Generalized Pascal triangles for binomial coefficients of finite words

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Classical Pascal triangle

$\binom{m}{k}$		k								
		0	1	2	3	4	5	6	7	$\cdots$
	0	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	2	1	2	1	0	0	0	0	0	
m	3	1	3	3	1	0	0	0	0	
	4	1	4	6	4	1	0	0	0	
	5	1	5	**10**	**10**	5	1	0	0	
	6	1	6	15	**20**	15	6	1	0	
	7	1	7	21	35	35	21	7	1	
	$\vdots$									$\ddots$

Usual binomial coefficients of integers:

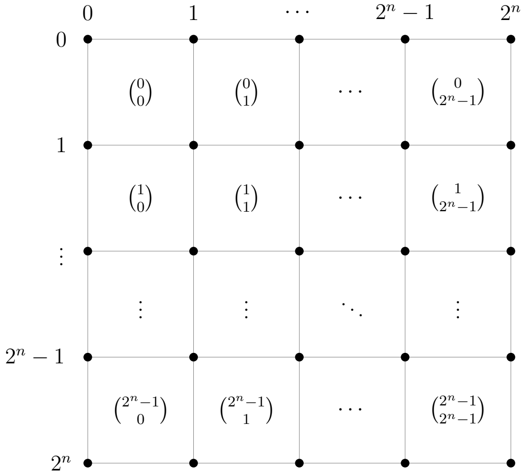
$$\binom{m}{k} = \frac{m!}{(m-k)!\,k!}$$

Pascal's rule:

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$$

A specific construction

- Grid: intersection between $\mathbb{N}^2$ and $[0, 2^n] \times [0, 2^n]$



$\mathbb{N}^2 \cap [0, 2^n]^2$

- Color the grid:
  Color the first 2^n rows and columns of the Pascal triangle

$$\left(\binom{m}{k} \bmod 2\right)_{0 \leq m,k < 2^n}$$

  in
  - white if $\binom{m}{k} \equiv 0 \bmod 2$
  - black if $\binom{m}{k} \equiv 1 \bmod 2$

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  - white if $\binom{m}{k} \equiv 0 \bmod 2$
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- Normalize by a homothety of ratio $1/2^n$
  (bring into $[0,1]\times[0,1]$)
  $\rightsquigarrow$ sequence belonging to $[0,1]\times[0,1]$

What happens for $n \in \{0, 1\}$

$n = 0$

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0 1

$$\binom{0}{0}$$

1

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$$1$$

1

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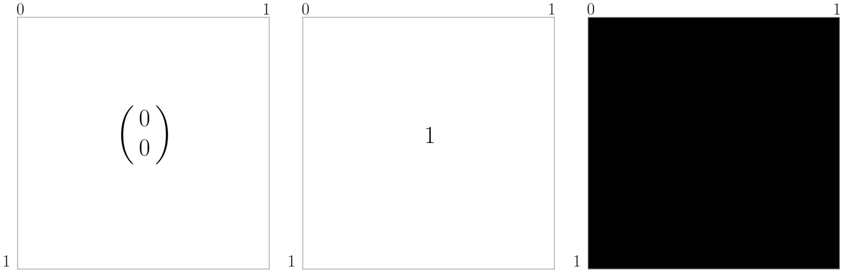

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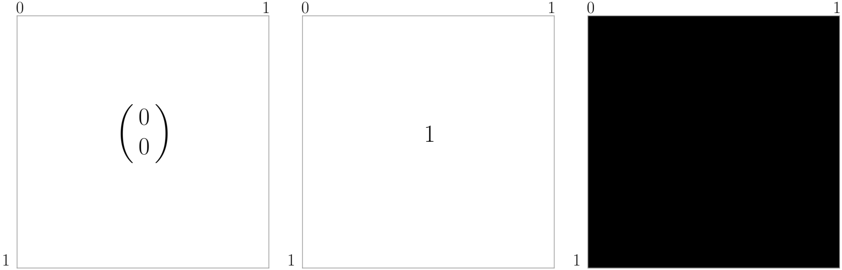

