cos\vartheta. \tag{81}$$

For a spheroid of revolution flattened at the poles F_ϑ vanishes at the poles and at the equator. Comparing F_r and F_ϑ at 45° for masses which have reached their critical configurations (maximum rotation compatible with steady rotation), we shall obtain an idea of the maximum effect of F_ϑ. If we take $\Gamma_1 = \frac{5}{3}$, the ratio of F_ϑ to F_r at 45° on the surface of the critical Roche's model is equal to $\frac{1}{5}$, and on the Maclaurin's spheroid it is equal to $\frac{1}{6.5}$. On the other hand, what happens in the external layers of the star is not likely to have a great bearing on the problem and in any case would not be expected to affect the proper period of the star, so that we should really compare the values of F_ϑ and F_r at some distance inside the surface. Actually, the ratio decreases rapidly with r. Thus, if we suppose ω_0 to be reasonably far from its critical value, the component of force along ϑ will be small, compared to the component along $\boldsymbol{r}$; and the transfer of energy from the pulsation to the currents due to F_ϑ will be slow, compared to the period of pulsation. Therefore, it is the stability, rather than the period, which will be affected.

However, if Γ_1 approaches $\frac{4}{3}$, equations (54) and (55) become equal at 45° and our approximation breaks down altogether, whatever the rotation. If Γ_1 becomes smaller than $\frac{4}{3}$, it is seen that the star becomes unstable at the poles along r and that currents will be set up rapidly in the rest of the star, increasing its instability.

Returning to equation (79), we see that, if its validity is assumed also when f and g are of the same order of magnitude, the contribution of rotation to σ^2 amounts to only about 10 per cent if $\Gamma_1 = \frac{5}{3}$. The corresponding change in the period will be about 5 per cent. However, as the central condensation decreases, the effect of rotation on the period increases. Thus for a polytrope n = 2, under similar circumstances the contribution of rotation to σ^2 can amount to 20 per cent.

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