Generalized Pascal triangles for binomial coefficients of finite words
Joint work with Julien Leroy (ULg) and Michel Rigo (ULg)

Manon Stipulanti (ULg) FRIA grantee

Computability in Europe (CiE)
Turku (Finland)
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Classical Pascal triangle

<table>
<thead>
<tr>
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<th>$k$</th>
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<th>3</th>
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<td>35</td>
<td>35</td>
<td>21</td>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

Usual binomial coefficients of integers:

\[
\binom{m}{k} = \frac{m!}{(m-k)! \cdot k!}
\]

Pascal’s rule:

\[
\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}
\]

Generalized Pascal triangles

Manon Stipulanti (ULg)
A specific construction

- Grid: intersection between $\mathbb{N}^2$ and $[0, 2^n] \times [0, 2^n]$
• Color the grid:
  Color the first $2^n$ rows and columns of the Pascal triangle

\[
\left( \binom{m}{k} \mod 2 \right)_{0 \leq m, k < 2^n}
\]

in

• white if $\binom{m}{k} \equiv 0 \mod 2$
• black if $\binom{m}{k} \equiv 1 \mod 2$
• Color the grid:
  Color the first $2^n$ rows and columns of the Pascal triangle

$$
\left\lfloor \left\lfloor \binom{m}{k} \mod 2 \right\rfloor \right\rfloor_{0 \leq m, k < 2^n}
$$

in

• white if $\binom{m}{k} \equiv 0 \mod 2$
• black if $\binom{m}{k} \equiv 1 \mod 2$

• Normalize by a homothety of ratio $1/2^n$
  $\sim$ sequence belonging to $[0,1] \times [0,1]$
The first six elements of the sequence

Generalized Pascal triangles

Manon Stipulanti (ULg)
The tenth element of the sequence
The Sierpiński gasket
The Sierpiński gasket

Generalized Pascal triangles
The Sierpiński gasket

Generalized Pascal triangles

Manon Stipulanti (ULg)
Folklore fact

The latter sequence converges to the Sierpiński gasket when $n$ tends to infinity (for the Hausdorff distance).
Folklore fact

The latter sequence converges to the Sierpiński gasket when $n$ tends to infinity (for the Hausdorff distance).

Definitions:

- **$\epsilon$-fattening** of a subset $S \subset \mathbb{R}^2$

  $$[S]_\epsilon = \bigcup_{x \in S} B(x, \epsilon)$$

- $(\mathcal{H}(\mathbb{R}^2), d_h)$ complete space of the non-empty compact subsets of $\mathbb{R}^2$ equipped with the **Hausdorff distance** $d_h$

  $$d_h(S, S') = \min\{\epsilon \in \mathbb{R}_{\geq 0} \mid S \subset [S']_\epsilon \text{ and } S' \subset [S]_\epsilon\}$$
**Remark**  
*(von Haeseler, Peitgen, Skordev, 1992)*

The sequence also converges for other modulos. For instance, the sequence converges when the Pascal triangle is considered modulo $p^s$ where $p$ is a prime and $s$ is a positive integer.
Replace usual binomial coefficients of integers by binomial coefficients of **finite words**
Definition: A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Binomial coefficient of words**

Let $u, v$ be two finite words.
The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Binomial coefficient of words
Let \( u, v \) be two finite words.
The binomial coefficient \( \binom{u}{v} \) of \( u \) and \( v \) is the number of times \( v \) occurs as a subsequence of \( u \) (meaning as a “scattered” subword).

Example: \( u = 101001 \quad v = 101 \)
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Binomial coefficient of words

Let $u, v$ be two finite words. The binomial coefficient $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

Example: $u = 101001$  \hspace{1cm} v = 101  \hspace{1cm} 1 \text{ occurrence}$
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Binomial coefficient of words**

Let $u, v$ be two finite words. The *binomial coefficient* $(^u_v)$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = 101001$  
$v = 101$  
2 occurrences
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet.*

**Binomial coefficient of words**

Let $u, v$ be two finite words. The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = 101001$  
$v = 101$  
3 occurrences
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet.*

**Binomial coefficient of words**

Let $u,v$ be two finite words. The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = 101001$  \quad $v = 101$  \quad 4 occurrences
**Definition:** A *finite word* is a finite sequence of letters belonging to a finite set called *alphabet*.

**Binomial coefficient of words**
Let $u, v$ be two finite words. The *binomial coefficient* $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

**Example:** $u = 101001$ $v = 101$ 5 occurrences
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Binomial coefficient of words
Let $u, v$ be two finite words. The binomial coefficient $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

Example: $u = 101001$  $v = 101$  6 occurrences
Definition: A finite word is a finite sequence of letters belonging to a finite set called alphabet.