$n = 1$

What happens for $n \in \{0, 1\}$

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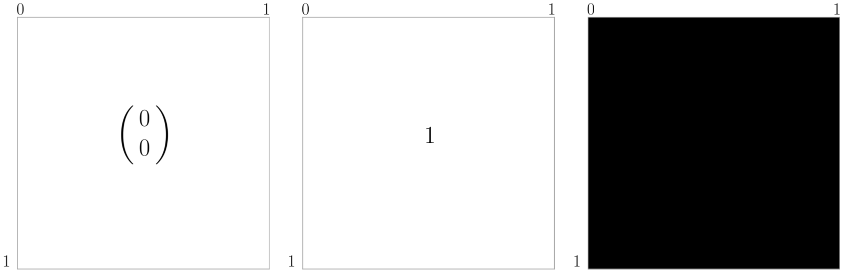

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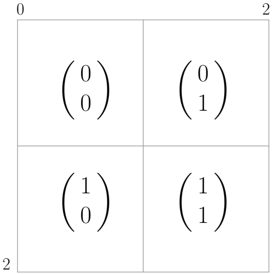
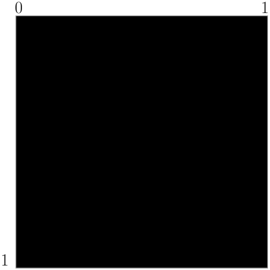


What happens for $n \in \{0, 1\}$

$n = 0$

$\binom{0}{0}$

1



$n = 1$

$\binom{0}{0}$	$\binom{0}{1}$
$\binom{1}{0}$	$\binom{1}{1}$

1	0
1	1

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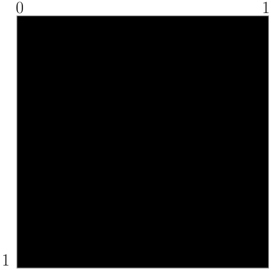

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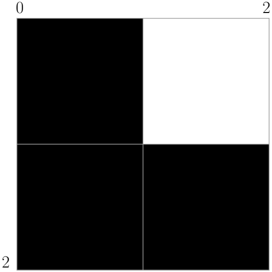
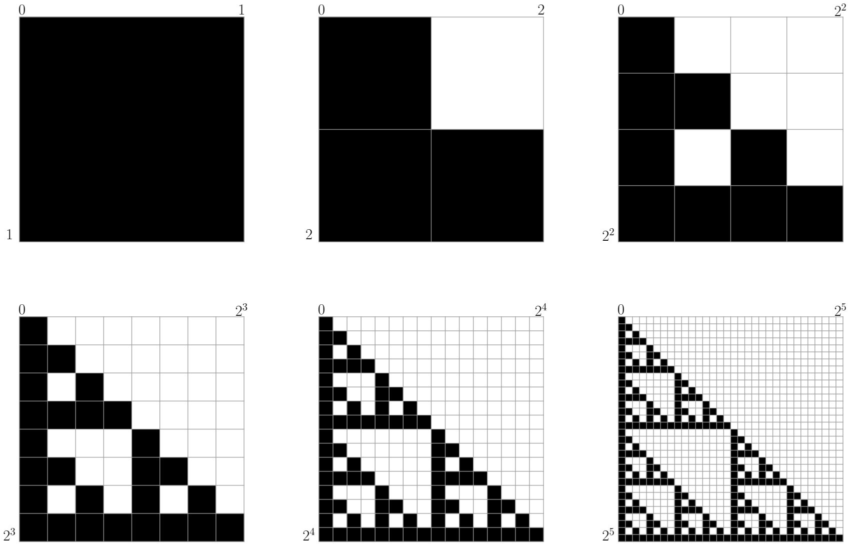
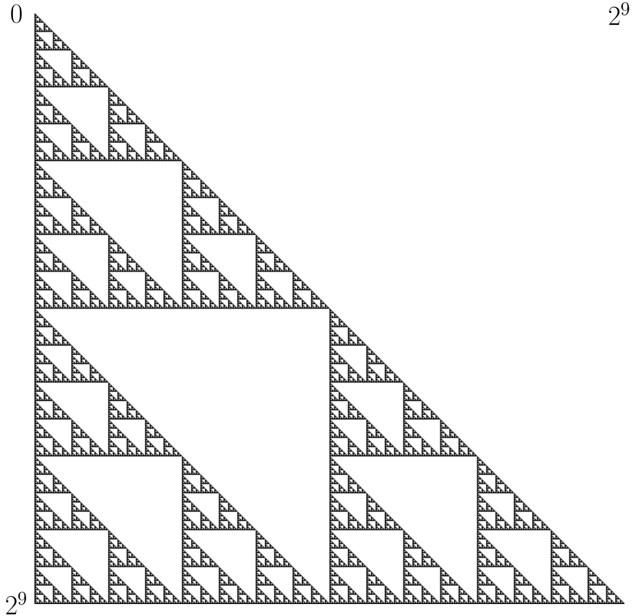
1	1



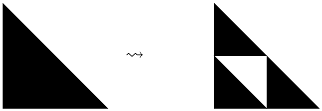
The first six elements of the sequence



The tenth element of the sequence



The Sierpiński gasket



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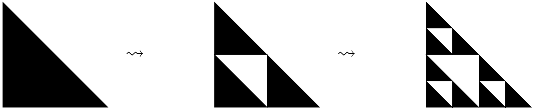

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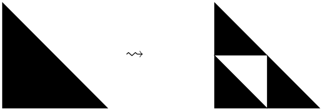

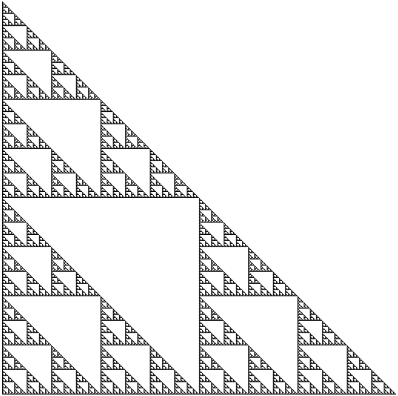

Folklore fact

The latter sequence converges to the Sierpiński gasket when n tends to infinity (for the Hausdorff distance).

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Definitions:

- ϵ-*fattening* of a subset $S \subset \mathbb{R}^2$

$$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the *Hausdorff distance* d_h

$$d_h(S, S') = \min\{\epsilon \in \mathbb{R}_{\geq 0} \mid S \subset [S']_\epsilon \quad \text{and} \quad S' \subset [S]_\epsilon\}$$

Remark (von Haeseler, Peitgen, Skordev, 1992)

The sequence also converges for other modulos.
For instance, the sequence converges when the Pascal triangle is considered modulo p^s where p is a prime and s is a positive integer.

New idea

Replace usual binomial coefficients of integers by binomial coefficients of finite words

<u>Definition</u>: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

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Example: $u = 101001$ $\quad v = 101$

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Example: $u = \mathbf{101}001$ $\qquad v = 101 \qquad$ 1 occurrence

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Example: $u = \mathbf{10}100\mathbf{1}$ $\quad v = 101 \quad$ 2 occurrences

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Example: $u = \mathbf{1}01\mathbf{0}0\mathbf{1}$ $\qquad v = 101 \qquad$ 3 occurrences

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Binomial coefficient of words

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Example: $u = \mathbf{1}010\mathbf{01}$ $\qquad v = 101 \qquad$ 4 occurrences

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Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 10\mathbf{1}\mathbf{0}0\mathbf{1}$ $\quad$ $v = 101$ $\quad$ 5 occurrences

Binomial coefficient of finite words

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Binomial coefficient of words

Let u, v be two finite words.
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Example: $u = 10\mathbf{1}\mathbf{0}\mathbf{0}\mathbf{1}$ $v = 101$ 6 occurrences

Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

Binomial coefficient of words

Let u, v be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of u and v is the number of times v occurs as a subsequence of u (meaning as a "scattered" subword).

Example: $u = 101001$ $\qquad v = 101$

$$\Rightarrow \binom{101001}{101} = 6$$

Remark:

Natural generalization of binomial coefficients of integers

With a one-letter alphabet $\{a\}$

$$\binom{a^m}{a^k} = \binom{\overbrace{a\cdots a}^{m \text{ times}}}{\underbrace{a\cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall\, m,k \in \mathbb{N}$$

Definitions:

- $\text{rep}_2(n)$ greedy base-2 expansion of $n \in \mathbb{N}_{>0}$ beginning by 1
- $\text{rep}_2(0) := \varepsilon$ where ε is the empty word

n		$\text{rep}_2(n)$
0		ε
1	1×2^0	1
2	$1 \times 2^1 + 0 \times 2^0$	10
3	$1 \times 2^1 + 1 \times 2^0$	11
4	$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$	100
5	$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$	101
6	$1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$	110
$\vdots$	$\vdots$	$\vdots$