Binomial coefficient of words

Let $u, v$ be two finite words. The binomial coefficient $\binom{u}{v}$ of $u$ and $v$ is the number of times $v$ occurs as a subsequence of $u$ (meaning as a “scattered” subword).

Example: $u = 101001$ $v = 101$

$$\Rightarrow \binom{101001}{101} = 6$$
Remark:
Natural generalization of binomial coefficients of integers

With a one-letter alphabet \( \{a\} \)

\[
\binom{a^m}{a^k} = \binom{\underbrace{a\cdots a}_{m \text{ times}}}{\underbrace{a\cdots a}_{k \text{ times}}} = \binom{m}{k} \quad \forall m, k \in \mathbb{N}
\]
Definitions:

- \( \text{rep}_2(n) \) greedy base-2 expansion of \( n \in \mathbb{N}_{>0} \) beginning by 1
- \( \text{rep}_2(0) := \varepsilon \) where \( \varepsilon \) is the empty word

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \text{rep}_2(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \varepsilon )</td>
</tr>
<tr>
<td>1</td>
<td>( 1 \times 2^0 )</td>
</tr>
<tr>
<td>2</td>
<td>( 1 \times 2^1 + 0 \times 2^0 )</td>
</tr>
<tr>
<td>3</td>
<td>( 1 \times 2^1 + 1 \times 2^0 )</td>
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<tr>
<td>4</td>
<td>( 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 )</td>
</tr>
<tr>
<td>5</td>
<td>( 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 )</td>
</tr>
<tr>
<td>6</td>
<td>( 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>
Generalized Pascal triangle in base 2

⇝ base-2 expansions ordered genealogically: first by length, then using the dictionary order

| \( \binom{u}{v} \) | \( \varepsilon \) | 1 | 10 | 11 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|
| \( \varepsilon \) | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| \( u \) | 11 | 1 | 2 | 0 | 1 | 0 | 0 | 0 |
| 100 | 1 | 1 | 2 | 0 | 1 | 0 | 0 | 0 |
| 101 | 1 | 2 | 1 | 1 | 0 | 1 | 0 | 0 |
| 110 | 1 | 2 | 2 | 1 | 0 | 0 | 1 | 0 |
| 111 | 1 | 3 | 0 | 3 | 0 | 0 | 0 | 1 |

Binomial coefficient of finite words:
\( \binom{u}{v} \)

Rule (not local):
\[
\binom{ua}{vb} = \binom{u}{vb} + \delta_{a,b} \binom{u}{v}
\]
Generalized Pascal triangle in base 2

⇝ base-2 expansions ordered genealogically: first by length, then using the dictionary order

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>1</th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
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<td>3</td>
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The classical Pascal triangle
Questions:

- After coloring and normalization can we expect the convergence to an analogue of the Sierpiński gasket?
- Could we describe this limit object?
- Grid: intersection between $\mathbb{N}^2$ and $[0, 2^n] \times [0, 2^n]$

\[
\begin{array}{cccccc}
0 & 1 & \cdots & 2^n - 1 & 2^n \\
\hline
0 & (\text{rep}_2(0)) & (\text{rep}_2(0)) & \cdots & (\text{rep}_2(0)) \\
1 & (\text{rep}_2(1)) & (\text{rep}_2(1)) & \cdots & (\text{rep}_2(1)) \\
\vdots & (\text{rep}_2(0)) & (\text{rep}_2(0)) & \cdots & (\text{rep}_2(0)) \\
2^n - 1 & (\text{rep}_2(2^n - 1)) & (\text{rep}_2(2^n - 1)) & \cdots & (\text{rep}_2(2^n - 1)) \\
2^n & \text{rep}_2(0) & \text{rep}_2(1) & \cdots & \text{rep}_2(2^n - 1) \\
\hline
\end{array}
\]
• Color the grid:
  Color the first $2^n$ rows and columns of the generalized Pascal triangle
  
  $$
  \left( \left( \binom{\text{rep}_2(m)}{\text{rep}_2(k)} \mod 2 \right) \right)_{0 \leq m, k < 2^n}
  $$

  in
  
  • white if $\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 0 \mod 2$
  
  • black if $\binom{\text{rep}_2(m)}{\text{rep}_2(k)} \equiv 1 \mod 2$

  • Normalize by a homothety of ratio $1/2^n$
  
  $\rightsquigarrow$ sequence belonging to $[0, 1] \times [0, 1]$
The first six elements of the sequence

Generalized Pascal triangles

Manon Stipulanti (ULg)
The tenth element of the sequence
A key result

Theorem [Leroy, Rigo, S., 2016]
The sequence of compact sets converges to a limit object $\mathcal{L}$.