Generalized Pascal triangle in base 2

$\binom{u}{v}$		v								
		ε	1	10	11	100	101	110	111	$\cdots$
	ε	1	0	0	0	0	0	0	0	
	1	1	1	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
u	11	1	**2**	0	**1**	0	0	0	0	
	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	1	3	0	**3**	0	0	0	1	
	$\vdots$									$\ddots$

Binomial coefficient of finite words: $\binom{u}{v}$

Rule (not local):

$$\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b}\binom{u}{v}$$

Generalized Pascal triangle in base 2

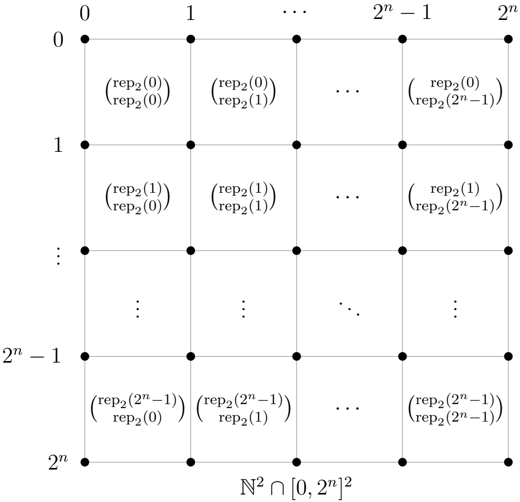
		v								
		ε	**1**	10	**11**	100	101	110	**111**	$\cdots$
	ε	**1**	0	0	0	0	0	0	0	
	1	**1**	**1**	0	0	0	0	0	0	
	10	1	1	1	0	0	0	0	0	
u	**11**	**1**	**2**	0	**1**	0	0	0	0	
	100	1	1	2	0	1	0	0	0	
	101	1	2	1	1	0	1	0	0	
	110	1	2	2	1	0	0	1	0	
	111	**1**	**3**	0	**3**	0	0	0	**1**	
	$\vdots$									$\ddots$

The classical Pascal triangle

Questions:

- After coloring and normalization can we expect the convergence to an analogue of the Sierpiński gasket?
- Could we describe this limit object ?

- Grid: intersection between $\mathbb{N}^2$ and $[0, 2^n] \times [0, 2^n]$



- Color the grid:
  Color the first 2^n rows and columns of the generalized Pascal triangle
  $$\left(\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \bmod 2\right)_{0\le m,k<2^n}$$
  in
  - white if $\binom{\mathrm{rep}_2(m)}{\mathrm{rep}_2(k)} \equiv 0 \bmod 2$
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- Normalize by a homothety of ratio $1/2^n$
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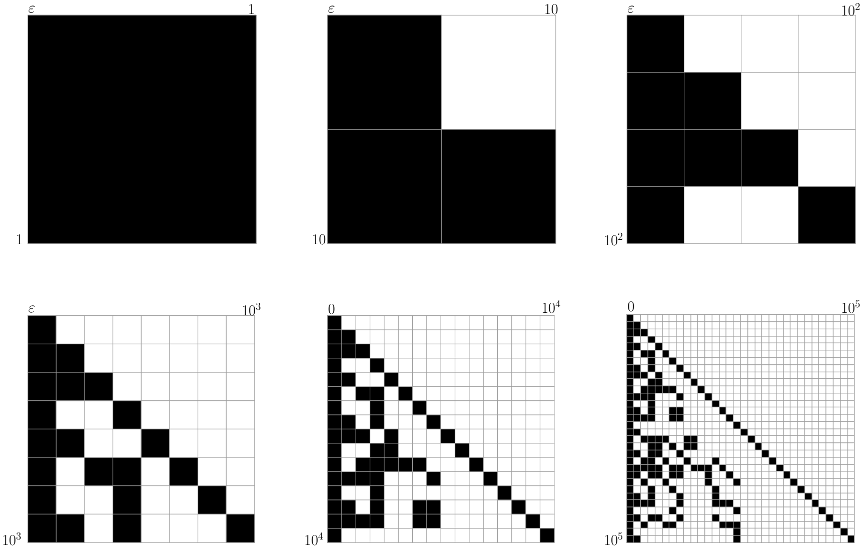
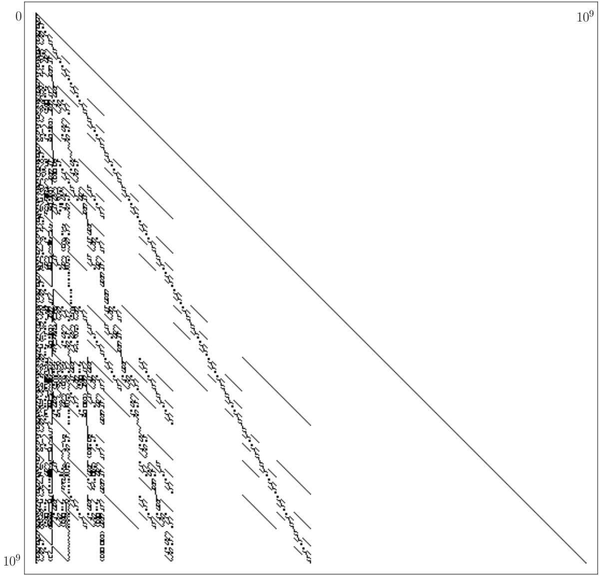
  $\rightsquigarrow$ sequence belonging to $[0,1]\times[0,1]$