“Simple” characterization of $\mathcal{L}$: topological closure of a union of segments described through a “simple” combinatorial property
Simplicity: coloring the cells of the grids regarding their parity

Extension

Everything still holds for binomial coefficients $\equiv r \mod p$ with

- base-2 expansions of integers
- $p$ a prime
- $r \in \{1, \ldots, p - 1\}$
Example with $p = 3$, $r = 2$

Left: binomial coefficients $\equiv 2 \mod 3$
Right: estimate of the corresponding limit object
## Counting positive binomial coefficients

### Generalized Pascal triangle in base 2

<table>
<thead>
<tr>
<th>$\binom{u}{v}$</th>
<th>$\varepsilon$</th>
<th>1</th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
<th>$n$</th>
<th>$S_2(n)$</th>
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<td>0</td>
<td>1</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

**Definition:** $S_2(n) = \# \left\{ m \in \mathbb{N} \mid \binom{\text{rep}_2(n)}{\text{rep}_2(m)} > 0 \right\}$ \quad \forall n \geq 0
The sequence \( (S_2(n))_{n \geq 0} \) in the interval \([0, 256]\)
• 2-\textit{kernel} of \( s = (s(n))_{n \geq 0} \)

\[
\mathcal{K}_2(s) = \{(s(n))_{n \geq 0}, (s(2n))_{n \geq 0}, (s(2n + 1))_{n \geq 0}, (s(4n))_{n \geq 0}, (s(4n + 1))_{n \geq 0}, (s(4n + 2))_{n \geq 0}, \ldots \}
\]

\[
= \{(s(2^i n + j))_{n \geq 0} \mid i \geq 0 \text{ and } 0 \leq j < 2^i \}
\]
Regularity in the sense of...

- **2-kernel** of \( s = (s(n))_{n \geq 0} \)

\[
\mathcal{K}_2(s) = \{(s(n))_{n \geq 0}, (s(2n))_{n \geq 0}, (s(2n + 1))_{n \geq 0}, (s(4n))_{n \geq 0}, (s(4n + 1))_{n \geq 0}, (s(4n + 2))_{n \geq 0}, \ldots \} \\
= \{(s(2^i n + j))_{n \geq 0} \mid i \geq 0 \text{ and } 0 \leq j < 2^i \}
\]

- **2-regular** if there exist

\[
(t_1(n))_{n \geq 0}, \ldots, (t_\ell(n))_{n \geq 0}
\]

s.t. each \((t(n))_{n \geq 0} \in \mathcal{K}_2(s)\) is a \(\mathbb{Z}\)-linear combination of the \(t_j\)'s
The sequence \((S_2(n))_{n \geq 0}\) satisfies, for all \(n \geq 0\),

\[
\begin{align*}
S_2(2n + 1) &= 3S_2(n) - S_2(2n) \\
S_2(4n) &= 2S_2(2n) - S_2(n) \\
S_2(4n + 2) &= 4S_2(n) - S_2(2n).
\end{align*}
\]
Theorem [Leroy, Rigo, S., 2017]

The sequence \( (S_2(n))_{n \geq 0} \) satisfies, for all \( n \geq 0 \),

\[
S_2(2n + 1) = 3S_2(n) - S_2(2n)
\]
\[
S_2(4n) = 2S_2(2n) - S_2(n)
\]
\[
S_2(4n + 2) = 4S_2(n) - S_2(2n).
\]

Corollary [Leroy, Rigo, S., 2017]

\( (S_2(n))_{n \geq 0} \) is 2-regular.
Theorem [Leroy, Rigo, S., 2017]
The sequence \((S_2(n))_{n \geq 0}\) satisfies, for all \(n \geq 0\),
\[
S_2(2n + 1) = 3S_2(n) - S_2(2n)
\]
\[
S_2(4n) = 2S_2(2n) - S_2(n)
\]
\[
S_2(4n + 2) = 4S_2(n) - S_2(2n).
\]

Corollary [Leroy, Rigo, S., 2017]
\((S_2(n))_{n \geq 0}\) is 2-regular.