The first six elements of the sequence



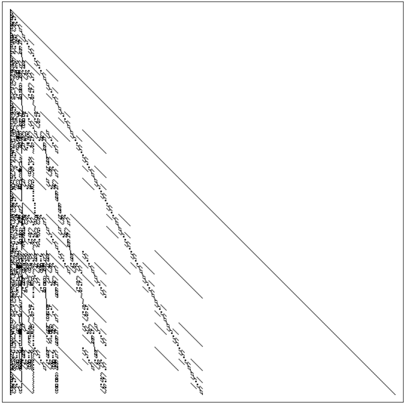
The tenth element of the sequence

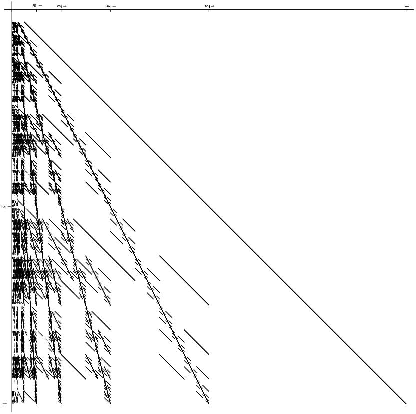


A key result

Theorem [Leroy, Rigo, S., 2016]

The sequence of compact sets converges to a limit object $\mathcal{L}$.





“Simple” characterization of $\mathcal{L}$: topological closure of a union of segments described through a “simple” combinatorial property

Extension modulo p

Simplicity: coloring the cells of the grids regarding their parity

Extension

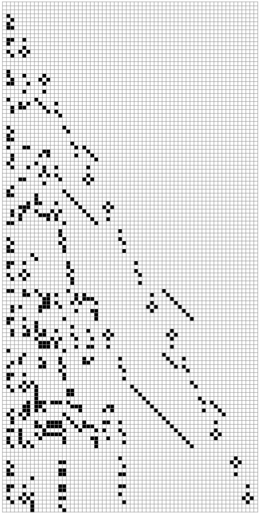
Everything still holds for binomial coefficients $\equiv r \bmod p$ with

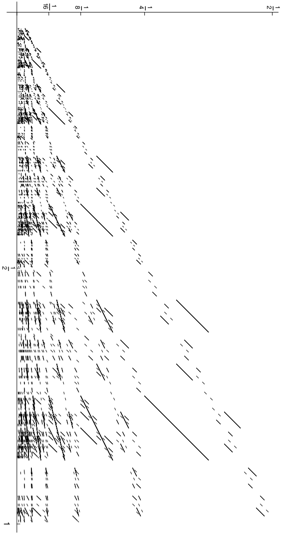
- base-2 expansions of integers
- p a prime
- $r \in \{1, \ldots, p-1\}$

Example with $p = 3$, $r = 2$

Left: binomial coefficients $\equiv 2 \bmod 3$

Right: estimate of the corresponding limit object





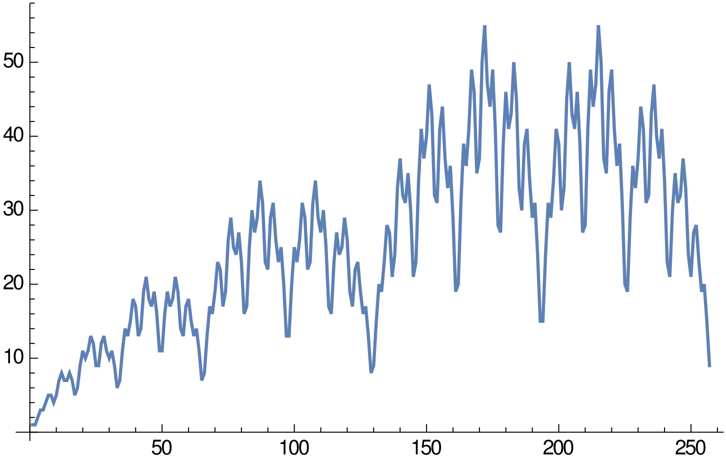
Counting positive binomial coefficients

Generalized Pascal triangle in base 2

$\binom{u}{v}$		v								n	$S_2(n)$
		ε	1	10	11	100	101	110	111		
	ε	1	0	0	0	0	0	0	0	0	**1**
	1	1	1	0	0	0	0	0	0	1	**2**
	10	1	1	1	0	0	0	0	0	2	**3**
u	11	1	2	0	1	0	0	0	0	3	**3**
	100	1	1	2	0	1	0	0	0	4	**4**
	101	1	2	1	1	0	1	0	0	5	**5**
	110	1	2	2	1	0	0	1	0	6	**5**
	111	1	3	0	3	0	0	0	1	7	**4**

Definition: $S_2(n) = \# \left\{ m \in \mathbb{N} \mid \binom{\text{rep}_2(n)}{\text{rep}_2(m)} > 0 \right\} \quad \forall n \geq 0$

The sequence $(S_2(n))_{n\geq 0}$ in the interval $[0, 256]$



Palindromic structure $\rightsquigarrow$ regularity

Regularity in the sense of...

- 2-*kernel* of $s = (s(n))_{n\geq 0}$

$$\begin{aligned}\mathcal{K}_2(s) =& \{(s(n))_{n\geq 0}, (s(2n))_{n\geq 0}, (s(2n+1))_{n\geq 0}, (s(4n))_{n\geq 0},\\ & (s(4n+1))_{n\geq 0}, (s(4n+2))_{n\geq 0}, \ldots\}\\ =& \{(s(2^i n + j))_{n\geq 0} \mid i \geq 0 \text{ and } 0 \leq j < 2^i\}\end{aligned}$$

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- 2-*regular* if there exist

$$(t_1(n))_{n\geq 0}, \ldots, (t_\ell(n))_{n\geq 0}$$

s.t. each $(t(n))_{n\geq 0} \in \mathcal{K}_2(s)$ is a $\mathbb{Z}$-linear combination of the t_j's

Theorem [Leroy, Rigo, S., 2017]

The sequence $(S_2(n))_{n\geq 0}$ satisfies, for all $n \geq 0$,

$$\begin{aligned} S_2(2n+1) &= 3\,S_2(n) - S_2(2n) \\ S_2(4n) &= 2\,S_2(2n) - S_2(n) \\ S_2(4n+2) &= 4\,S_2(n) - S_2(2n). \end{aligned}$$

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Corollary [Leroy, Rigo, S., 2017]

$(S_2(n))_{n\geq 0}$ is 2-regular.

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Corollary [Leroy, Rigo, S., 2017]

$(S_2(n))_{n\geq 0}$ is 2-regular.