\(\Rightarrow\) Matrix representation to compute \((S_2(n))_{n \geq 0}\) easily
### The Fibonacci case

**Definitions:**
- **Fibonacci sequence** \((F(n))_{n \geq 0}^n\): \(F(0) = 1\), \(F(1) = 2\) and
  \[
  F(n + 2) = F(n + 1) + F(n) \quad \forall n \geq 0
  \]
- \(\text{rep}_F(n)\) greedy Fibonacci representation of \(n \in \mathbb{N}_{>0}\) beginning by 1
- \(\text{rep}_F(0) := \varepsilon\) where \(\varepsilon\) is the empty word

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\text{rep}_F(n))</th>
<th>Evitability</th>
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</tr>
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</tr>
<tr>
<td>3</td>
<td>(1 \times F(2) + 0 \times F(1) + 0 \times F(0))</td>
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</tr>
<tr>
<td>4</td>
<td>(1 \times F(2) + 0 \times F(1) + 1 \times F(0))</td>
<td>101</td>
</tr>
<tr>
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Generalized Pascal triangle in base Fibonacci

⇝ Fibonacci representations ordered genealogically

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<th>101</th>
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<th>1001</th>
<th>1010</th>
<th>$n$</th>
<th>$S_F(n)$</th>
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**Definition:** $S_F(n) = \# \left\{ m \in \mathbb{N} \mid (\text{rep}_F(n))_{\text{rep}_F(m)} > 0 \right\} \quad \forall n \geq 0$
The sequence \((S_F(n))_{n \geq 0}\) in the interval \([0, 233]\)
2-kernel $\mathcal{K}_2(s)$ of a sequence $s$

- Select all the nonnegative integers whose base-2 expansion (with leading zeroes) ends with $w \in \{0, 1\}^*$
- Evaluate $s$ at those integers
- Let $w$ vary in $\{0, 1\}^*$

<table>
<thead>
<tr>
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<th>$\text{rep}_2(n)$</th>
<th>$s(n)$</th>
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<td>1</td>
<td>$s(1)$</td>
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<tr>
<td>5</td>
<td>101</td>
<td>$s(5)$</td>
</tr>
</tbody>
</table>

$F$-kernel $\mathcal{K}_F(s)$ of a sequence $s$

- Select all the nonnegative integers whose Fibonacci representation (with leading zeroes) ends with $w \in \{0, 1\}^*$
- Evaluate $s$ at those integers
- Let $w$ vary in $\{0, 1\}^*$

<table>
<thead>
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<td>$s(5)$</td>
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Property of the sequence \((S_F(n))_{n \geq 0}\)

\[ s = (s(n))_{n \geq 0} \text{ is } F\text{-regular if there exist } \]
\[ (t_1(n))_{n \geq 0}, \ldots, (t_\ell(n))_{n \geq 0} \]
\[ \text{s.t. each } (t(n))_{n \geq 0} \in \mathcal{K}_F(s) \text{ is a } \mathbb{Z}\text{-linear combination of the } t_j\text{'s} \]
Property of the sequence \( (S_F(n))_{n \geq 0} \)

\[ s = (s(n))_{n \geq 0} \] is \( F \)-regular if there exist

\[ (t_1(n))_{n \geq 0}, \ldots, (t_\ell(n))_{n \geq 0} \]

s.t. each \( (t(n))_{n \geq 0} \in \mathcal{K}_F(s) \) is a \( \mathbb{Z} \)-linear combination of the \( t_j \)'s

**Proposition [Leroy, Rigo, S., 2017]**

\( (S_F(n))_{n \geq 0} \) is \( F \)-regular.
Property of the sequence \((S_F(n))_{n \geq 0}\)

\[ s = (s(n))_{n \geq 0} \] is \(F\)-regular if there exist

\[ (t_1(n))_{n \geq 0}, \ldots, (t_{\ell}(n))_{n \geq 0} \]

s.t. each \((t(n))_{n \geq 0} \in \mathcal{K}_F(s)\) is a \(\mathbb{Z}\)-linear combination of the \(t_j\)’s

**Proposition [Leroy, Rigo, S., 2017]**

\((S_F(n))_{n \geq 0}\) is \(F\)-regular.

In the literature, not so many sequences that have this kind of property
Conclusion

Done:

- Generalized Pascal triangle and generalized Sierpiński gasket in base 2
- Regularity of \((S_2(n))_{n \geq 0}\), summatory function and asymptotics
- Regularity of \((S_F(n))_{n \geq 0}\), summatory function and asymptotics
- Extension to any integer base \(b \geq 2\): regularity of \((S_b(n))_{n \geq 0}\), summatory function and asymptotics

To do:

- Generalized Pascal triangle and generalized Sierpiński gasket: convergence for integer bases, Fibonacci numeration system, etc.
- Study of \(S\): extension to other numeration systems