$\rightsquigarrow$ Matrix representation to compute $(S_2(n))_{n\geq 0}$ easily

The Fibonacci case

Definitions:

- Fibonacci sequence $(F(n))_{n\geq 0}$: $F(0)=1$, $F(1)=2$ and
$$F(n+2)=F(n+1)+F(n)\ \forall\, n\geq 0$$
- $\text{rep}_F(n)$ greedy Fibonacci representation of $n\in\mathbb{N}_{>0}$ beginning by 1
- $\text{rep}_F(0):=\varepsilon$ where ε is the empty word

n		$\text{rep}_F(n)$	Evitability
0		ε	
1	$1\times F(0)$	1	
2	$1\times F(1)+0\times F(0)$	10	
3	$1\times F(2)+0\times F(1)+0\times F(0)$	100	No factor
4	$1\times F(2)+0\times F(1)+1\times F(0)$	101	11
5	$1\times F(3)+0\times F(2)+0\times F(1)+0\times F(0)$	1000	
6	$1\times F(3)+0\times F(2)+0\times F(1)+1\times F(0)$	1001	
$\vdots$	$\vdots$	$\vdots$	

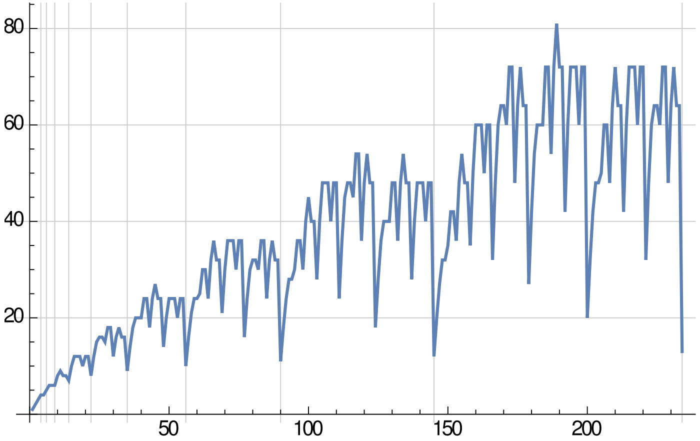
Generalized Pascal triangle in base Fibonacci

$\rightsquigarrow$ Fibonacci representations ordered genealogically

	$\binom{u}{v}$	v									
		ε	1	10	100	101	1000	1001	1010	n	$S_F(n)$
	ε	1	0	0	0	0	0	0	0	0	**1**
	1	1	1	0	0	0	0	0	0	1	**2**
	10	1	1	1	0	0	0	0	0	2	**3**
u	100	1	1	2	1	0	0	0	0	3	**4**
	101	1	2	1	0	1	0	0	0	4	**4**
	1000	1	1	3	3	0	1	0	0	5	**5**
	1001	1	2	2	1	2	0	1	0	6	**6**
	1010	1	2	3	1	1	0	0	1	7	**6**

Definition: $S_F(n) = \#\left\{ m \in \mathbb{N} \mid \binom{\text{rep}_F(n)}{\text{rep}_F(m)} > 0 \right\} \quad \forall n \geq 0$

The sequence $(S_F(n))_{n\geq 0}$ in the interval $[0, 233]$



F-kernel and F-regularity

2-kernel $\mathcal{K}_2(s)$ of a sequence s

- Select all the nonnegative integers whose base-2 expansion (with leading zeroes) ends with $w \in \{0,1\}^*$
- Evaluate s at those integers
- Let w vary in $\{0,1\}^*$

$\mathbf{w = 0}$

n	$\text{rep}_2(n)$	$s(n)$
0	ε	$\mathbf{s(0)}$
1	1	$s(1)$
2	**10**	$\mathbf{s(2)}$
3	11	$s(3)$
4	**100**	$\mathbf{s(4)}$
5	101	$s(5)$

F-kernel $\mathcal{K}_F(s)$ of a sequence s

- Select all the nonnegative integers whose Fibonacci representation (with leading zeroes) ends with $w \in \{0,1\}^*$
- Evaluate s at those integers
- Let w vary in $\{0,1\}^*$

n	$\text{rep}_F(n)$	$h(n)$
0	ε	$\mathbf{s(0)}$
1	1	$s(1)$
2	**10**	$\mathbf{s(2)}$
3	**100**	$\mathbf{s(3)}$
4	101	$s(4)$
5	**1000**	$\mathbf{s(5)}$

Property of the sequence $(S_F(n))_{n\geq 0}$

$s = (s(n))_{n\geq 0}$ is F-*regular* if there exist

$$(t_1(n))_{n\geq 0}, \ldots, (t_\ell(n))_{n\geq 0}$$

s.t. each $(t(n))_{n\geq 0} \in \mathcal{K}_F(s)$ is a $\mathbb{Z}$-linear combination of the t_j's

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Proposition [Leroy, Rigo, S., 2017]

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In the literature, not so many sequences that have this kind of property